The Classification Program for Counting Problems III

Planar Dichotomy Theorems

Heng Guo

Queen Mary, University of London

Simons Institute

Jan 28, 2016

Contents

#CSP

- Counting Constraint Satisfaction Problems:

#CSP

- Counting Constraint Satisfaction Problems:
  - V a set of variables and C a set of constraints.
  - C can be also viewed as hyperedges.

#CSP

- Counting Constraint Satisfaction Problems:
  - V a set of variables and C a set of constraints.
  - C can be also viewed as hyperedges.
- **Name** #CSP($\mathcal{F}$)

  Instance A bipartite graph $G = (V, C, E)$ and a mapping $\pi : C \to \mathcal{F}$

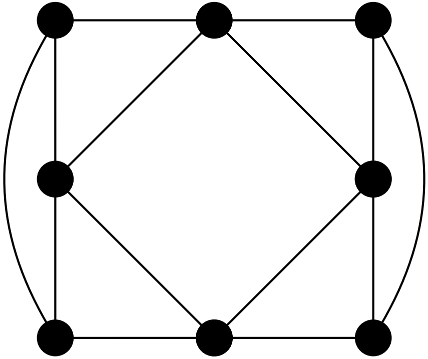
  Output The quantity:

$$\sum_{\sigma: V \to \{0,1\}} \prod_{c \in C} f_c \left(\sigma \mid_{N(c)} \right),$$

where $N(c)$ are the neighbors of c and $f_c = \pi(c) \in \mathcal{F}$.

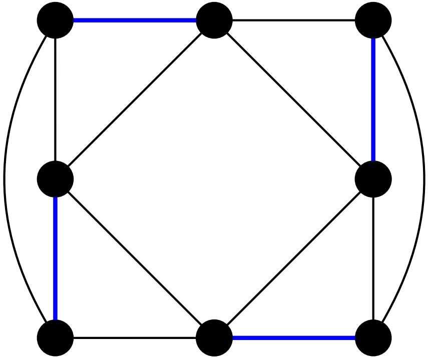
Counting Perfect Matchings

Perfect Matchings



Counting Perfect Matchings

Perfect Matchings



Holant Problems

- #PM is provably not expressible in vertex assignment models.
  (see e.g. [Freedman, Lovász, and Schrijver 07])

Holant Problems

- #PM is provably not expressible in vertex assignment models.

  (see e.g. [Freedman, Lovász, and Schrijver 07])

- Edge-coloring models — edges are variables and vertices are functions.

Holant Problems

- #PM is provably not expressible in vertex assignment models.
  (see e.g. [Freedman, Lovász, and Schrijver 07])
- Edge-coloring models — edges are variables and vertices are functions.
- **Name** Holant($\mathcal{F}$)

  Instance A graph $G = (V, E)$ and a mapping $\pi : V \to \mathcal{F}$

  Output The quantity:

$$\sum_{\sigma : E \to \{0,1\}} \prod_{v \in V} f_v \left(\sigma \mid_{E(v)} \right),$$

  where $E(v)$ are the incident edges of v and $f_v = \pi(v) \in \mathcal{F}$.

- More general than #CSP:

$$\#\text{CSP}(\mathcal{F}) \equiv_T \text{Holant}(\mathcal{EQ} \cup \mathcal{F}),$$

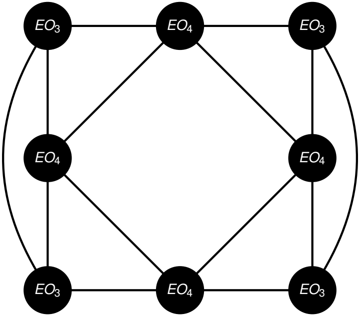
  where $\mathcal{EQ} = \{=_1, =_2, =_3, \dots\}$ is the set of equalities of all arities.
- Equivalent formulation: Tensor network contraction . . .
- Pl-Holant$(\mathcal{F})$ denotes the version where instances are all planar.

#PM as a Holant

- Put functions ExactOne (EO) on nodes (edges are variables).

#PM as a Holant

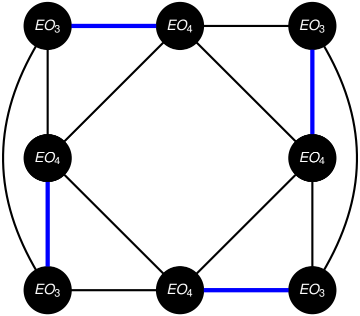
- Put functions ExactOne (EO) on nodes (edges are variables).



#PM as a Holant

- Put functions ExactOne (EO) on nodes (edges are variables).
- #PM is then the partition function:

$$\#\mathrm{PM} = \sum_{\sigma : E \to \{0,1\}} \prod_{v \in V} \mathrm{EO}_d(\sigma|_{E(v)}).$$



Counting Perfect Matchings in Planar Graphs

- Counting Perfect Matchings (#PM) is #**P**-hard [Valiant 79] in general graphs.

Counting Perfect Matchings in Planar Graphs

- Counting Perfect Matchings (#PM) is #**P**-hard [Valiant 79] in general graphs.
- However, for planar graphs, there is a polynomial time algorithm [Kastelyn 61 & 67, Temperley and Fisher 61].

Counting Perfect Matchings in Planar Graphs

- Counting Perfect Matchings (#PM) is #**P**-hard [Valiant 79] in general graphs.
- However, for planar graphs, there is a polynomial time algorithm [Kastelyn 61 & 67, Temperley and Fisher 61].
  - The FKT algorithm is via Pfaffian orientations of planar graphs.

Holographic Algorithms

Valiant introduced holographic algorithms to extend the reach of FKT algorithms [Valiant 04]:

Holographic Algorithms

Valiant introduced holographic algorithms to extend the reach of FKT algorithms [Valiant 04]:

- Matchgates: functions expressible by perfect matchings (via planar gadgets).

Holographic Algorithms

Valiant introduced holographic algorithms to extend the reach of FKT algorithms [Valiant 04]:

- Matchgates: functions expressible by perfect matchings (via planar gadgets).
- Holographic Transformation: a change of basis.

Holographic Transformation

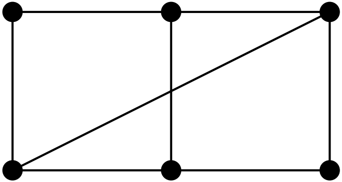
For a 2-by-2 nonsingular matrix T, two functions f and g of arities m and n respectively, Valiant's Holant theorem states

$$\text{Holant}(f \mid g) = \text{Holant}(fT^{\otimes m} \mid (T^{-1})^{\otimes n} g).$$

Note that $\text{Holant}(f) = \text{Holant}(f \mid =_2)$.

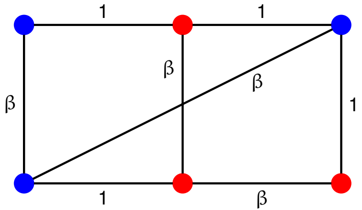
Ising Model

Edge interaction $\begin{bmatrix} \beta & 1 \\ 1 & \beta \end{bmatrix}$



Ising Model

Edge interaction $\begin{bmatrix} \beta & 1 \\ 1 & \beta \end{bmatrix}$



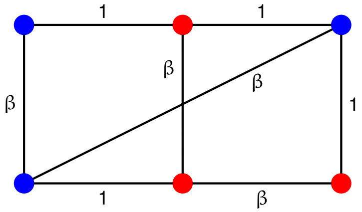
Configuration $\sigma : V \to \{0, 1\}$

$$w(\sigma) = \beta^4$$

$$\Pr(\sigma) \sim w(\sigma)$$

Ising Model

Edge interaction $\begin{bmatrix} \beta & 1 \\ 1 & \beta \end{bmatrix}$



Partition function (normalizing factor):

$$Z_G(\beta) = \sum_{\sigma : V \to \{0,1\}} w(\sigma)$$

where $w(\sigma) = \beta^{m(\sigma)}$, $m(\sigma)$ is the number of monochromatic edges under σ.

