Finitary Coloring

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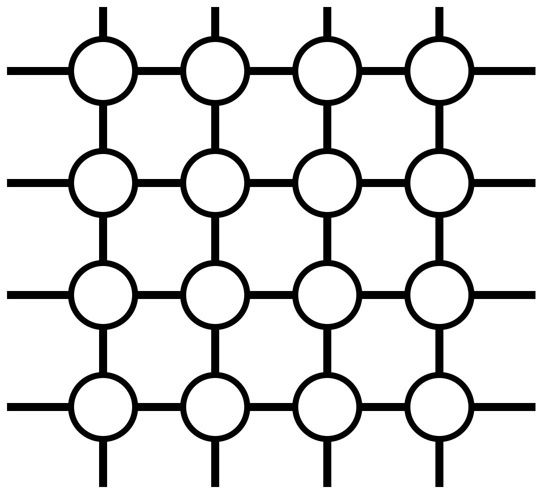
(Proper) q-coloring of graph G:

labelling of vertices with colors 1,...,q giving adjacent vertices different colors

E.g. Z^2



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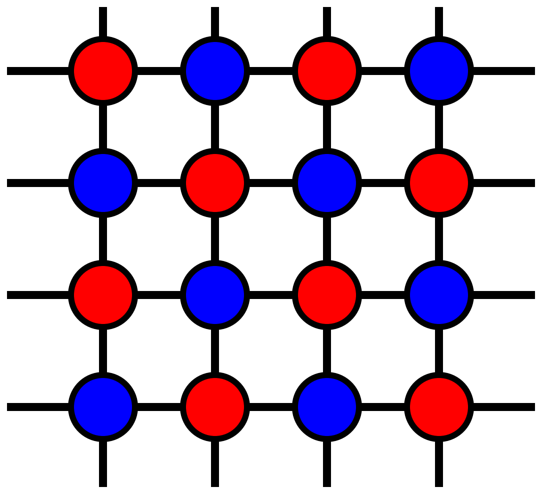


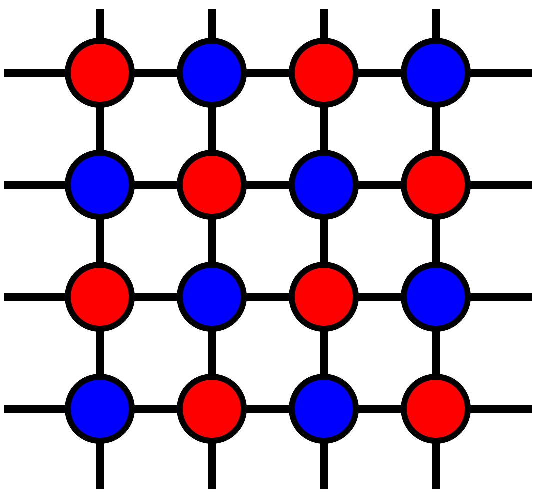
2-colouring

Can we color with no "central authority" - each vertex is an identical "independent agent"?

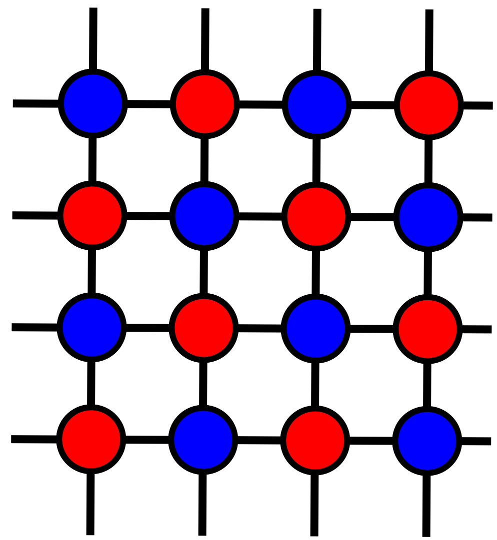
Random stationary colorings?

I.i.d. colors clearly impossible.
Approximate independence?
Trivial tails? Decay of correlations? Mixing?

E.g. Z^2



with prob 1/2



with prob 1/2

2-coloring (but long-range correlations)

Definition: a process $X=(X_v)_{v \in Z^d}$ is a *finitary factor of an iid process* (*ffiid*) if:

1. $X=f(U)$ for $U=(U_v)$ an iid process on Z^d
2. f is translation-equivariant:
   $f(T\,u) = T\,f(u)$ for any translation T
3. f is finitary:
   X_0 is determined by U on $[-R,R]$^d
   for some random $R < \infty$,
   (the coding radius)

I.e.: $\exists\, r=r(u)$ s.t.
$u=u'$ on $[-r,r]^d \Rightarrow f(u)_0=f(u')_0.$
$R:=r(U)$

Question: does there exist an ffiid coloring?
If so, how small can we make coding radius R?

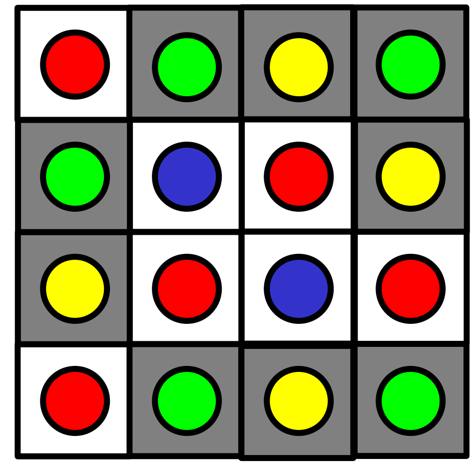
Not with q=2 colors on Z^d .
Not with $R \equiv 0$ (i.i.d.!).

Application:
Network of machines.
Colors represent updating schedules/
communication frequencies
(neighbors must not conflict).
Can the machines choose colors
locally, in **distributed** fashion? How locally?

E.g.

