Thresholds in random CSPs

Nike Sun (Berkeley)

Counting complexity and phase transitions
Simons Institute, Berkeley
28 January 2016

Plan for the talk

Introduction: random k-SAT model

Threshold conjecture, Friedgut's theorem

Statistical physics viewpoint of random CSPs

Replica symmetry (RS) vs. replica symmetry breaking (RSB)

One-step replica symmetry breaking (1RSB)

Graphical models for clusters

Credits (a non-exhaustive list)

(physics) Florent Krząkała, Stephan Mertens, Marc Mézard, Andrea Montanari, Giorgio Parisi, Federico Ricci-Tersenghi, Guilhem Semerjian, Lenka Zdeborová, Riccardo Zecchina

(combinatorial cluster model) Alfredo Braunstein, Elitza Maneva, Marc Mézard, Elchanan Mossel, Giorgio Parisi, Alistair Sinclair, Martin Wainwright, Riccardo Zecchina

(upper bound) Silvio Franz, Francesco Guerra, Michele Leone, Dmitry Panchenko, Michel Talagrand, Fabio Toninelli

(lower bound) Dimitris Achlioptas, Amin Coja-Oghlan, Jian Ding, Cris Moore, Assaf Naor, Konstantinos Panagiotou, Yuval Peres, Allan Sly, Daniel Vilenchik

Random CSPs; and the random k-SAT model

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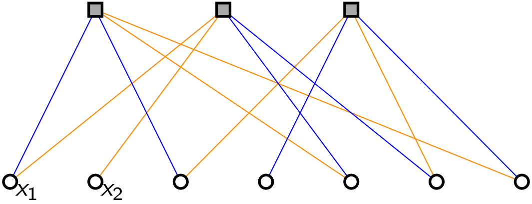

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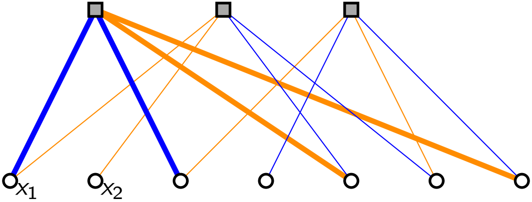

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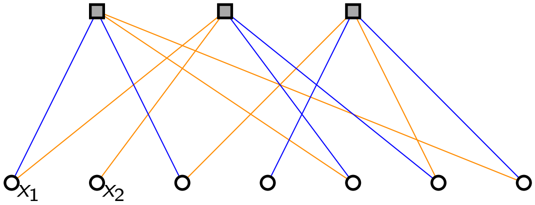

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Computational question: decide if there exists any variable assignment $\underline{x} \in \{+, -\}^n$ satisfying all clauses.

Constraint satisfaction problems

SAT is a **constraint satisfaction problem** (**CSP**).

A general CSP is a set of variables subject to some constraints: the question is to decide whether there exists some variable assignment satisfying all constraints.

For a large class of CSPs, including SAT, best known algorithms have exponential runtime on worst-case instances, motivating interest in *average-case* behavior.

One direction is to investigate the typical behavior for models of *random CSPs*, as the system size becomes large. This line of research has been pursued since the 1980s.

Formal definition of k-SAT

A k-SAT problem is specified by a boolean formula

clause of width $k = 4$

$$
\begin{aligned}
& (+x_1 \text{ OR } +x_3 \text{ OR } -x_5 \text{ OR } -x_7) \\
\text{AND } & (-x_1 \text{ OR } -x_2 \text{ OR } +x_5 \text{ OR } +x_6) \\
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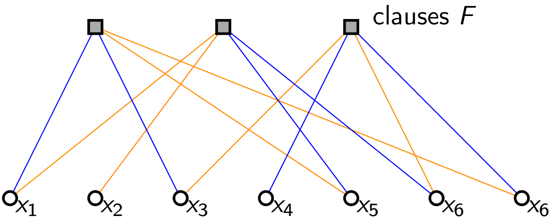
Equivalently, a factor graph with colored edges:



