FPTAS for #BIS with degree bounds on one side

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Outline

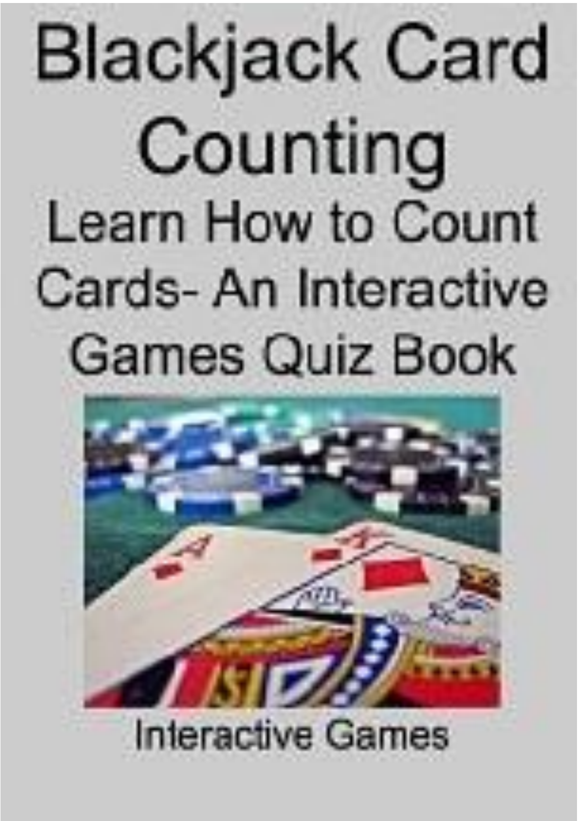
- Counting and probability distribution
- Correlation decay approach
- FPTAS for #BIS with degree bounds on one side

Counting Problems

- SAT: Is there a satisfying assignment for a given a CNF formula?
- Counting SAT: How many?
- Counting Colorings of a graph
- Counting Independent sets of a graph
- Counting perfect matchings of a bipartite graph (Permanent)
- ……