$\exists$ an ffiid 4-coloring of Z^2 with $P(R>r) \leq e^{-cr}$:

Label each vertex
black/white indep. w.p. $\frac{1}{2}$



Percolation theory:

$\frac{1}{2} < p_c^{site}(Z^2) \Rightarrow$ black/white clusters finite
and diam(cluster at 0) has exp tails

Checkerboard
white clusters in red/blue
black clusters in green/yellow

(starting from NE corner)

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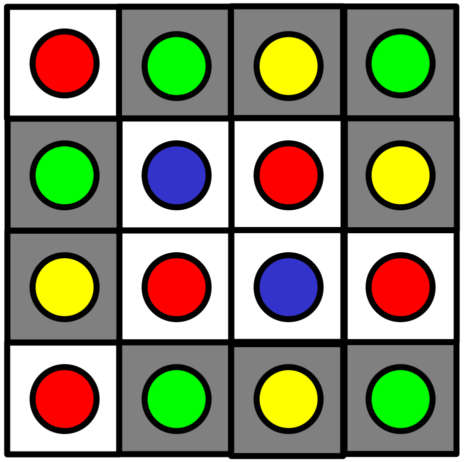

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(isometry-equivariant version: use argmax of iid U[0,1] rvs)

$\exists$ an ffiid 4-coluring of Z^2 with $P(R>r) \leq e^{-cr}$:

Can we do better? R bounded?

Other d?

Other numbers of colors?

Theorem: For Z^d, ~~$d \geq 2$~~: $d=1$

$\exists$ ffiid 3-coloring with $P(R>r) < r^{-a}$ ~~$\exists$ ffiid 3-coloring with $P(R>r) < r^{-a}$~~ (power law)

~~Any ffiid 3-coloring has $E(R^2) = \infty$~~

$\exists$ ffiid ~~4~~ 3-coloring with $P(R>r) < 1 / \underbrace{e^{e^{\cdot^{\cdot^{\cdot^{e}}}}}}_{cr}$

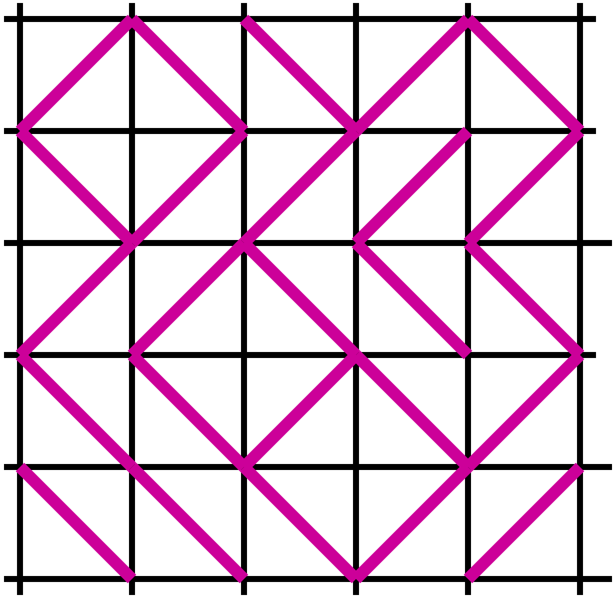
Any ffiid q-coloring has $P(R>r) > 1 / \underbrace{e^{e^{\cdot^{\cdot^{\cdot^{e}}}}}}_{Cr}$

(tower law)

($a, c, C \in (0,\infty)$ depending on d, q)

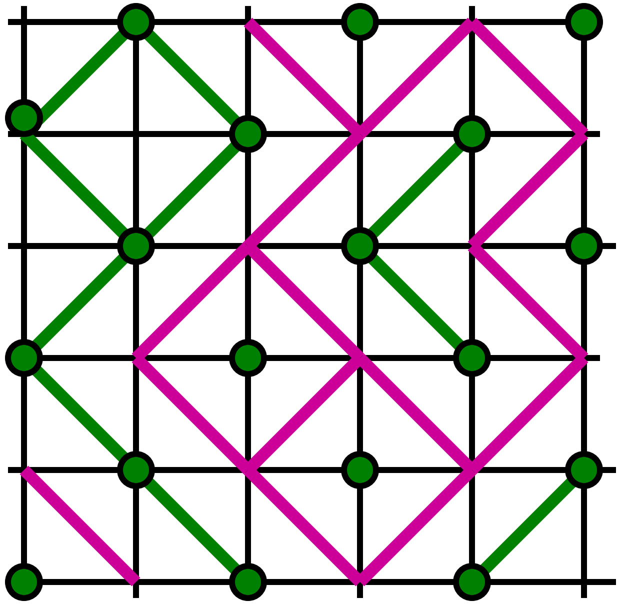
Proofs…

3-coloring Z^2 with power law tails:



Draw / or \ w.p. ½ in each square

3-coloring Z^2 with power law tails:



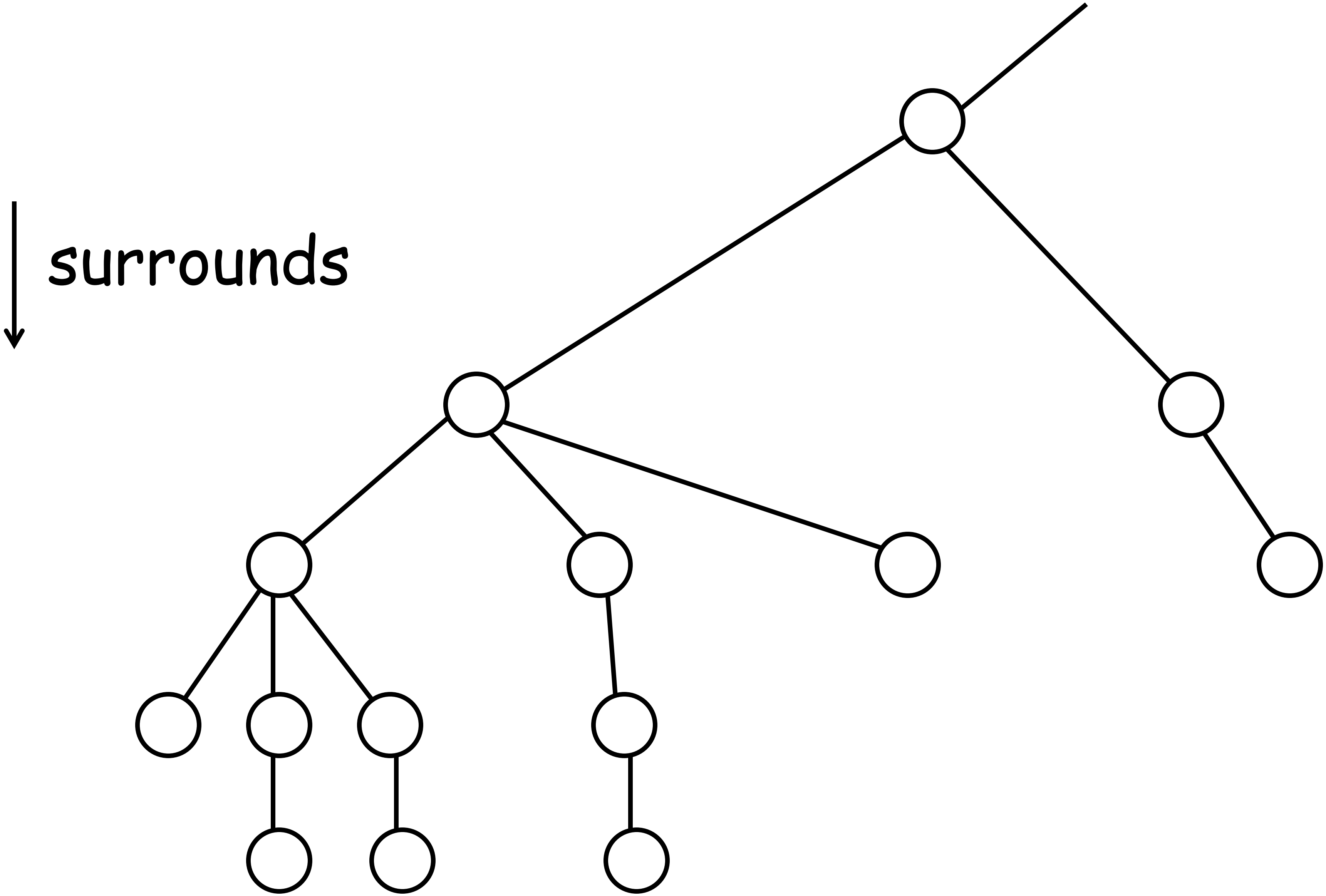
Draw / or \ w.p. $\frac{1}{2}$ in each square

On even sub-lattice, see critical bond percolation (clusters finite, power law tails)

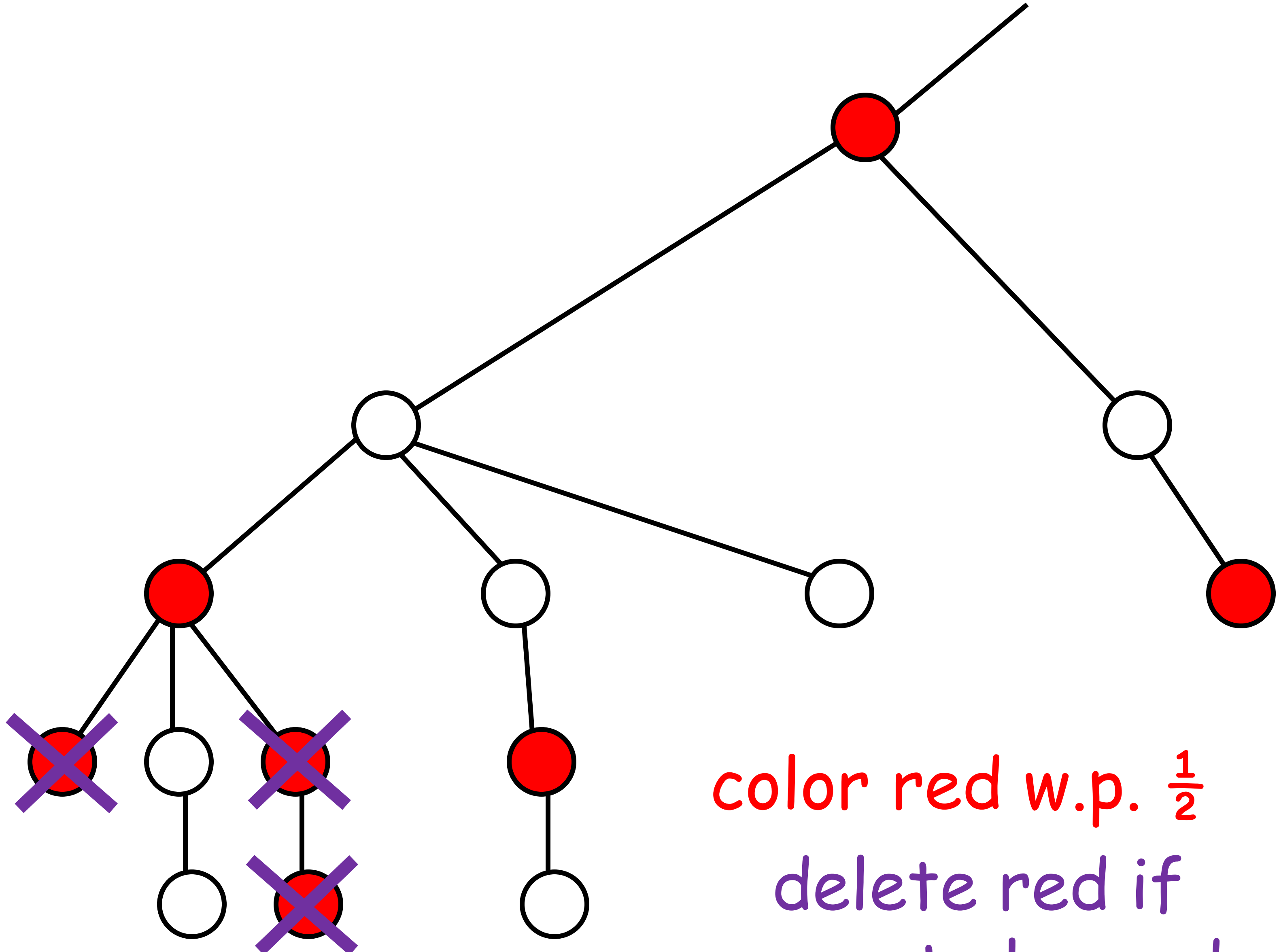
On odd sub-lattice, see dual perc. config.