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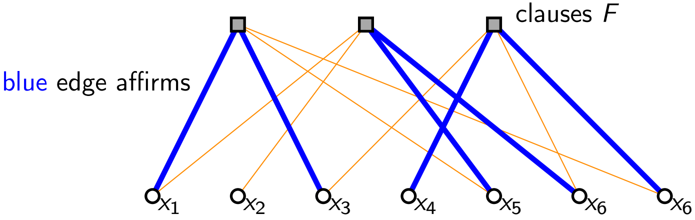

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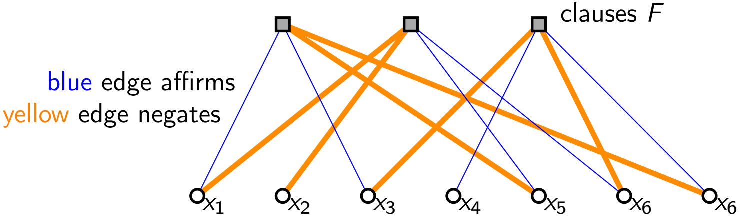

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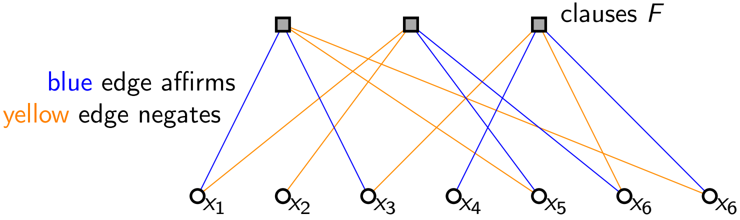
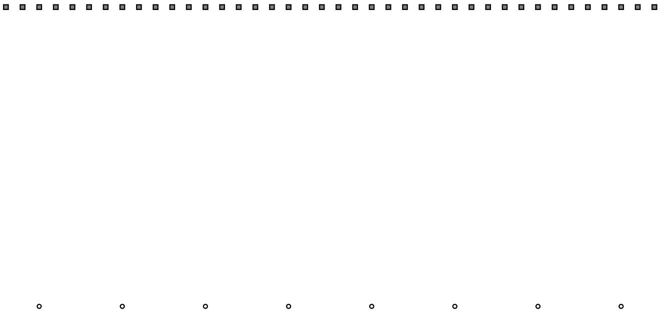


Random k-SAT at clause density α

set F of $m \sim \text{Poisson}(n\alpha)$ clauses



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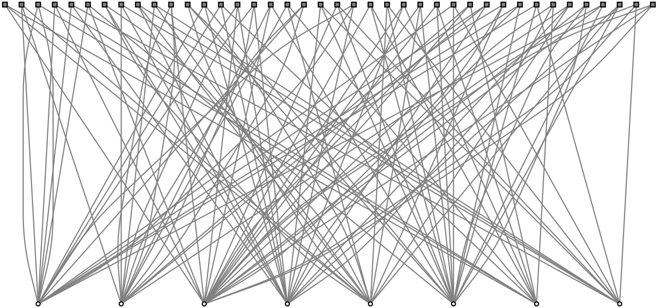

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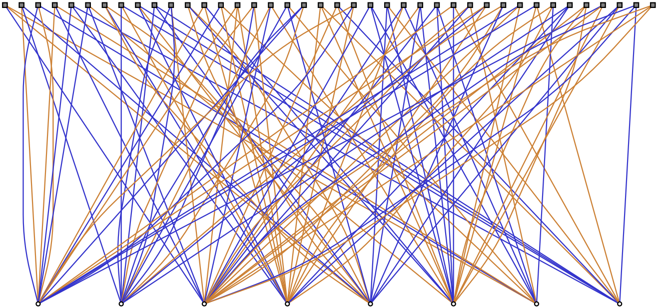