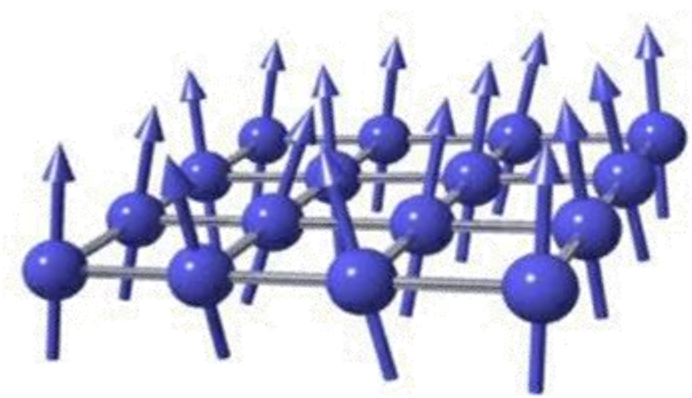
Counting Problems

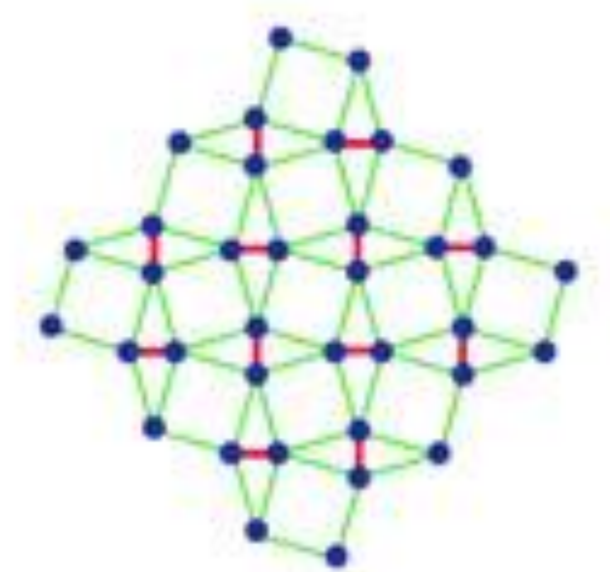
- Probability



Counting Problems

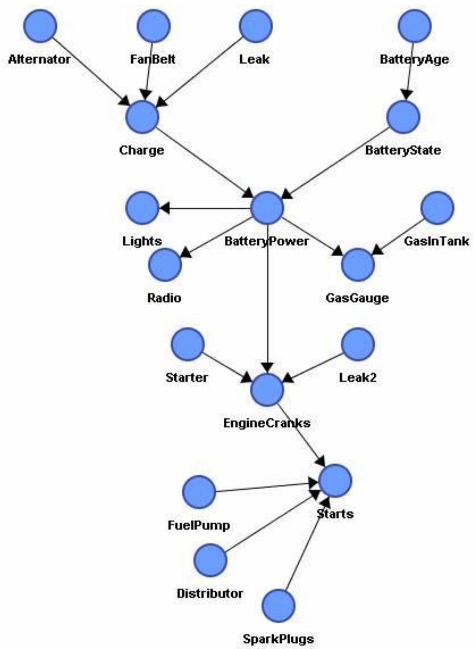
- Probability
- Partition function on statistical physics

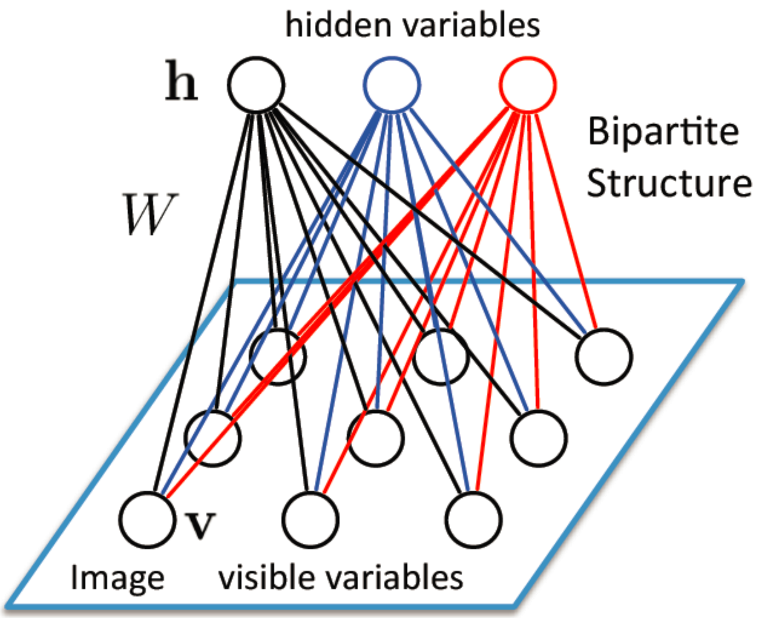




Counting Problems

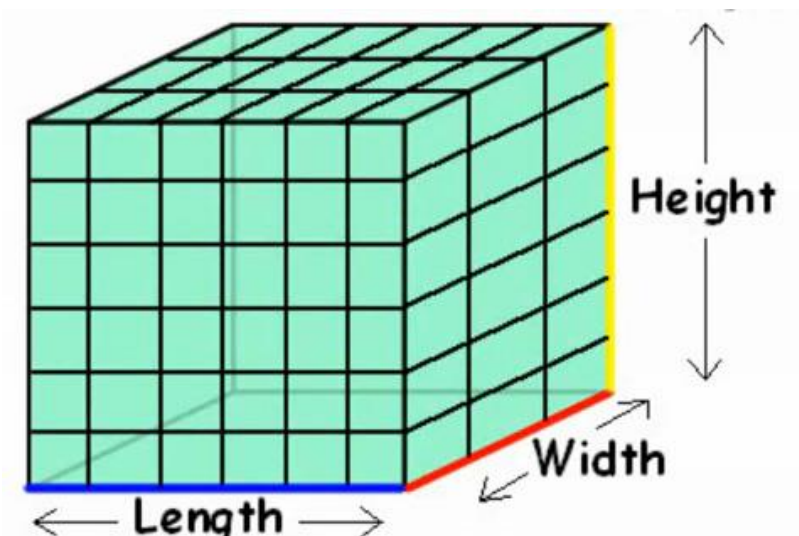
- Probability
- Partition function on statistical physics
- Inference on Graphical Models