Each cluster is surrounded by a cluster and vice versa. We'll give each cluster a color.

Adjacency graph of clusters:



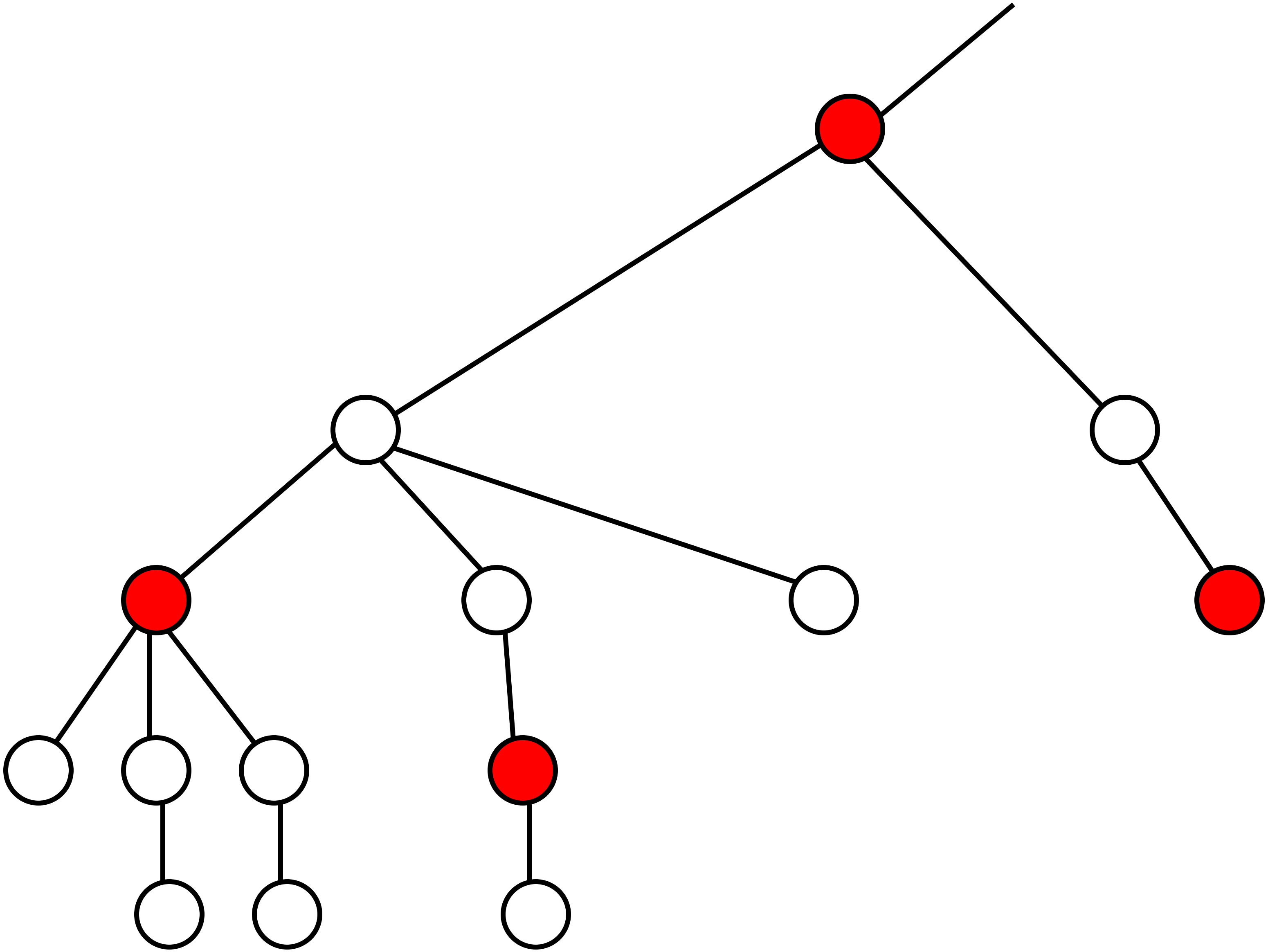
Adjacency graph of clusters:



color red w.p. $\frac{1}{2}$

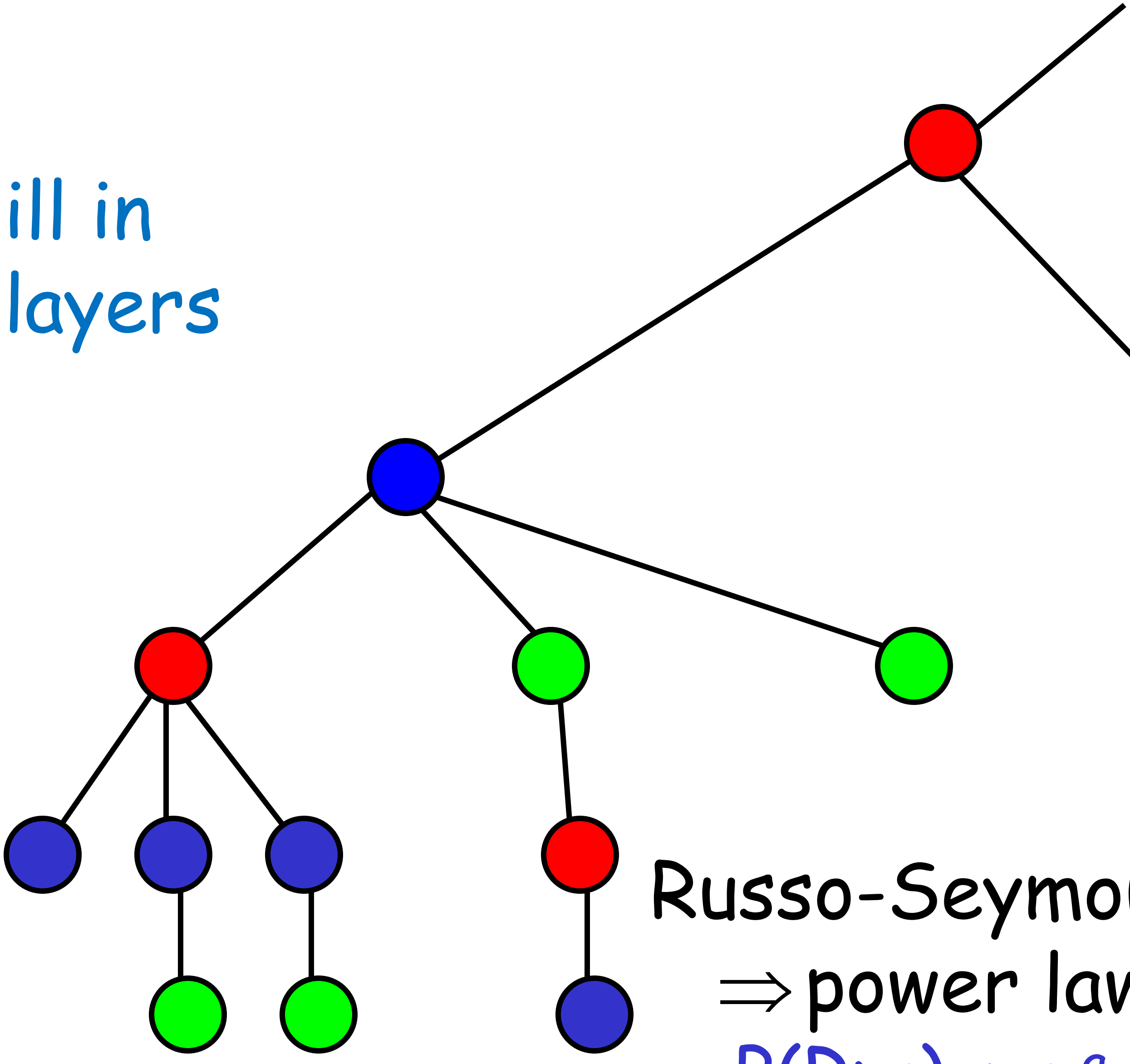
delete red if parent also red

Adjacency graph of clusters:



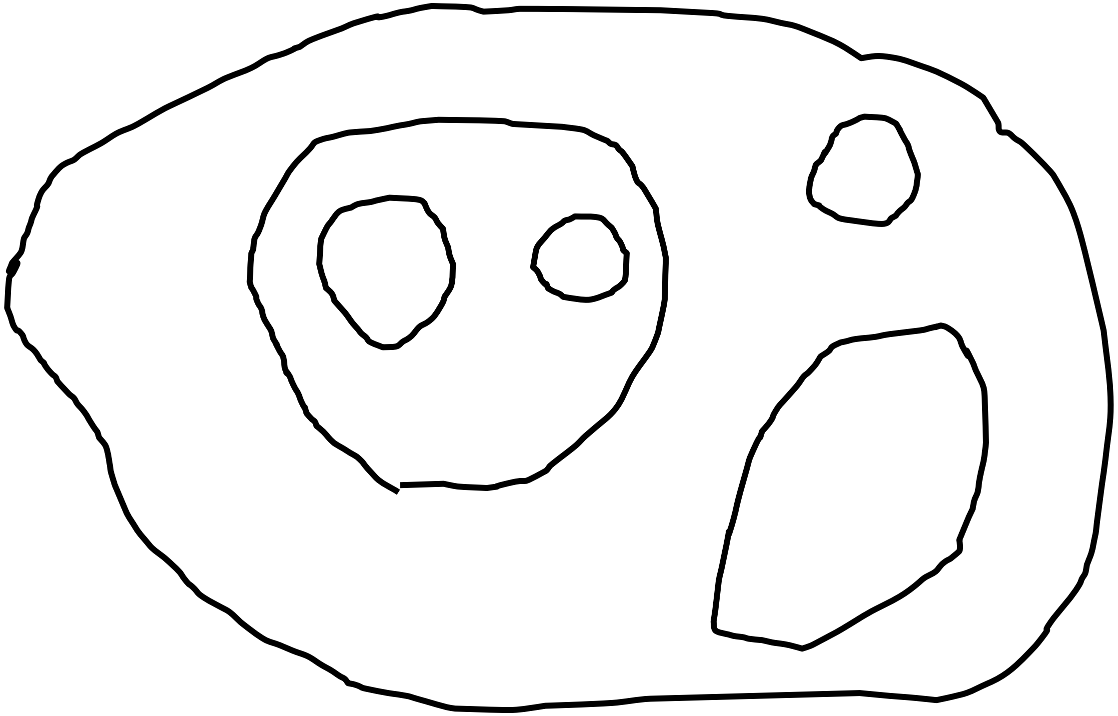
Adjacency graph of clusters:

fill in
in layers



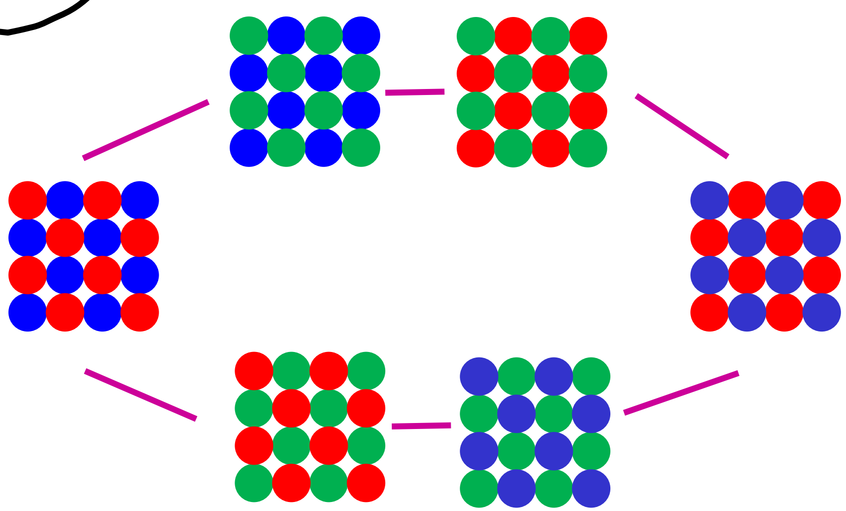
Russo-Seymour-Welsh
$\Rightarrow$ power law bound
$P(R>r) < r^{-a}$ $(a \ll 1)$

3-coloring Z^d with power law tails:

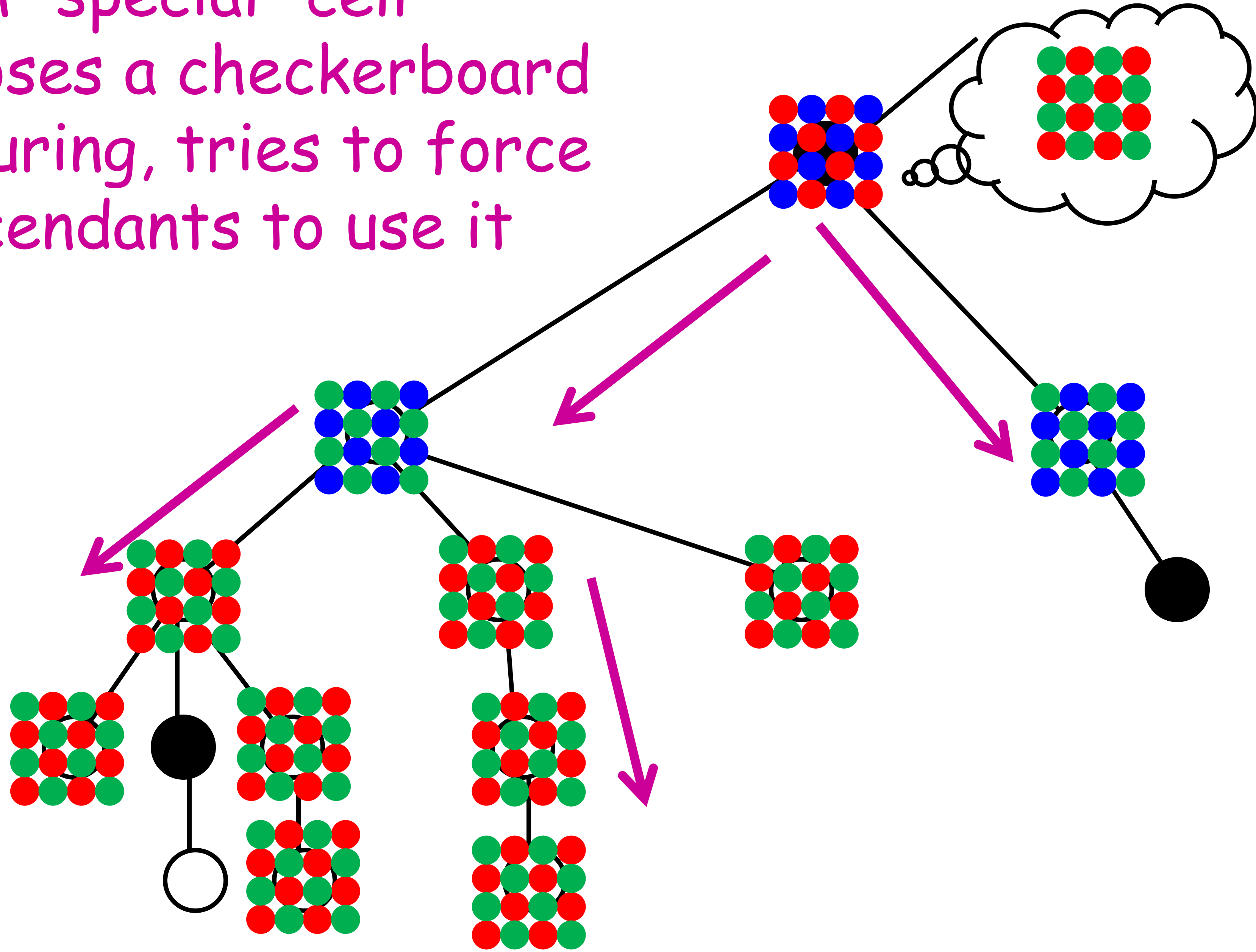


Hierarchical construction of partition of Z^d with tree structure, power tails

Color each cell with a checkerboard:



Each "special" cell chooses a checkerboard colouring, tries to force descendants to use it



Proof of lower bound $E(R^2) = \infty$ for 3-coloring on Z^2:

in fact, any stationary 3-coloring has slowly decaying (power law) correlations.