Planar Ising is Tractable

- Recall that $[f_0, f_1, \ldots, f_d]$ is a symmetric function f where entries are listed according to Hamming weights.

Planar Ising is Tractable

- Recall that $[f_0, f_1, \dots, f_d]$ is a symmetric function f where entries are listed according to Hamming weights.
  - $=_d$: $[1, 0, \dots, 0, 1]$;
  - EO_d : $[0, 1, 0, \dots, 0]$;
  - Ising function : $[\beta, 1, \beta]$.

Planar Ising is Tractable

- Recall that $[f_0, f_1, \dots, f_d]$ is a symmetric function f where entries are listed according to Hamming weights.
  - $=_d : [1, 0, \dots, 0, 1]$;
  - $\mathrm{EO}_d : [0, 1, 0, \dots, 0]$;
  - Ising function : $[\beta, 1, \beta]$.
- Vertices can be viewed as $=_d$ functions (d is the degree).

Planar Ising is Tractable

- Recall that $[f_0, f_1, \ldots, f_d]$ is a symmetric function f where entries are listed according to Hamming weights.
  - $=_d : [1, 0, \ldots, 0, 1]$;
  - $\mathrm{EO}_d : [0, 1, 0, \ldots, 0]$;
  - Ising function : $[\beta, 1, \beta]$.
- Vertices can be viewed as $=_d$ functions (d is the degree).
- Ising is then

$$\mathrm{Holant}(=_1, =_2, \ldots, =_d, \ldots \mid [\beta, 1, \beta])$$

Planar Ising is Tractable (Cont.)

- Do a transformation of $H = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$. (Notice that $H = H^{-1}$.)

Planar Ising is Tractable (Cont.)

- Do a transformation of $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. (Notice that $H = H^{-1}$.)
  - On the vertex side:

$$
\begin{aligned}
(=_d)H^{\otimes d} &= \left([1,0]^{\otimes d} + [0,1]^{\otimes d}\right) H^{\otimes d} \\
&= ([1,0]H)^{\otimes d} + ([0,1]H)^{\otimes d} = [1,1]^{\otimes d} + [1,-1]^{\otimes d} \\
&= [1,0,1,0,\ldots,0,1]
\end{aligned}
$$

Planar Ising is Tractable (Cont.)

- Do a transformation of $H = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$. (Notice that $H = H^{-1}$.)
  - On the vertex side:

$$\begin{aligned}(=_d)H^{\otimes d} &= \left([1,0]^{\otimes d} + [0,1]^{\otimes d}\right) H^{\otimes d} \\ &= ([1,0]H)^{\otimes d} + ([0,1]H)^{\otimes d} = [1,1]^{\otimes d} + [1,-1]^{\otimes d} \\ &= [1,0,1,0,\dots,0,1]\end{aligned}$$

  - On the edge side:

$$\begin{aligned}H^{\otimes 2}\left((\beta-1)\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]^{\otimes 2} + (\beta-1)\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]^{\otimes 2} + \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]^{\otimes 2}\right) &= [\beta-1,0,\beta-1] + [2,0,0] \\ &= [\beta+1,0,\beta-1]\end{aligned}$$

Both of the two functions above are matchgates.

Complexity Classifications

Counting problems with local constraints are usually classified into:

1. **P**-time solvable over general graphs;
2. #**P**-hard over general graphs but **P**-time solvable over planar graphs;
3. #**P**-hard over planar graphs.

Complexity Classifications

Counting problems with local constraints are usually classified into:

1. **P**-time solvable over general graphs;
2. #**P**-hard over general graphs but **P**-time solvable over planar graphs;
3. #**P**-hard over planar graphs.

Category (2) is always captured by holographic algorithms with matchgates.

Examples include:

- Tutte polynomials [Vertigan 91], [Vertigan 05].
- 2-Spin systems [Kowalczyk 10], [Cai, Kowalczyk, Williams 12].
- Boolean #CSP [Cai, Lu, Xia 10], [G. and Williams 13], [Cai, Fu 16].

Pl-Holant is different

Let $\mathcal{F}$ be a set of symmetric complex-weighted Boolean functions.

Pl-Holant$(\mathcal{F})$ is #**P**-hard unless [Cai, Fu, G., Williams 15]

PI-Holant is different

Let $\mathcal{F}$ be a set of symmetric complex-weighted Boolean functions.

PI-Holant$(\mathcal{F})$ is #**P**-hard unless [Cai, Fu, G., Williams 15]

1. Holant$(\mathcal{F})$ is tractable;

PI-Holant is different

Let $\mathcal{F}$ be a set of symmetric complex-weighted Boolean functions.

PI-Holant$(\mathcal{F})$ is #**P**-hard unless [Cai, Fu, G., Williams 15]

1. Holant$(\mathcal{F})$ is tractable;
2. there exists a holographic transformation under which $\mathcal{F}$ is matchgate,

Pl-Holant is different

Let $\mathcal{F}$ be a set of symmetric complex-weighted Boolean functions.

Pl-Holant$(\mathcal{F})$ is #**P**-hard unless [Cai, Fu, G., Williams 15]

1. Holant$(\mathcal{F})$ is tractable;
2. there exists a holographic transformation under which $\mathcal{F}$ is matchgate,
3. $\mathcal{F}$ defines a special class of problems to count orientations.

PI-Holant is different

Let $\mathcal{F}$ be a set of symmetric complex-weighted Boolean functions.

PI-Holant$(\mathcal{F})$ is #**P**-hard unless [Cai, Fu, G., Williams 15]

1. Holant$(\mathcal{F})$ is tractable;
2. there exists a holographic transformation under which $\mathcal{F}$ is matchgate,
3. $\mathcal{F}$ defines a special class of problems to count orientations.

Category (1) is characterized in [Cai, G., Williams 13].

PI-Holant is different

Let $\mathcal{F}$ be a set of symmetric complex-weighted Boolean functions.

PI-Holant$(\mathcal{F})$ is #**P**-hard unless [Cai, Fu, G., Williams 15]

1. Holant$(\mathcal{F})$ is tractable;
2. there exists a holographic transformation under which $\mathcal{F}$ is matchgate,
3. $\mathcal{F}$ defines a special class of problems to count orientations.

Category (1) is characterized in [Cai, G., Williams 13].

Category (3) is not captured by holographic algorithms with matchgates!

Contents

Tractable Cases

- Affine $\mathscr{A}$: $\chi_{\mathbf{x}A=0} \cdot \mathtt{i}^{\mathbf{x}B\mathbf{x}^{\mathtt{T}}}$.

Tractable Cases

- Affine $\mathcal{A}$: $\chi_{\mathbf{x}A=0} \cdot \mathtt{i}^{\mathbf{x}B\mathbf{x}^{\mathtt{T}}}$.
- Product-type $\mathcal{P}$: products of weighted equalities and disequalities.

Tractable Cases

- Affine $\mathcal{A}$: $\chi_{\mathbf{x}A=0} \cdot \mathtt{i}^{\mathbf{x}B\mathbf{x}^{\mathtt{T}}}$.
- Product-type $\mathcal{P}$: products of weighted equalities and disequalities.
- Matchgates $\mathcal{M}$. (Only tractable on planar graphs.)