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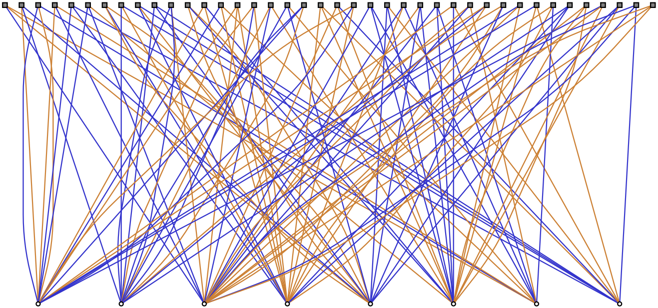

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set E of random edges, each clause degree k (here $k = 3$)
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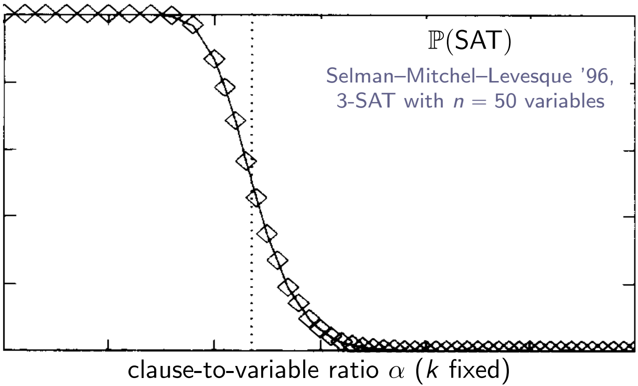
— altogether forms a **random k-SAT instance $\mathscr{G}$**:
an 'average-case' version of k-SAT

Threshold conjecture

SAT threshold conjecture

SAT threshold conjecture. *For each fixed k (with $k \geqslant 2$), random k-SAT has a sharp satisfiability threshold* $\alpha_{\mathsf{sat}}(k)$*:*



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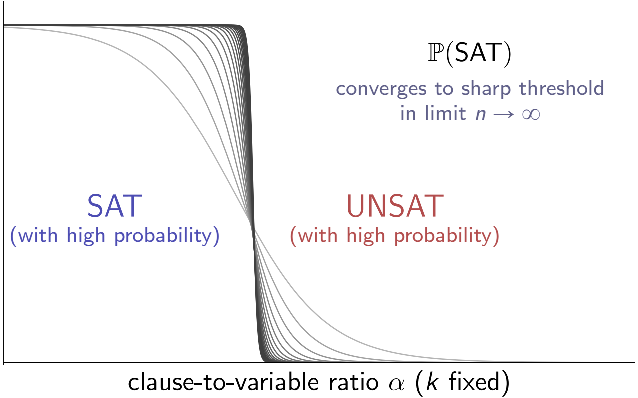

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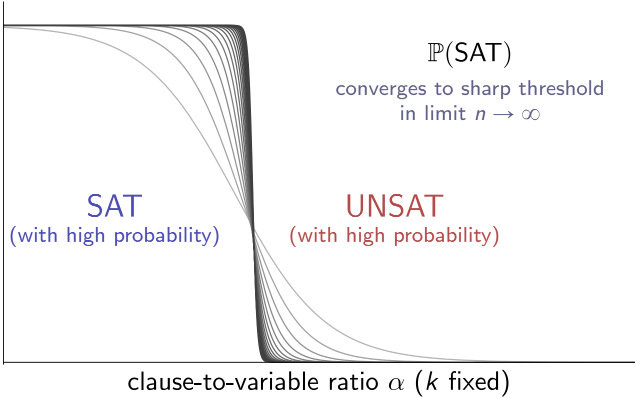

Since early '90s, known for $k = 2$, open for $k \geqslant 3$.