0	2	1	2	0	1
2	1	0	1	2	0
1	2	1	0	1	2
2	0	2	1	2	0

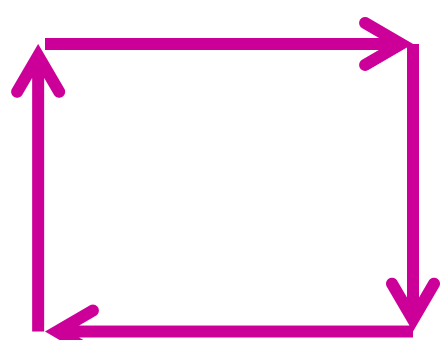
colouring

→

6	5	4	5	6	7
5	4	3	4	5	6
4	5	4	3	4	5
5	6	5	4	5	6

height function

height change around contour must be 0



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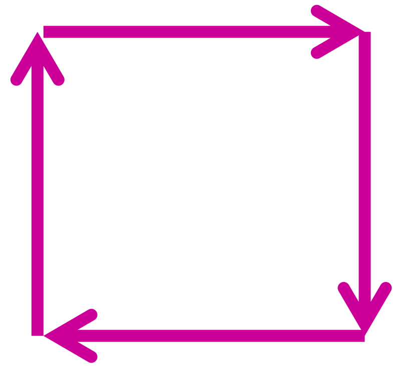


tail triviality $\Rightarrow$ Var(change along side) large

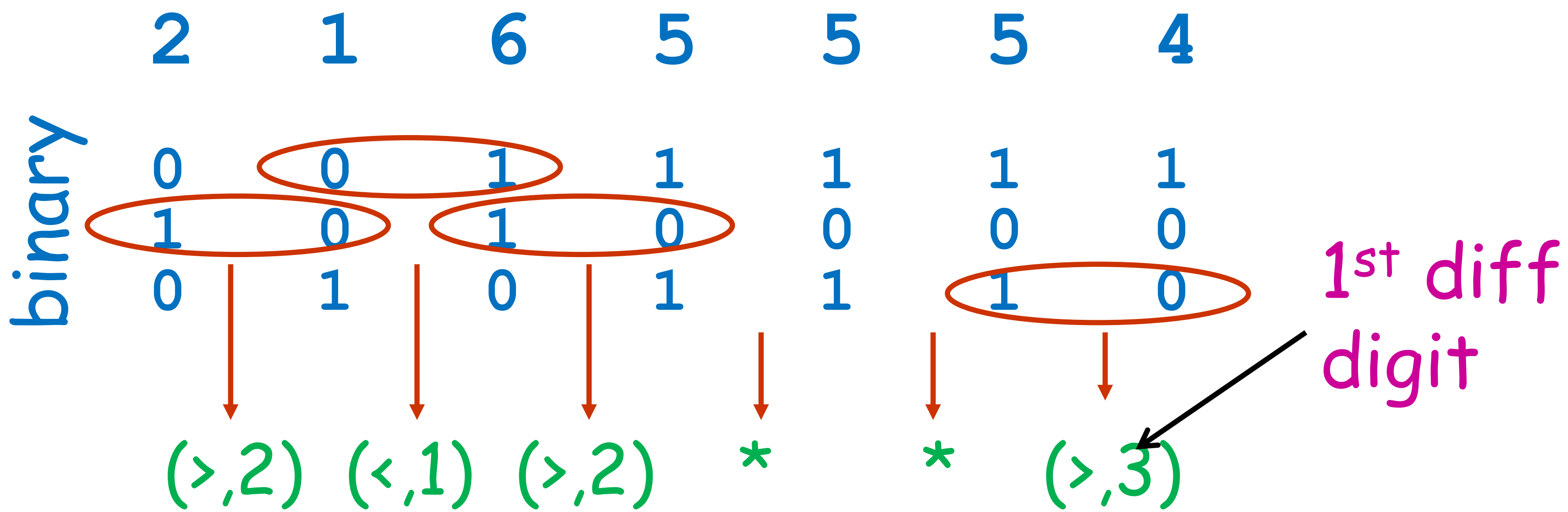
(Lemma: $(Y_i)_{i \in Z}$ ±1-valued, stationary, right-tail-trivial, not deterministic $\Rightarrow \limsup_{n\to\infty} \mathrm{Var} \sum_1^n Y_i = \infty$ a.s.)

fast decay of correlations $\Rightarrow$
changes along sides approx independent

$\Rightarrow$ contradiction.

Tower colouring on Z with *some* # of colors:

Reduction 2^n-labelling → (2n+1)-labelling
(essentially Cole, Vishkin, 1986)



Get a color-clash only where original sequence had one.

Doing this k times, starting from i.i.d., get P(clash) < 1/tower(ck), 6+1 colors, $R \leq \lceil k/2 \rceil$.

"Stitch together" these almost-colorings for different k to get ffiid 6-coloring of Z