Tractable Cases

- Affine $\mathcal{A}$: $\chi_{\mathbf{x}A=0} \cdot \mathtt{i}^{\mathbf{x}B\mathbf{x}^{\mathtt{T}}}$.
- Product-type $\mathcal{P}$: products of weighted equalities and disequalities.
- Matchgates $\mathcal{M}$. (Only tractable on planar graphs.)
- Vanishing $\mathcal{V}$: always return 0.

Vanishing

Vanishing

- $[1, \mathtt{i}]$ is vanishing.

Vanishing

- $[1, \mathtt{i}]$ is vanishing.
- Any degenerate signature containing more than half $[1, \mathtt{i}]$'s is vanishing. For example,

$$f = [1, \mathtt{i}] \otimes [1, \mathtt{i}] \otimes [0, 1].$$

Vanishing

- $[1, \mathtt{i}]$ is vanishing.
- Any degenerate signature containing more than half $[1, \mathtt{i}]$'s is vanishing. For example,

$$f = [1, \mathtt{i}] \otimes [1, \mathtt{i}] \otimes [0, 1].$$

- However, such signatures are not symmetric. We need to introduce an operation of symmetrization.

Symmetrization

Let S_n be the symmetric group of degree n. Then for positive integers t and n with $t \leqslant n$ and unary signatures $v, v_1, \dots, v_{n-t}$, we define

$$\mathrm{Sym}_n^t(v; v_1, \dots, v_{n-t}) = \sum_{\pi \in S_n} \bigotimes_{k=1}^{n} u_{\pi(k)},$$

where the ordered sequence

$$(u_1, u_2, \dots, u_n) = (\underbrace{v, \dots, v}_{t \text{ copies}}, v_1, \dots, v_{n-t}).$$

Examples

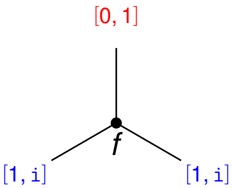
For example,

$$\mathrm{Sym}_3^2([1, \mathtt{i}]; [0, 1]) = 2[0, 1] \otimes [1, \mathtt{i}] \otimes [1, \mathtt{i}] + 2[1, \mathtt{i}] \otimes [0, 1] \otimes [1, \mathtt{i}] + 2[1, \mathtt{i}] \otimes [1, \mathtt{i}] \otimes [0, 1]$$
$$= 2[0, 1, 2\mathtt{i}, -3].$$

Examples

For example,

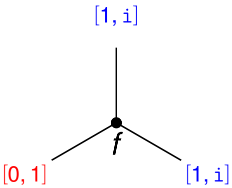
$$\mathrm{Sym}_3^2([1,\mathtt{i}];[0,1]) = 2[0,1]\otimes[1,\mathtt{i}]\otimes[1,\mathtt{i}] + 2[1,\mathtt{i}]\otimes[0,1]\otimes[1,\mathtt{i}] + 2[1,\mathtt{i}]\otimes[1,\mathtt{i}]\otimes[0,1]$$
$$= 2[0,1,2\mathtt{i},-3].$$



Examples

For example,

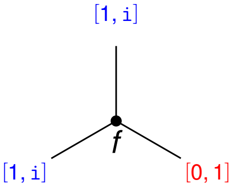
$$\mathrm{Sym}_3^2([1,\mathtt{i}];[0,1]) = 2[0,1]\otimes[1,\mathtt{i}]\otimes[1,\mathtt{i}] + 2[1,\mathtt{i}]\otimes[0,1]\otimes[1,\mathtt{i}] + 2[1,\mathtt{i}]\otimes[1,\mathtt{i}]\otimes[0,1]$$
$$= 2[0,1,2\mathtt{i},-3].$$



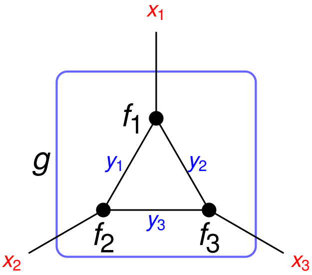
Examples

For example,

$$\mathrm{Sym}_3^2([1,\mathtt{i}];[0,1]) = 2[0,1]\otimes[1,\mathtt{i}]\otimes[1,\mathtt{i}] + 2[1,\mathtt{i}]\otimes[0,1]\otimes[1,\mathtt{i}] + 2[1,\mathtt{i}]\otimes[1,\mathtt{i}]\otimes[0,1]$$
$$= 2[0,1,2\mathtt{i},-3].$$

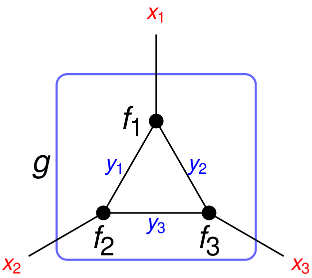


Gadget Construction



$$g(x_1, x_2, x_3) = \sum_{y_1, y_2, y_3} f_1(x_1, y_1, y_2) \cdot f_2(x_2, y_1, y_3) \cdot f_3(x_3, y_2, y_3)$$

Gadget Construction



$$g(x_1, x_2, x_3) = \sum_{y_1, y_2, y_3} f_1(x_1, y_1, y_2) \cdot f_2(x_2, y_1, y_3) \cdot f_3(x_3, y_2, y_3)$$

Maximal tractable cases should be closed under gadget construction.

All of $\mathcal{A}$, $\mathcal{P}$, $\mathcal{M}$, $\mathcal{V}$ do.

A Universal Approach

Indeed, algorithms for $\mathcal{A}$, $\mathcal{P}$, $\mathcal{M}$, $\mathcal{V}$ can be described in a uniform way:

A Universal Approach

Indeed, algorithms for $\mathcal{A}$, $\mathcal{P}$, $\mathcal{M}$, $\mathcal{V}$ can be described in a uniform way:

1. There exists a succinct representation (polynomial size) of any function in $\mathcal{F}$.

A Universal Approach

Indeed, algorithms for $\mathcal{A}$, $\mathcal{P}$, $\mathcal{M}$, $\mathcal{V}$ can be described in a uniform way:

1. There exists a succinct representation (polynomial size) of any function in $\mathcal{F}$.
   - Affine $\mathcal{A}$: $\chi_{\mathbf{x}A=0} \cdot \mathtt{i}^{\mathbf{x}B\mathbf{x}^{\mathsf{T}}}$.
   - Product-type $\mathcal{P}$: products of weighted equalities and disequalities.
   - Matchgates $\mathcal{M}$. (Only tractable on planar graphs.)
   - Vanishing $\mathcal{V}$: always return 0.

A Universal Approach

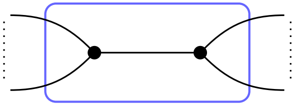
Indeed, algorithms for $\mathcal{A}$, $\mathcal{P}$, $\mathcal{M}$, $\mathcal{V}$ can be described in a uniform way:

1. There exists a succinct representation (polynomial size) of any function in $\mathcal{F}$.
2. This representation can be updated efficiently with the following two basic operations:

A Universal Approach

Indeed, algorithms for $\mathcal{A}$, $\mathcal{P}$, $\mathcal{M}$, $\mathcal{V}$ can be described in a uniform way:

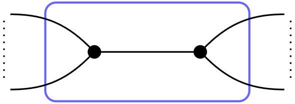
1. There exists a succinct representation (polynomial size) of any function in $\mathcal{F}$.
2. This representation can be updated efficiently with the following two basic operations:

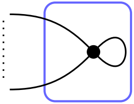


A Universal Approach

Indeed, algorithms for $\mathcal{A}$, $\mathcal{P}$, $\mathcal{M}$, $\mathcal{V}$ can be described in a uniform way:

1. There exists a succinct representation (polynomial size) of any function in $\mathcal{F}$.
2. This representation can be updated efficiently with the following two basic operations:

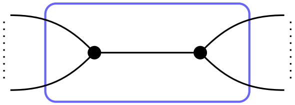


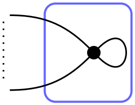


A Universal Approach

Indeed, algorithms for $\mathcal{A}$, $\mathcal{P}$, $\mathcal{M}$, $\mathcal{V}$ can be described in a uniform way:

1. There exists a succinct representation (polynomial size) of any function in $\mathcal{F}$.
2. This representation can be updated efficiently with the following two basic operations:





Repeat above until one vertex is left.