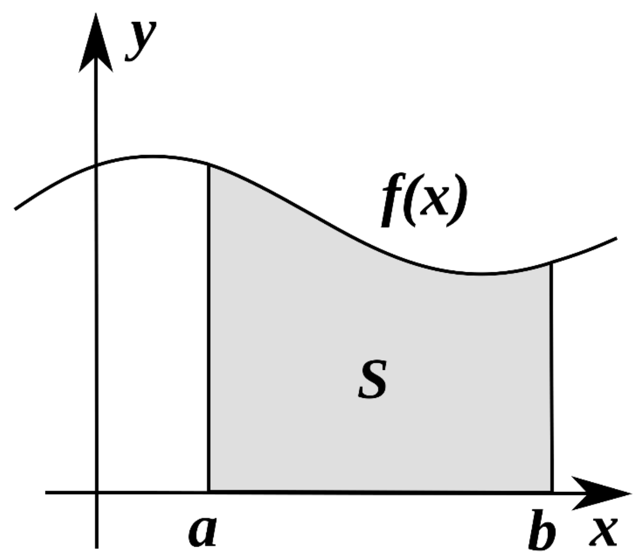




Counting Problems

- Probability
- Partition function on statistical physics
- Inference on Graphical Models
- Volume estimation and numerical integration





Counting Problems

- Probability
- Partition function on statistical physics
- Inference on Graphical Models
- Volume estimation and numerical integration
- Query on probabilistic database
- Optimization on stochastic model
- ……

Approximate Counting

- Let $\epsilon > 0$ be an approximation parameter and Z be the correct counting number of the instance, the algorithm returns a number Z' such that $(1-\epsilon)Z \leq Z' \leq (1+\epsilon)Z$, in time ploy($n$,1/$\epsilon$).
- Fully polynomial-time approximation scheme (FPTAS).
- Fully polynomial-time randomized approximation scheme (FPRAS) is its randomized version.

Counting vs Distribution

- IS(G): the set of independent sets of graph G
- X is chosen from IS(G) uniformly at random
- $P_G(v)$: the probability that v is not in X
- $\Pr(X = \emptyset) = \frac{1}{|IS(G)|}$
- $\Pr(X = \emptyset) = P_{G_1}(v_1) P_{G_2}(v_2) \ldots P_{G_n}(v_n)$, where $G_1 = G, G_{i+1} = G_i - v_i$