($k = 2$) Goerdt '92, '96, Chvátal–Reed '92, de la Vega '92

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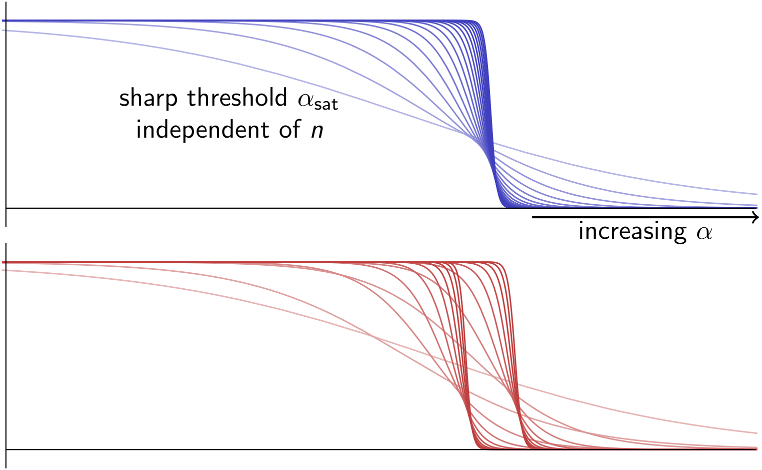

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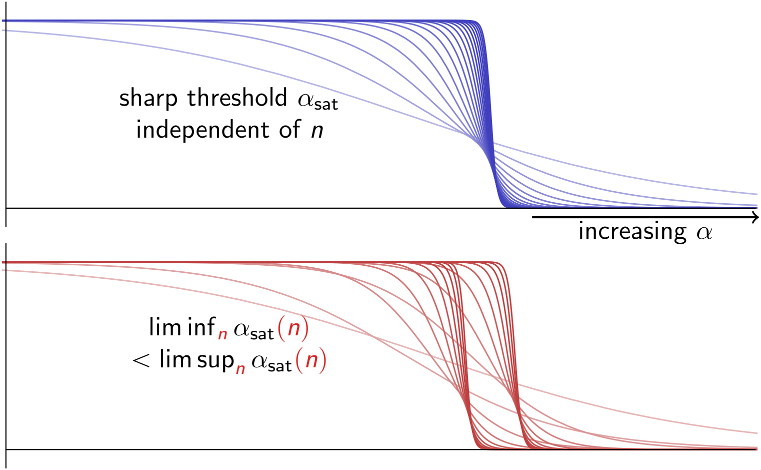

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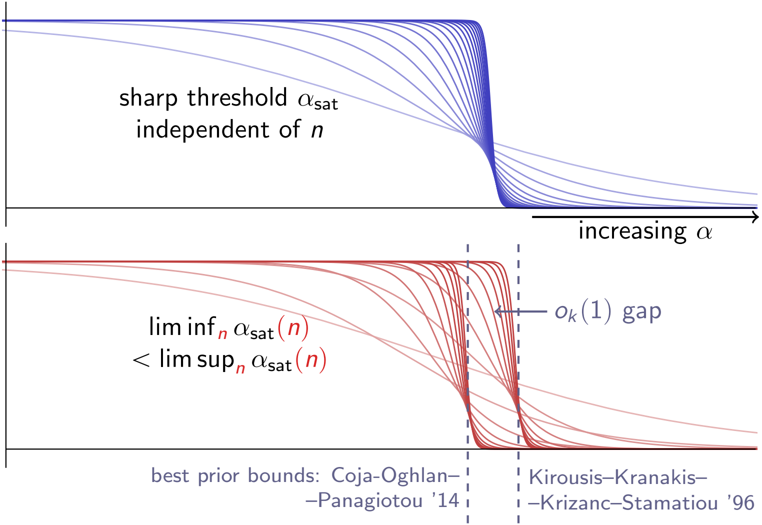