The resulting nullary function is the Holant value.

$\mathcal{F}$-transformable

$$\text{Holant}(f) = \text{Holant}(f \mid =_2)$$

$\mathcal{F}$-transformable

$$\begin{aligned}\mathrm{Holant}(f) &= \mathrm{Holant}(f \mid =_2) \\ &= \mathrm{Holant}(f T^{\otimes n} \mid (T^{-1})^{\otimes 2} =_2)\end{aligned}$$

$\mathcal{F}$-transformable

$$\begin{aligned} \text{Holant}(f) &= \text{Holant}(f \mid =_2) \\ &= \text{Holant}(fT^{\otimes n} \mid (T^{-1})^{\otimes 2} =_2) \\ &\leqslant_{\mathrm{T}} \text{Holant}(fT^{\otimes n}, (T^{-1})^{\otimes 2} =_2) \end{aligned}$$

$\mathcal{F}$-transformable

$$
\begin{aligned}
\text{Holant}(f) &= \text{Holant}(f \mid =_2) \\
&= \text{Holant}(f T^{\otimes n} \mid (T^{-1})^{\otimes 2} =_2) \\
&\leqslant_T \text{Holant}(f T^{\otimes n}, (T^{-1})^{\otimes 2} =_2)
\end{aligned}
$$

- f is $\mathcal{F}$-transformable if there exists T such that $\{f T^{\otimes n}, (T^{-1})^{\otimes 2} =_2\} \subset \mathcal{F}$.

$\mathcal{F}$-transformable

$$
\begin{aligned}
\text{Holant}(f) &= \text{Holant}(f \mid =_2) \\
&= \text{Holant}(fT^{\otimes n} \mid (T^{-1})^{\otimes 2} =_2) \\
&\leqslant_T \text{Holant}(fT^{\otimes n}, (T^{-1})^{\otimes 2} =_2) \\
&\leqslant_T \text{Holant}(\mathcal{F})
\end{aligned}
$$

- f is $\mathcal{F}$-transformable if there exists T such that $\{fT^{\otimes n}, (T^{-1})^{\otimes 2} =_2\} \subset \mathcal{F}$.

$\mathcal{F}$-transformable

$$\begin{aligned}
\text{Holant}(f) &= \text{Holant}(f \mid =_2) \\
&= \text{Holant}(fT^{\otimes n} \mid (T^{-1})^{\otimes 2} =_2) \\
&\leqslant_T \text{Holant}(fT^{\otimes n}, (T^{-1})^{\otimes 2} =_2) \\
&\leqslant_T \text{Holant}(\mathcal{F})
\end{aligned}$$

- f is $\mathcal{F}$-transformable if there exists T such that $\{fT^{\otimes n}, (T^{-1})^{\otimes 2} =_2\} \subset \mathcal{F}$.
- If $\mathcal{F}$ is tractable, then so is $\mathcal{F}$-transformable.

$\mathcal{F}$-transformable

$$\begin{aligned}
\text{Holant}(f) &= \text{Holant}(f \mid =_2) \\
&= \text{Holant}(fT^{\otimes n} \mid (T^{-1})^{\otimes 2} =_2) \\
&\leqslant_T \text{Holant}(fT^{\otimes n}, (T^{-1})^{\otimes 2} =_2) \\
&\leqslant_T \text{Holant}(\mathcal{F})
\end{aligned}$$

- f is $\mathcal{F}$-transformable if there exists T such that $\{fT^{\otimes n}, (T^{-1})^{\otimes 2} =_2\} \subset \mathcal{F}$.
- If $\mathcal{F}$ is tractable, then so is $\mathcal{F}$-transformable.
- $\mathcal{V}$ is closed under such transformations, but $\mathcal{A}$, $\mathcal{P}$, $\mathcal{M}$ are not.

$\mathcal{F}$-transformable

$$\begin{aligned} \text{Holant}(f) &= \text{Holant}(f \mid =_2) \\ &= \text{Holant}(f T^{\otimes n} \mid (T^{-1})^{\otimes 2} =_2) \\ &\leqslant_T \text{Holant}(f T^{\otimes n}, (T^{-1})^{\otimes 2} =_2) \\ &\leqslant_T \text{Holant}(\mathcal{F}) \end{aligned}$$

- f is $\mathcal{F}$-transformable if there exists T such that $\{f T^{\otimes n}, (T^{-1})^{\otimes 2} =_2\} \subset \mathcal{F}$.
- If $\mathcal{F}$ is tractable, then so is $\mathcal{F}$-transformable.
- $\mathcal{V}$ is closed under such transformations, but $\mathcal{A}$, $\mathcal{P}$, $\mathcal{M}$ are not.
- $\mathcal{A}$, $\mathcal{P}$, $\mathcal{M}$-transformables and $\mathcal{V}$ are the main tractable classes for Pl-Holant.

New Planar Tractable Case

Counting Orientations, (equivalent to normal Holant via $\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$)

where two types of nodes are allowed:

1. Exactly one edge coming in;
2. All edges coming in or going out (either a sink or a source).

Moreover, we require that the gcd of the degrees of type 2 nodes is at least 5.

Then the problem is tractable for planar graphs.

Contents

General Proof Strategy

- Prove the dichotomy for a single function first.
  Induction on the arity.

General Proof Strategy

- Prove the dichotomy for a single function first.
  Induction on the arity.
  - Base cases: arity-3 or 4.
  - Arity-reduction.

General Proof Strategy

- Prove the dichotomy for a single function first.

  Induction on the arity.

  - Base cases: arity-3 or 4.
  - Arity-reduction.

- Prove that different tractable cases cannot mix together.

General Proof Strategy

- Prove the dichotomy for a single function first.
  Induction on the arity.
  - Base cases: arity-3 or 4.
  - Arity-reduction.
- Prove that different tractable cases cannot mix together.
- We will show next that most arity-4 functions are #**P**-hard.

Signature Matrices

The signature matrix of a symmetric arity 4 signature $f = [f_0, f_1, f_2, f_3, f_4]$ is

$$M_f = \begin{bmatrix} f_0 & f_1 & f_1 & f_2 \\ f_1 & f_2 & f_2 & f_3 \\ f_1 & f_2 & f_2 & f_3 \\ f_2 & f_3 & f_3 & f_4 \end{bmatrix}.$$

Signature Matrices

The signature matrix of a symmetric arity 4 signature $f = [f_0, f_1, f_2, f_3, f_4]$ is

$$M_f = \begin{bmatrix} f_0 & f_1 & f_1 & f_2 \\ f_1 & f_2 & f_2 & f_3 \\ f_1 & f_2 & f_2 & f_3 \\ f_2 & f_3 & f_3 & f_4 \end{bmatrix}.$$

For asymmetric signatures,

$$M_g = \begin{bmatrix} g^{0000} & g^{0010} & g^{0001} & g^{0011} \\ g^{0100} & g^{0110} & g^{0101} & g^{0111} \\ g^{1000} & g^{1010} & g^{1001} & g^{1011} \\ g^{1100} & g^{1110} & g^{1101} & g^{1111} \end{bmatrix},$$

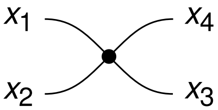
rows indexed by $(x_1, x_2) \in \{0,1\}^2$ and columns by $(x_4, x_3) \in \{0,1\}^2$.