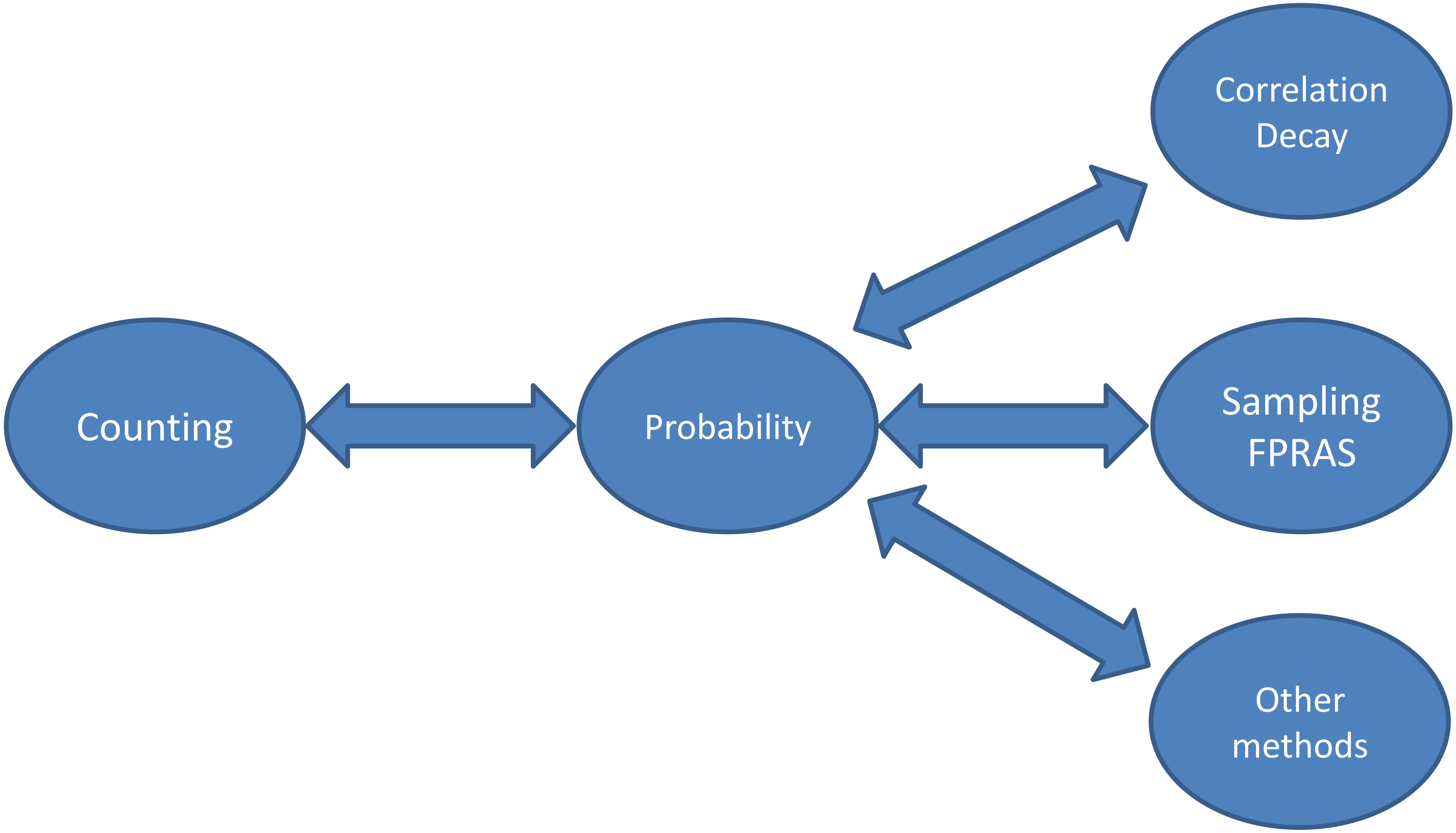
Counting vs Distribution

- $\frac{1}{|IS(G)|} = P_{G_1}(v_1)P_{G_2}(v_2)\ldots P_{G_n}(v_n)$
- If we can compute (estimate) $P_G(v)$, we can (approximately) compute |IS(G)|.

- FPRAS: Estimate $P_G(v)$ by sampling
- FPTAS: Approximately compute $P_G(v)$ directly and deterministically