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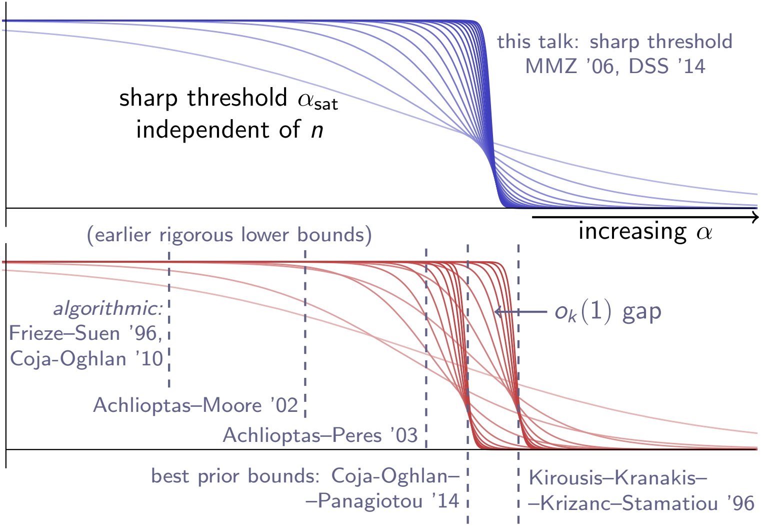

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Less is understood for *sparse* models like random k-SAT.

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Extensive physics literature proposes a class of sparse random CSPs exhibiting the **same** qualitative behavior — ‘1RSB’.

Krząkała–Montanari–Ricci-Tersenghi–Semerjian–Zdeborová ’07, Zdeborová–Krząkała ’07, Montanari–Ricci-Tersenghi–Semerjian ’08

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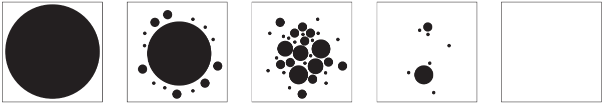

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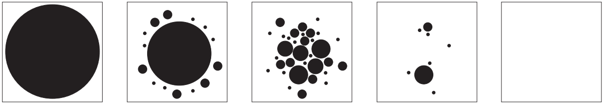
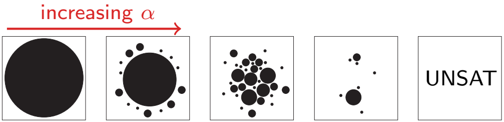


KMRSZ '07

Structural phenomena have been linked to algorithmic barriers.

e.g. Achlioptas–Coja-Oghlan '08, Sly '10, Gamarnik–Sudan '13, Rahman–Virag '14

The 1RSB threshold



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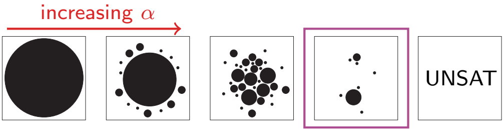

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On the basis of this *structural assumption*, one can derive an explicit conjecture $\alpha_{\text{sat}} = \alpha_{\star}$. This is the *1RSB threshold formula*. Similar formulas can be derived in other models.

derivation for random k-SAT: Mertens–Mézard–Zecchina '06

Moment method and 1RSB

In prior literature, best bounds on α_{sat} are by moment method on Z (number of solutions), with increasingly sophisticated truncation/conditioning to handle the non-concentration of Z.

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The predicted threshold value $\alpha_\star$ is a complicated function — makes it (highly) unlikely that a rigorous determination of α_{sat} can be made without relying on the physics insight.

Replica symmetry breaking

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The (random) measure ν is an example of a *graphical model* (or *factor model/Gibbs measure/Markov random field*).

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We say ν is *RS* if faraway variables are 'nearly independent' *(correlation decay)*.

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Otherwise, ν has *long-range dependencies* and it is RSB. In this case $\text{overlap}(\underline{X}^1, \underline{X}^2)$ has a non-trivial distribution.

failure of correlation decay is a key source of difficulty in the analysis

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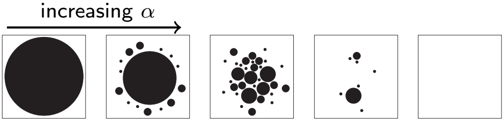
Random k-SAT exhibits both RS and RSB regimes:



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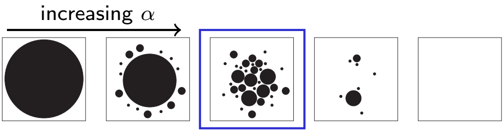