Signature Matrices

$$M_g = \begin{bmatrix} g^{0000} & g^{0010} & g^{0001} & g^{0011} \\ g^{0100} & g^{0110} & g^{0101} & g^{0111} \\ g^{1000} & g^{1010} & g^{1001} & g^{1011} \\ g^{1100} & g^{1110} & g^{1101} & g^{1111} \end{bmatrix}$$

rows indexed by $(x_1, x_2) \in \{0, 1\}^2$ and

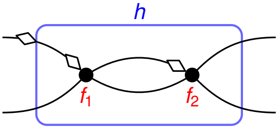
columns by $(x_4, x_3) \in \{0, 1\}^2$.



Signature Matrices

We wrote the signature matrices in this way so that

$$M_h = M_{f_1} M_{f_2}$$



Redundant signature matrices

- $\mathrm{RM}_4(\mathbb{C})$: 4-by-4 redundant matrices

$$M_f = \begin{bmatrix} f_0 & f_1 & f_1 & f_2' \\ f_1' & f_2 & f_2 & f_3 \\ f_1' & f_2 & f_2 & f_3 \\ f_2'' & f_3' & f_3' & f_4 \end{bmatrix}$$

Redundant signature matrices

- $\mathrm{RM}_4(\mathbb{C})$: 4-by-4 redundant matrices

$$M_f = \begin{bmatrix} f_0 & f_1 & f_1 & f_2' \\ f_1' & f_2 & f_2 & f_3 \\ f_1' & f_2 & f_2 & f_3 \\ f_2'' & f_3' & f_3' & f_4 \end{bmatrix}$$

Redundant signature matrices

- $\mathrm{RM}_4(\mathbb{C})$: 4-by-4 redundant matrices

$$M_f = \begin{bmatrix} f_0 & f_1 & f_1 & f_2' \\ f_1' & f_2 & f_2 & f_3 \\ f_1' & f_2 & f_2 & f_3 \\ f_2'' & f_3' & f_3' & f_4 \end{bmatrix}$$

Redundant signature matrices

- $\mathrm{RM}_4(\mathbb{C})$: 4-by-4 redundant matrices

$$M_f = \begin{bmatrix} f_0 & f_1 & f_1 & f_2' \\ f_1' & f_2 & f_2 & f_3 \\ f_1' & f_2 & f_2 & f_3 \\ f_2'' & f_3' & f_3' & f_4 \end{bmatrix}$$

- Compressed signature matrix $\widetilde{M_f}$

$$\begin{bmatrix} f_0 & f_1 & f_1 & f_2' \\ f_1' & f_2 & f_2 & f_3 \\ f_1' & f_2 & f_2 & f_3 \\ f_2'' & f_3' & f_3' & f_4 \end{bmatrix} \to \begin{bmatrix} f_0 & 2f_1 & f_2' \\ f_1' & 2f_2 & f_3 \\ f_2'' & 2f_3' & f_4 \end{bmatrix}$$

This operation is a semi-group isomorphism between $\mathrm{RM}_4(\mathbb{C})$ and $\mathbb{C}^{3\times 3}$.

Redundant signature matrices

- $\mathrm{RM}_4(\mathbb{C})$: 4-by-4 redundant matrices

$$M_f = \begin{bmatrix} f_0 & f_1 & f_1 & f_2' \\ f_1' & f_2 & f_2 & f_3 \\ f_1' & f_2 & f_2 & f_3 \\ f_2'' & f_3' & f_3' & f_4 \end{bmatrix}$$

- Compressed signature matrix $\widetilde{M_f}$

$$\begin{bmatrix} f_0 & f_1 & f_1 & f_2' \\ f_1' & f_2 & f_2 & f_3 \\ f_1' & f_2 & f_2 & f_3 \\ f_2'' & f_3' & f_3' & f_4 \end{bmatrix} \to \begin{bmatrix} f_0 & 2f_1 & f_2' \\ f_1' & 2f_2 & f_3 \\ f_2'' & 2f_3' & f_4 \end{bmatrix}$$

This operation is a semi-group isomorphism between $\mathrm{RM}_4(\mathbb{C})$ and $\mathbb{C}^{3\times 3}$.

- If $M_h = M_{f_1} M_{f_2}$ then $\widetilde{M_h} = \widetilde{M_{f_1}}\,\widetilde{M_{f_2}}$.

Non-singular Compressed Matrix means Hardness

Lemma

Let f be an arity 4 signature with complex weights. If M_f is redundant and $\widetilde{M_f}$ is nonsingular, then Pl-Holant(f) *is* #**P**-*hard.*

Outline

We will show the lemma in 3 steps.

Outline

We will show the lemma in 3 steps.

#PL-4-REG-EO is #**P**-hard.

Outline

We will show the lemma in 3 steps.

#PL-4-REG-EO is #**P**-hard.

#PL-4-REG-EO $\leqslant$ Pl-Holant($\textit{id}$)

Outline

We will show the lemma in 3 steps.

#PL-4-REG-EO is #**P**-hard.

#PL-4-REG-EO $\leqslant$ Pl-Holant(id)

Pl-Holant(id) $\leqslant$ Pl-Holant(f)

The identity of $\mathrm{RM}_4(\mathbb{C})$

The identity element of $\mathrm{RM}_4(\mathbb{C})$ corresponds to an arity 4 signature *id* with

$$M_{id} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$\widetilde{M_{id}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The identity is hard.

Recall that Pl-Holant($[3, 0, 1, 0, 3]$) is equivalent to counting Eulerian Orientations in planar 4-regular graphs (via $\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$), which is #**P**-hard.

The identity is hard.

Recall that Pl-Holant$([3,0,1,0,3])$ is equivalent to counting Eulerian Orientations in planar 4-regular graphs (via $\left[\begin{smallmatrix}1 & 1\\ i & -i\end{smallmatrix}\right]$), which is #**P**-hard.

We will show next Pl-Holant$([3,0,1,0,3]) \leqslant_T$ Pl-Holant(id).

Approximating $[1, 0, \frac{1}{3}, 0, 1]$

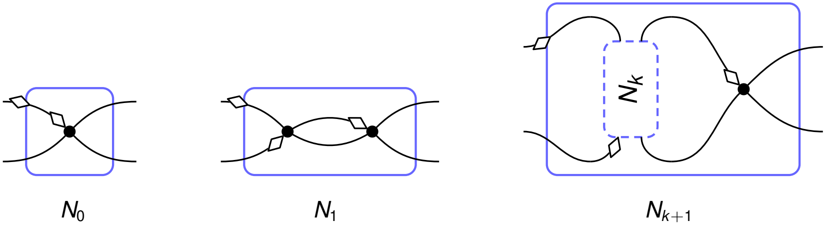
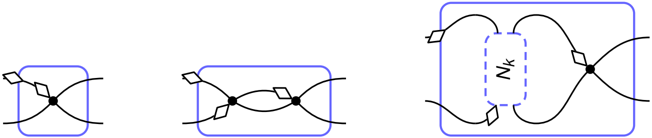


Figure: Recursive construction to approximate $[1, 0, \frac{1}{3}, 0, 1]$. Vertices are assigned *id*.