Counting vs Distribution

Counting vs Distribution



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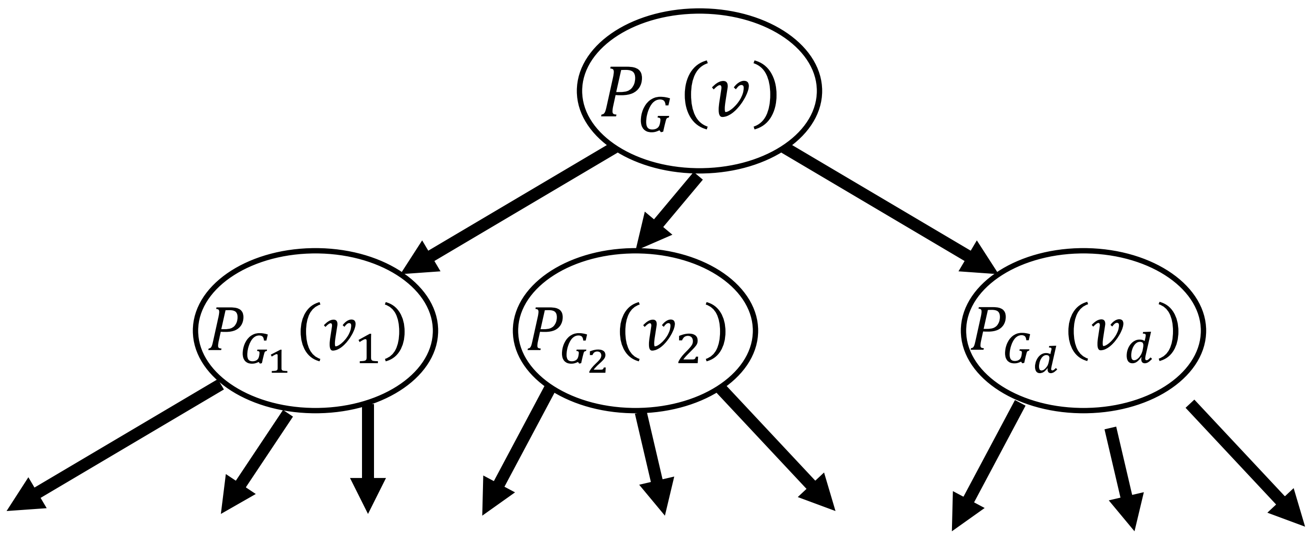
Computation Tree

- Relate the probability $P_G(v)$ to these of its neighbors in smaller instances.

- $P_G(v) = \mathrm{f}(\mathrm{P}_{\mathrm{G}_1}(v_1), \mathrm{P}_{\mathrm{G}_2}(v_2), \ldots, \mathrm{P}_{\mathrm{G}_\mathrm{d}}(v_\mathrm{d}))$