$\Rightarrow 6^d$-coloring on Z^d

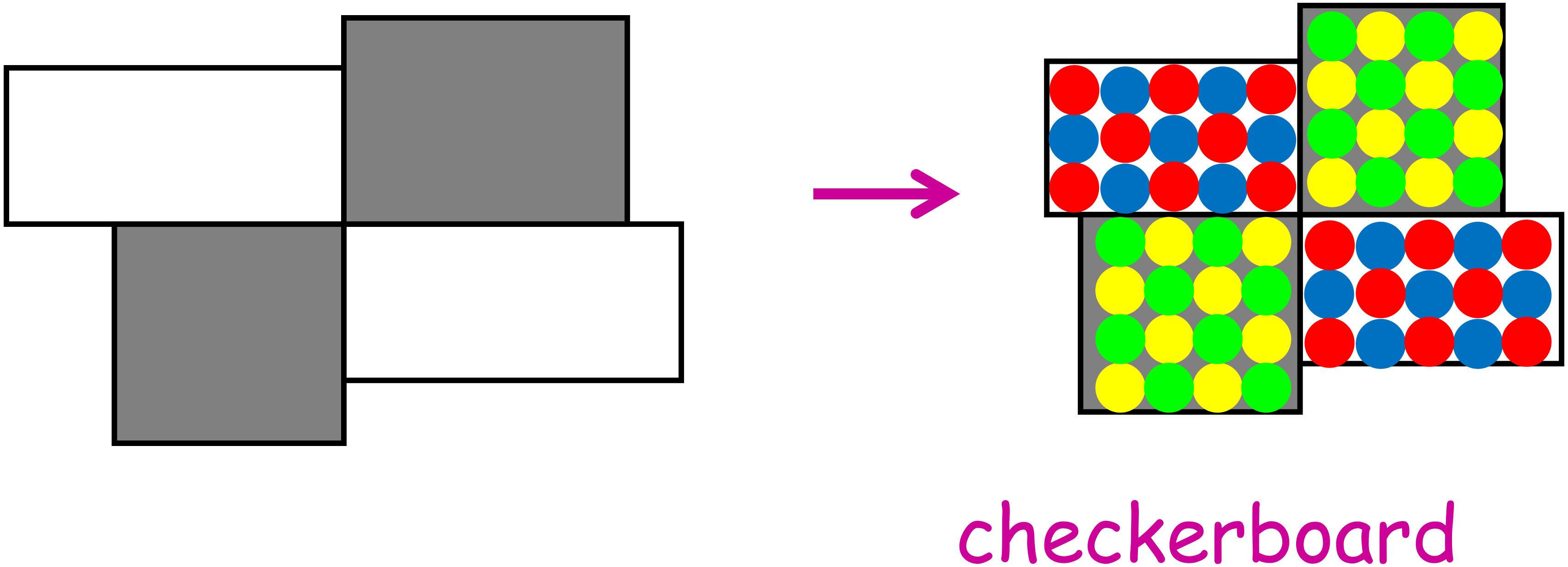
Reduce # colors to degree+1 by elimination:

5 3 6 2 1 6 2 1 6 5

5 3 1 2 1 3 2 1 2 5

Reduction to 4 colors on Z^d:

long-range colorings, maximal indep sets,
local modifications $\Rightarrow$
tower-ffiid 2-labelling with bounded clusters



Proof of lower bound P(R>r) > 1/tower(Cr)

Key step: (essentially M. Naor 1991; arguably Ramsey 1930)

For (U_i) i.i.d., $f: R^r \to \{1, \dots, q\}$,

$$\mathbb{P}[f(U_1, \dots, U_r) = f(U_2, \dots, U_{r+1})] > \frac{1}{\underbrace{2^{2^{2^{\cdot^{\cdot^{\cdot^{(4q)}}}}}}}_{\text{height } r}}$$

height r

(essentially tight!)

Proof that for (U_i) i.i.d. and $f: R^r \to \{1, \ldots, q\}$,

$$P[f(U_1, \ldots, U_r) = f(U_2, \ldots, U_{r+1})] > 0:$$

with positive probability

Induction on r. r = 1 easy (i.i.d.).

r ≥ 2:

$$S(u_1, \ldots, u_{r-1}) := \{a : f(u_1, \ldots, u_{r-1}, U_r) = a \textbf{ wpp}\}.$$

S takes $\leq 2^q$ values;

induction $\Rightarrow S(U_1, \ldots, U_{r-1}) = S(U_2, \ldots, U_r)$ wpp.

So $\exists$ a,A s.t. wpp:

$$S(U_1, \ldots, U_{r-1}) = A = S(U_2, \ldots, U_r)$$

$$\Cup$$

$$f(U_1, \ldots, U_r) = a = f(U_2, \ldots, U_{r+1}) \qquad \square$$

Beyond colouring:

A shift of finite type is a subset of $\{1,\ldots,q\}^{Z^d}$ determined by insisting that all k-boxes lie in some given $A \subset \{1,\ldots,q\}^{[0,k]^d}$

E.g.: q-coloring on Z: $A=\{(x,y): x \neq y\}$

Theorem: For d=1 and any shift of finite type S, either:

1. there is *no* ffiid process in S ("periodicity" obstruction) (e.g. 2-colouring)

2. $\exists$ ffiid process in S with $P(R>r) < 1/\text{tower}(cr)$ any ffiid process in S has $P(R>r) > 1/\text{tower}(Cr)$ (e.g. 3-colouring)

or

3. some *constant* sequence lies in S (so ffiid with "R=0") (e.g. $q=1$, $S=\{1\}^{Z}$)

For a shift of finite type in $d \geq 2$, can have:

No ffiid process (e.g. 2-col)

Power law ffiid process (e.g. 3-col)

Tower law ffiid process (e.g. 4-col)

Constant (R=0) process possible
(e.g. "no restriction")

<u>Q:</u> Is any other behaviour possible?

Beyond finitary factors:

Process $X=(X_i)_{i\in Z}$

X is a k-block factor if $X_i=g(U_{i+1},\ldots,U_{i+k})$, (U_i) iid

(Ffiid process with $R\leq k \Leftrightarrow$ (2k+1)-block factor)

Theorem $\Rightarrow$ no block factor colourings.

Stationary process X is k-dependent if

$$(\ldots,X_{-2},X_{-1}) \perp\!\!\!\perp (X_k,X_{k+1},\ldots)$$

k-block factor $\Rightarrow$ stationary, (k-1)-dependent

$\Leftrightarrow$? (Ibragimov, Linnik, 1965)

No! (Aaronson, Gilat, Keane, de Valk, 1989)

Longstanding Q: "Natural" counterexample?

Coloring leads to an answer!

Thm (H., Liggett): $\exists$ 1-dependent 4-coloring!