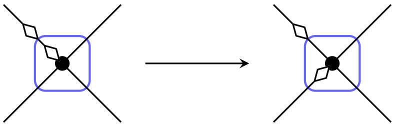
N_0 N_1 N_{k+1}

We claim that the signature matrix M_{N_k} of Gadget N_k is

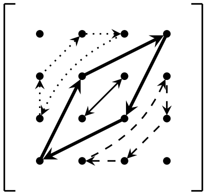
$$M_{N_k} = \begin{bmatrix} 1 & 0 & 0 & a_k \\ 0 & a_{k+1} & a_{k+1} & 0 \\ 0 & a_{k+1} & a_{k+1} & 0 \\ a_k & 0 & 0 & 1 \end{bmatrix},$$

where $a_k = \frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^k$.

Rotation of the Signature Matrix



(d) Counterclockwise Rotation



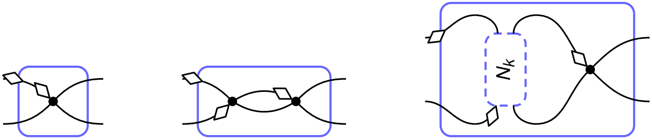
(e) Movement of Entries

Entires of Hamming weight 1 are in the dotted cycle.

Entires of Hamming weight 2 are in the two solid cycles.

Entries of Hamming weight 3 are in the dashed cycle.

Approximating $[1, 0, \frac{1}{3}, 0, 1]$



N_0 N_1 N_{k+1}

$$M_{N_{k+1}} = \begin{bmatrix} 1 & 0 & 0 & a_{k+1} \\ 0 & a_k & a_{k+1} & 0 \\ 0 & a_{k+1} & a_k & 0 \\ a_{k+1} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It is easy to verify that $\frac{a_k + a_{k+1}}{2} = a_{k+2}$.

Approximating $[1, 0, \frac{1}{3}, 0, 1]$

- We can realize $M_{N_k} = \begin{bmatrix} 1 & 0 & 0 & a_k \\ 0 & a_{k+1} & a_{k+1} & 0 \\ 0 & a_{k+1} & a_{k+1} & 0 \\ a_k & 0 & 0 & 1 \end{bmatrix}$ where $a_k = \frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^k$,

  and our target is $\begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 0 \\ 1/3 & 0 & 0 & 1 \end{bmatrix}$.

Approximating $[1, 0, \frac{1}{3}, 0, 1]$

- We can realize $M_{N_k} = \begin{bmatrix} 1 & 0 & 0 & a_k \\ 0 & a_{k+1} & a_{k+1} & 0 \\ 0 & a_{k+1} & a_{k+1} & 0 \\ a_k & 0 & 0 & 1 \end{bmatrix}$ where $a_k = \frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^k$,

  and our target is $\begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 0 \\ 1/3 & 0 & 0 & 1 \end{bmatrix}$.

- If we can reduce the error below 3^{-n},

  then we can recover the exact value.

Approximating $[1, 0, \frac{1}{3}, 0, 1]$

- We can realize $M_{N_k} = \begin{bmatrix} 1 & 0 & 0 & a_k \\ 0 & a_{k+1} & a_{k+1} & 0 \\ 0 & a_{k+1} & a_{k+1} & 0 \\ a_k & 0 & 0 & 1 \end{bmatrix}$ where $a_k = \frac{1}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^k$,

  and our target is $\begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 0 \\ 1/3 & 0 & 0 & 1 \end{bmatrix}$.

- If we can reduce the error below 3^{-n},

  then we can recover the exact value.

- It suffices to do $k = 4n$.

Back to the Lemma

Lemma

Let f be an arity 4 signature with complex weights. If M_f is redundant and $\widetilde{M_f}$ is nonsingular, then we have

$$\operatorname{Holant}(id) \leqslant_T \operatorname{Holant}(f).$$

Therefore $\operatorname{Holant}(f)$ *is* #**P**-*hard.*

We will show it by interpolation.

Sequential Construction

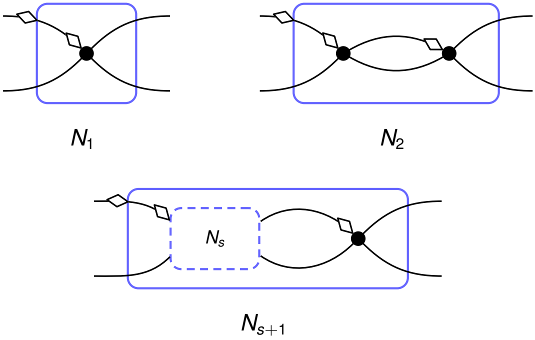


Figure: Recursive construction to interpolate *id*. The vertices are assigned f. $M_{N_s} = (M_f)^s$. Diamonds indicates the most significant bit and the bits are ordered counterclockwise.

Interpolation

Suppose that *id* appears n times in Ω.

Interpolation

Suppose that *id* appears n times in Ω.

Replace *id* by N_s to get Ω_s.

$M_{N_s} = (M_f)^s$.

Interpolation

Suppose that *id* appears n times in Ω.

Replace *id* by N_s to get Ω_s.

$M_{N_s} = (M_f)^s$.

By the Jordan normal form of $\widetilde{M_f}$, there exists $T, \Lambda \in \mathbb{C}^{3\times 3}$ such that

$$\widetilde{M_f} = T \Lambda T^{-1} = T \begin{bmatrix} \lambda_1 & b_1 & 0 \\ 0 & \lambda_2 & b_2 \\ 0 & 0 & \lambda_3 \end{bmatrix} T^{-1},$$

where $b_1, b_2 \in \{0, 1\}$.

Interpolation

Suppose that *id* appears n times in Ω.

Replace *id* by N_s to get Ω_s.

$M_{N_s} = (M_f)^s$.

By the Jordan normal form of $\widetilde{M_f}$, there exists $T, \Lambda \in \mathbb{C}^{3\times 3}$ such that

$$\widetilde{M_f} = T\Lambda T^{-1} = T \begin{bmatrix} \lambda_1 & b_1 & 0 \\ 0 & \lambda_2 & b_2 \\ 0 & 0 & \lambda_3 \end{bmatrix} T^{-1},$$

where $b_1, b_2 \in \{0, 1\}$.

Here we will only deal with the case that $\lambda_1 = \lambda_2 = \lambda_3 \neq 0$ and $b_1 = b_2 = 1$.

Sleight of Hand

We have

$$(\widetilde{M_f})^s = T(\Lambda)^s T^{-1},$$

where

$$\Lambda = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}.$$

Notice

$$\widetilde{M_{id}} = T\widetilde{M_{id}}T^{-1}.$$

We will consider new instances where each occurrence of *id* (or N_s) is replaced by three signatures whose compressed matrices are T, $\widetilde{M_{id}}$ (or Λ^s), and T^{-1} respectively.

Stratification

We stratify all assignments to Λ^s according to:

- $(0,0)$ or $(2,2)$ i many times;
- $(1,1)$ j many times;
- $(0,1)$ k many times;
- $(1,2)$ ℓ many times;
- $(0,2)$ m many times.

Any other assignment contributes a factor 0.

In Ω only $(0,0)$, $(1,1)$ $(2,2)$ contributes a 1.