An example of independent sets

- $P_G(v) = \frac{Z(G-v)}{Z(G)} = \frac{Z(G-v)}{Z(G-v)+Z(G-v-N(v))}$

  $= \frac{1}{1+Z(G-v-N(v))/Z(G-v)}$

- $\frac{Z(G-v-N(v))}{Z(G-v)} = P_{G-v}(v_1 v_2 \cdots v_d)$

  $= P_{G_1}(v_1) P_{G_2}(v_2\ ) \ldots P_{G_d}(v_d)$

  where $G_1 = G - v, G_{i+1} = G_i - v_i$

- $P_G(v) = \frac{1}{1+P_{G_1}(v_1)P_{G_2}(v_2\ )\ldots P_{G_d}(v_d)}$

Computation Tree

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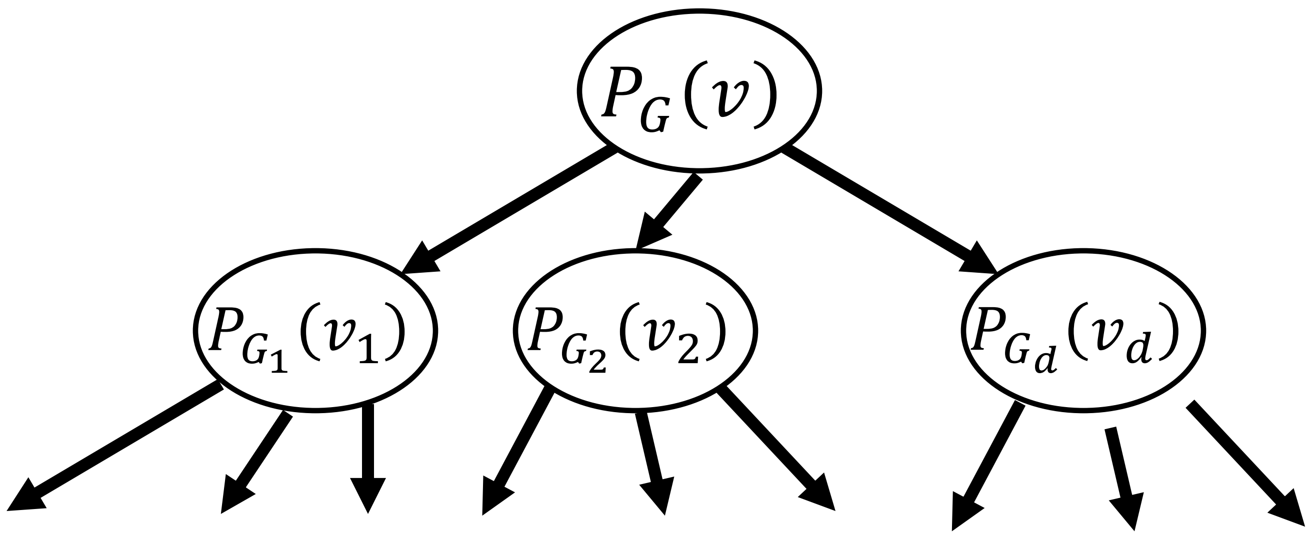

Correlation Decay

Truncate the computation tree at depth L, we can compute an estimation $P_G^L(v)$.

The system is called of exponential correlation decay if

$$\left|P_G^L(v) - P_G(v)\right| \leq \exp(-L).$$

We can estimate $P_G(v)$ by set $L = O(\log n + \log\frac{1}{\epsilon})$

Computational Efficient Correlation Decay

- M-based depth:
  - $L_M(root) = 0$;
  - $L_M(u) = L_M(v) + \lceil \log_M(d + 1) \rceil$, if u is one of the d children of v.
- Exponential correlation decay with respect to M-based depth.
- Computational efficient correlation decay supports FPTAS for general graph.

Proof Sketch for Correlation Decay

- $p = f(p_1, p_2, \dots, p_d)$

- Estimate the error for one recursive step:

$$\epsilon = \frac{\partial f}{\partial p_1}\epsilon_1 + \frac{\partial f}{\partial p_2}\epsilon_2 + \cdots + \frac{\partial f}{\partial p_d}\epsilon_d$$

$$|\epsilon| \leq \left(\left|\frac{\partial f}{\partial p_1}\right| + \left|\frac{\partial f}{\partial p_2}\right| + \cdots + \left|\frac{\partial f}{\partial p_d}\right|\right)\max(|\epsilon_i|)$$

- $\left(\left|\frac{\partial f}{\partial p_1}\right| + \left|\frac{\partial f}{\partial p_2}\right| + \cdots + \left|\frac{\partial f}{\partial p_d}\right|\right) < 1$?

Potential Function

- This may not be correct stepwise. We use a potential function to amortize it.
- Let $\phi: R^+ \to R^+$ be a bijective function. $q = \phi(p), q_i = \phi(p_i)$.
- $q = \phi\left(f(\phi^{-1}(q_1), \phi^{-1}(q_2), \dots, \phi^{-1}(q_d))\right)$.
- Then we show that the error for q is stepwise decreased by a constant factor.
- The main difficulty is to find the potential function ϕ.

Some examples

- Independent set [Weitz 06]
- Graph coloring [Gamarnik, Katz 07][L., Yin 13]
- Anti-Ferromagnetic 2-spin system [Li, L., Yin 12,13] [Sinclair, Srivastava, Thurley 12]
- Ferromagnetic 2-spin system [Guo, L. 15]
- Edge covers [Lin, Liu, L. 14] [Liu, L., Zhang 14]
- Monotone CNF [Liu, L. 15][L. Yang, Zhang 16] [Bezakova, Galanis, Goldberg, Guo,Stefankovic 16]
- Hyper graph matching [Liu, L. 15] [Yin, Zhao 15]
- …

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Counting Independent Sets (#IS)

- NP-hard (reduced from maximum IS)
- FPRAS for maximum degree 3 and then 4 [Dyer, Greenhill 00][Luby, Vigoda 97]
- NP-hard if Δ achieves 25 (reduced from Max Cut) [Dyer, Frieze, Jerrum 02]
- FPTAS for $\Delta \leq 5$ [Weitz 06]
- NP-hard if $\Delta = 6$ (reduced from Max Cut) [Sly 10]

Correlation decay for #IS

- $P_G(v) = \frac{1}{1+P_{G_1}(v_1)P_{G_2}(v_2\ )\ldots P_{G_d}(v_d)}$
- Define: $\mathrm{R_G(v)} = \frac{1-P_G(v)}{P_G(v)}$
- $\mathrm{R_G(v)} = \prod_{i=1}^{d} \frac{1}{1+\mathrm{R}_{\mathrm{G}_i}(\mathrm{v}_i)}$
- Define: $f(\boldsymbol{x}) = \prod_{i=1}^{d} \frac{1}{1+x_i}$

Correlation decay for #IS (cont.)

- $f(\boldsymbol{x}) = \prod_{i=1}^{d} \frac{1}{1+x_i}$
- Find $\Phi(\cdot)$ such that $\sum_i^d \left|\frac{\partial f}{\partial x_i}\right| \frac{\Phi(f)}{\Phi(x_i)} < 1$
- Choose $\Phi(x) = \frac{1}{\sqrt{x(1+x)}}$
- $\sum_i^d \left|\frac{\partial f}{\partial x_i}\right| \frac{\Phi(f)}{\Phi(x_i)} = \sqrt{\frac{f}{1+f}} \sum_i \sqrt{\frac{x_i}{1+x_i}}$
- Symmetrize x_i by Jensen's inquality and get a function with a single variable
- It works!

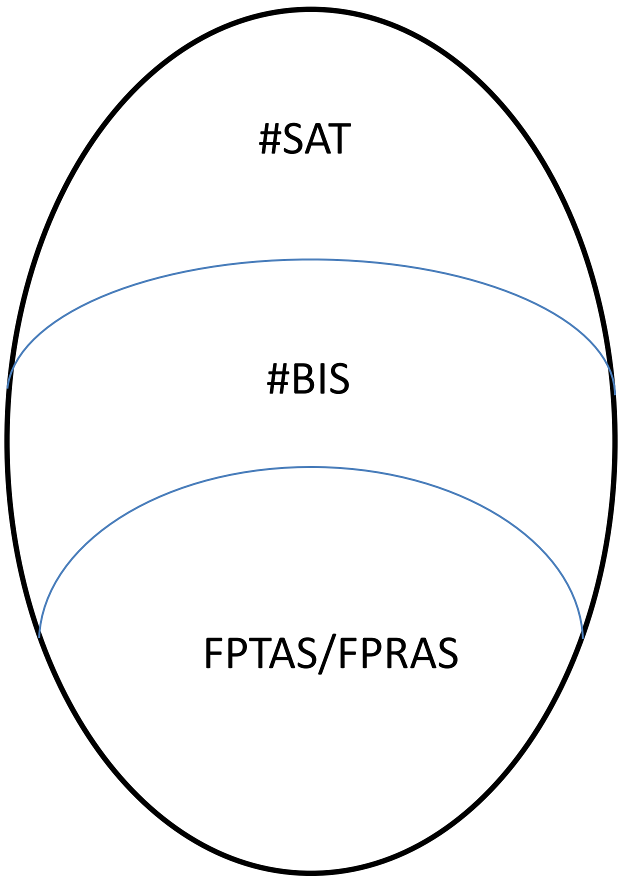
#BIS

- Counting independent sets for bipartite graphs.
- Local MCMCs mixes slowly even on bipartite graphs.
- Max IS and Max-CUT are easy for bipartite graphs.
- #BIS with a maximum degree of 6 is already as hard as general #BIS (#BIS-hard). [CGGGJSV14]
- No algorithmic evidence to distinguish #BIS from #IS.