Stratification

Let $c_{ijk\ell m}$ be the sum over all such assignments of the products of evaluations (including the contributions from T and T^{-1}) on Ω_s.

$$\mathrm{Holant}_\Omega = \sum_{i+j=n} \frac{c_{ij000}}{2^j}.$$

The value of the Holant on Ω_s, for $s \geqslant 1$, is

$$\begin{aligned}\mathrm{Holant}_{\Omega_s} &= \sum_{i+j+k+\ell+m=n} \lambda^{(i+j)s} \left(s\lambda^{s-1}\right)^{k+\ell} \left(s(s-1)\lambda^{s-2}\right)^m \left(\frac{c_{ijk\ell m}}{2^{j+k+m}}\right) \\ &= \lambda^{ns} \sum_{i+j+k+\ell+m=n} s^{k+\ell+m}(s-1)^m \left(\frac{c_{ijk\ell m}}{\lambda^{k+\ell+2m}2^{j+k+m}}\right).\end{aligned}$$

Rank Deficiency

However, the linear system is rank deficient.

Rank Deficiency

However, the linear system is rank deficient.

A simple example:

$$\begin{cases} x + y + z = 1 \\ x + y + 2z = 2 \end{cases}$$

Rank Deficiency

However, the linear system is rank deficient.

A simple example:

$$\begin{cases} x + y + z = 1 \\ x + y + 2z = 2 \end{cases} \qquad \Rightarrow$$

Rank Deficiency

However, the linear system is rank deficient.

A simple example:

$$\begin{cases} x + y + z = 1 \\ x + y + 2z = 2 \end{cases} \Rightarrow \begin{cases} x + y = 0 \\ z = 1 \end{cases}$$

Rank Deficiency

However, the linear system is rank deficient.

A simple example:

$$\begin{cases} x + y + z = 1 \\ x + y + 2z = 2 \end{cases} \Rightarrow \begin{cases} x + y = 0 \\ z = 1 \end{cases}$$

Our system:

$$\text{Holant}_{\Omega_s} = \lambda^{ns} \sum_{i+j+k+\ell+m=n} s^{k+\ell+m}(s-1)^m \left(\frac{c_{ijk\ell m}}{\lambda^{k+\ell+2m} 2^{j+k+m}} \right).$$

Rank Deficiency

We define new unknowns for any $q, m \geqslant 0$ and $q + m \leqslant n$,

$$x_{q,m} = \sum_{i+j=n-m-q,k+\ell=q} \left(\frac{c_{ijk\ell m}}{\lambda^{k+\ell+2m} 2^{j+k+m}} \right)$$

The Holant of Ω, which equals to $\sum_{i+j=n} \frac{c_{ij000}}{2^j}$, now becomes $x_{0,0}$.

This new linear system is

$$\text{Holant}_{\Omega_s} = \lambda^{ns} \sum_{q+m \leqslant n} s^{q+m} (s-1)^m x_{q,m}.$$

Let $\alpha_{q,m} = s^{q+m}(s-1)^m$ be the coefficients.

Rank Deficiency

The new system is still rank deficient.

Observe that

$$s^{q+m}(s-1)^m = s^{q-1+m}(s-1)^m + s^{q-2+m+1}(s-1)^{m+1}.$$

Therefore

$$\alpha_{q,m}x_{q,m} = \alpha_{q-1,m}x_{q,m} + \alpha_{q-2,m+1}x_{q,m}.$$

More new unknowns

We recursively define new variables

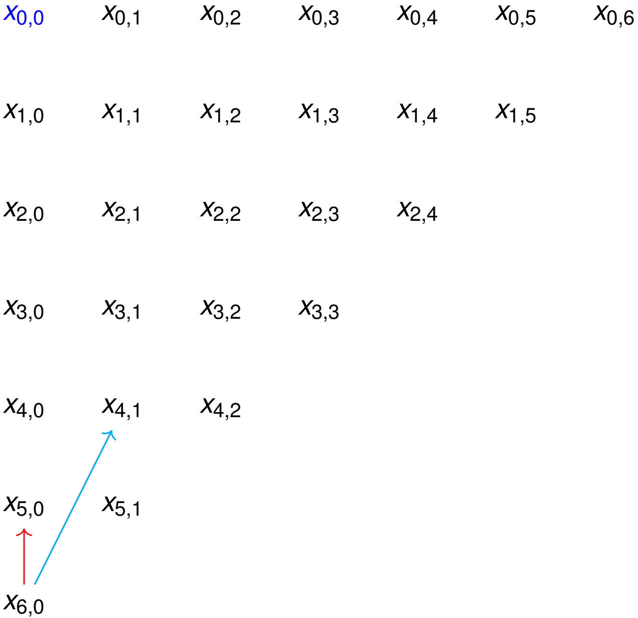
$$x_{q-1,m} \leftarrow x_{q,m} + x_{q-1,m}$$

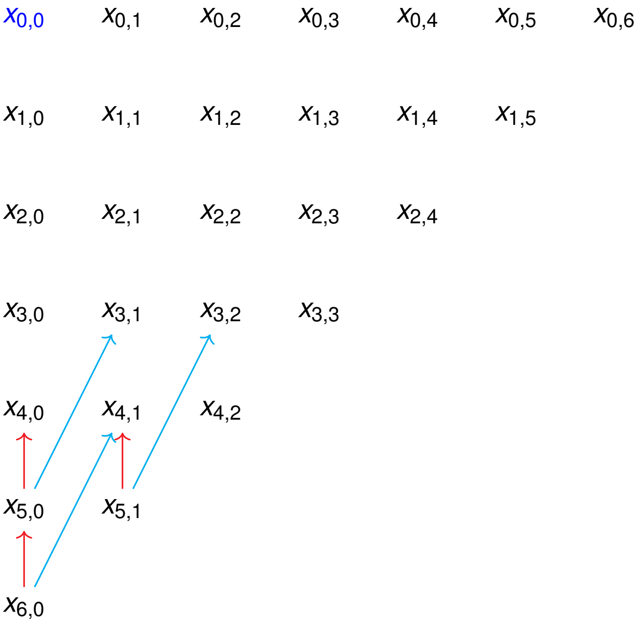
$$x_{q-2,m+1} \leftarrow x_{q,m} + x_{q-2,m+1}$$

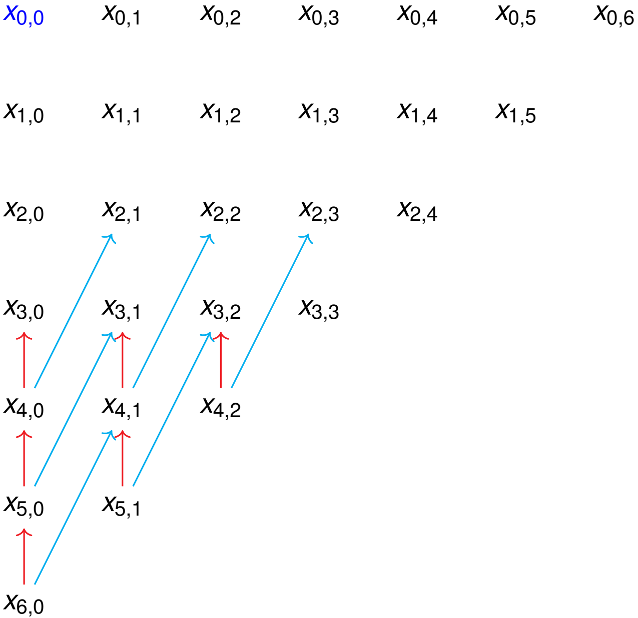
from $q = n$ down to 2.

$$
\begin{array}{ccccccc}
x_{0,0} & x_{0,1} & x_{0,2} & \cdots & x_{0,n-2} & x_{0,n-1} & x_{0,n} \\
x_{1,0} & x_{1,1} & x_{1,2} & \cdots & x_{1,n-2} & x_{1,n-1} & \\
x_{2,0} & x_{2,1} & x_{2,2} & \cdots & x_{2,n-2} & & \\
\vdots & \vdots & \vdots & & & & \\
x_{n-2,0} & x_{n-2,1} & x_{n-2,2} & & & & \\
x_{n-1,0} & x_{n-1,1} & & & & & \\
x_{n,0} & & & & & &
\end{array}
$$

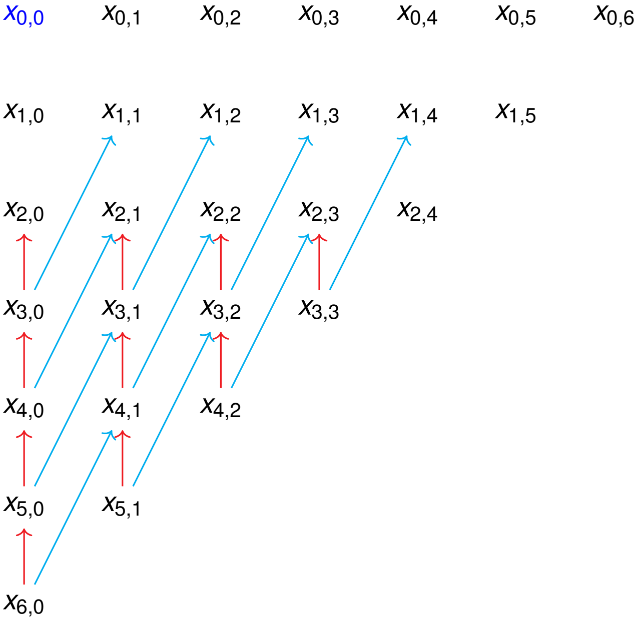
$x_{0,0}$	$x_{0,1}$	$x_{0,2}$	$x_{0,3}$	$x_{0,4}$	$x_{0,5}$	$x_{0,6}$
$x_{1,0}$	$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$	$x_{1,5}$	
$x_{2,0}$	$x_{2,1}$	$x_{2,2}$	$x_{2,3}$	$x_{2,4}$		
$x_{3,0}$	$x_{3,1}$	$x_{3,2}$	$x_{3,3}$			
$x_{4,0}$	$x_{4,1}$	$x_{4,2}$				
$x_{5,0}$	$x_{5,1}$					
$x_{6,0}$						

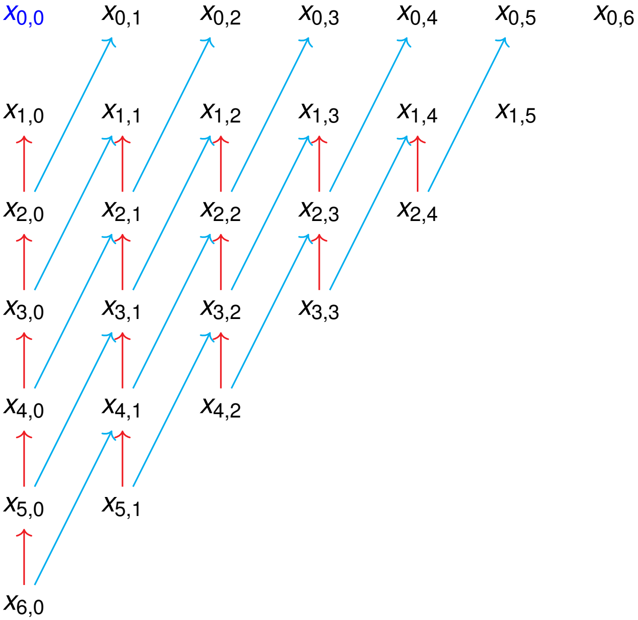

$x_{0,0}$ $x_{0,1}$ $x_{0,2}$ $x_{0,3}$ $x_{0,4}$ $x_{0,5}$ $x_{0,6}$
$x_{1,0}$ $x_{1,1}$ $x_{1,2}$ $x_{1,3}$ $x_{1,4}$ $x_{1,5}$
$x_{2,0}$ $x_{2,1}$ $x_{2,2}$ $x_{2,3}$ $x_{2,4}$
$x_{3,0}$ $x_{3,1}$ $x_{3,2}$ $x_{3,3}$
$x_{4,0}$ $x_{4,1}$ $x_{4,2}$
$x_{5,0}$ $x_{5,1}$
$x_{6,0}$


$x_{0,0}$ $x_{0,1}$ $x_{0,2}$ $x_{0,3}$ $x_{0,4}$ $x_{0,5}$ $x_{0,6}$
$x_{1,0}$ $x_{1,1}$ $x_{1,2}$ $x_{1,3}$ $x_{1,4}$ $x_{1,5}$
$x_{2,0}$ $x_{2,1}$ $x_{2,2}$ $x_{2,3}$ $x_{2,4}$
$x_{3,0}$ $x_{3,1}$ $x_{3,2}$ $x_{3,3}$
$x_{4,0}$ $x_{4,1}$ $x_{4,2}$
$x_{5,0}$ $x_{5,1}$
$x_{6,0}$



$x_{0,0}$ $x_{0,1}$ $x_{0,2}$ $x_{0,3}$ $x_{0,4}$ $x_{0,5}$ $x_{0,6}$
$x_{1,0}$ $x_{1,1}$ $x_{1,2}$ $x_{1,3}$ $x_{1,4}$ $x_{1,5}$
$x_{2,0}$ $x_{2,1}$ $x_{2,2}$ $x_{2,3}$ $x_{2,4}$
$x_{3,0}$ $x_{3,1}$ $x_{3,2}$ $x_{3,3}$
$x_{4,0}$ $x_{4,1}$ $x_{4,2}$
$x_{5,0}$ $x_{5,1}$
$x_{6,0}$


$x_{0,0}$ $x_{0,1}$ $x_{0,2}$ $x_{0,3}$ $x_{0,4}$ $x_{0,5}$ $x_{0,6}$
$x_{1,0}$ $x_{1,1}$ $x_{1,2}$ $x_{1,3}$ $x_{1,4}$ $x_{1,5}$
$x_{2,0}$ $x_{2,1}$ $x_{2,2}$ $x_{2,3}$ $x_{2,4}$
$x_{3,0}$ $x_{3,1}$ $x_{3,2}$ $x_{3,3}$
$x_{4,0}$ $x_{4,1}$ $x_{4,2}$
$x_{5,0}$ $x_{5,1}$
$x_{6,0}$



$x_{0,0}$ $x_{0,1}$ $x_{0,2}$ $x_{0,3}$ $x_{0,4}$ $x_{0,5}$ $x_{0,6}$
$x_{1,0}$ $x_{1,1}$ $x_{1,2}$ $x_{1,3}$ $x_{1,4}$ $x_{1,5}$
$x_{2,0}$ $x_{2,1}$ $x_{2,2}$ $x_{2,3}$ $x_{2,4}$
$x_{3,0}$ $x_{3,1}$ $x_{3,2}$ $x_{3,3}$
$x_{4,0}$ $x_{4,1}$ $x_{4,2}$
$x_{5,0}$ $x_{5,1}$
$x_{6,0}$

Concluding Remarks

- Guessing tractable cases is usually the first step towards a dichotomy.

Concluding Remarks

- Guessing tractable cases is usually the first step towards a dichotomy.
  - The next step is trying to show anything different is hard.

Concluding Remarks

- Guessing tractable cases is usually the first step towards a dichotomy.
  - The next step is trying to show anything different is hard.
- However, there always are some new tractable cases each time we extend our setting. Is there more?

Concluding Remarks

- Guessing tractable cases is usually the first step towards a dichotomy.
  - The next step is trying to show anything different is hard.
- However, there always are some new tractable cases each time we extend our setting. Is there more?
- Future directions:
  - H-minor free graphs.
  - Higher domains.
  - ...

Thank You!