#BIS

- Conjectured to be of intermediate complexity.
- Plays a similar role as the Unique Game for optimization problems.
- A large number of other problems are proved to have the same complexity as #BIS (#BIS-equivalent) or at least as hard as #BIS (#BIS-hard)

Some approximation trichotomies



- Boolean CSP [Dyer, Goldberg, Jerrum 10]
- Degree bounded Boolean CSP [Dyer, Goldberg, Jalsenius, Richerby 12]
- Conservative weighted Boolean CSPs[Bulatov, Dyer, Goldberg, Jerrum, McQuillan 13]

Our new algorithm

- FPTAS for #BIS when the maximum degree for one side is no larger than 5. [Liu, L. 2015]
- No restriction for the degrees on the other side.
- Identical to Weitz's algorithm for general #IS.
- Combine two recursion steps into one, and work with this two-layer recursion instead.

Two-layer recursion

- $\mathrm{R_G(v)} = \prod_{i=1}^{d} \frac{1}{1+\mathrm{R}_{\mathrm{G}_i}(\mathrm{v}_i)}$
- $\mathrm{R}_{\mathrm{G}_i}(\mathrm{v_i}) = \prod_{j=1}^{d_i} \frac{1}{1+\mathrm{R}_{\mathrm{G}_{ij}}(\mathrm{v}_{ij})}$
- $\mathrm{R_G(v)} = \prod_{i=1}^{d} \left(1 + \prod_{j=1}^{d_i} \frac{1}{1+\mathrm{R}_{\mathrm{G}_{ij}}(\mathrm{v}_{ij})}\right)^{-1}$
- Define $f(\boldsymbol{x})$=$\prod_{i=1}^{d} \left(1 + \prod_{j=1}^{d_i} \frac{1}{1+x_{ij}}\right)^{-1}$

Choice of the Potential Function

Recall: $f(\boldsymbol{x}) = \prod_{i=1}^{d} \left(1 + \prod_{j=1}^{d_i} \frac{1}{1+x_{ij}}\right)^{-1}$

$$\sum_{i,j} \left|\frac{\partial f}{\partial x_{ij}}\right| \frac{\Phi(f)}{\Phi(x_{ij})}$$

$$= \Phi(f) f \sum_i \frac{\prod_{j=1}^{d_i} \frac{1}{1+x_{ij}}}{1 + \prod_{j=1}^{d_i} \frac{1}{1+x_{ij}}} \sum_j \frac{1}{\Phi(x_{ij})(1+x_{ij})}$$

Choose: $\Phi(x) = \frac{1}{(1+x)Log(1+x)}$

$$Log \prod_{j=1}^{d_i} (1 + x_{ij})$$

- $\Phi(x) = \frac{1}{(1+x)Log(1+x)}$
- $s_i \triangleq \prod_{j=1}^{d_i} \frac{1}{(1+x_{ij})}$
- $f(\boldsymbol{x}) = \prod_{i=1}^{d} \frac{1}{1+s_i}$
- $\sum_{i,j} \left| \frac{\partial f}{\partial x_{ij}} \right| \frac{\Phi(f)}{\Phi(x_{ij})} = -\Phi(f) f \sum_i \frac{s_i}{1+s_i} Log\,(s_i)$
- Symmetrize s_i and get a function with a single variable
- It works!

Adaptive Truncation

- $\mathrm{R_G(v)} = \prod_{i=1}^{d}\left(1 + \prod_{i=1}^{d_i}\frac{1}{1+\mathrm{R}_{\mathrm{G}_{ij}}(\mathrm{v}_{ij})}\right)^{-1}$
- d is bounded by 4 but there is no bound for d_i
- Some d_i may even be super constant
- Computational efficient correlation decay

Analogy with MCMC

- Algorithm design part is kind of standard.
- The over framework for the analysis is the same.
- A number of techniques and tools have been developed to the analysis.
- However, it is still very challenging to prove and many problems remains open.

Thank You!

Q & A