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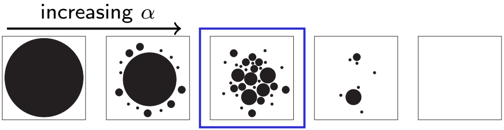

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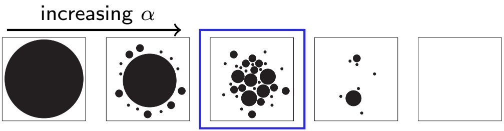

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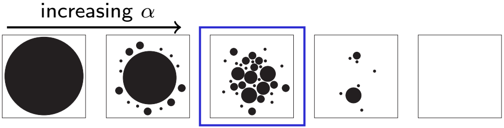

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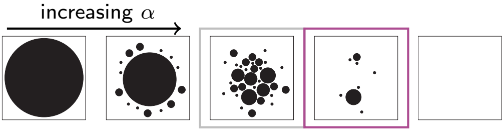

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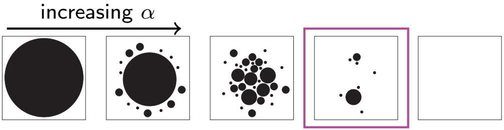

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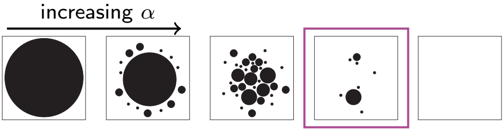

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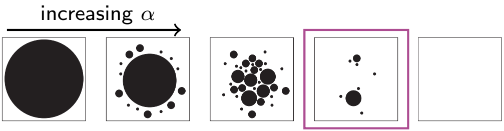

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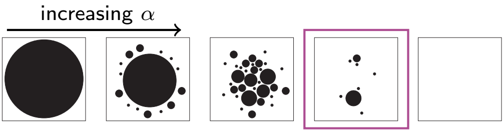

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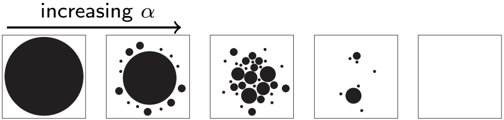
Clusters of solutions

Why does the solution space **SOL** exhibit clustering?



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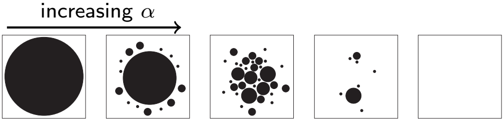

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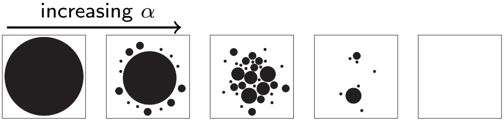

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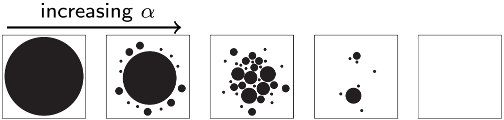

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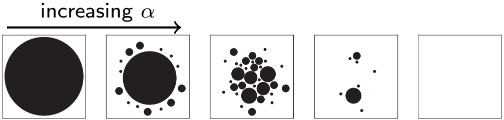

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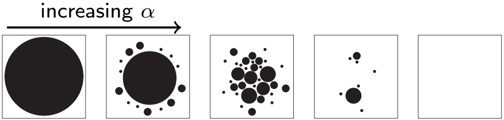

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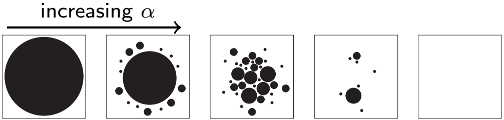

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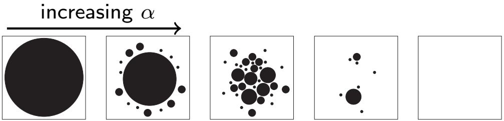

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Such clustering is a generic feature of sparse random CSPs.

Condensation, 1RSB, and cluster encodings

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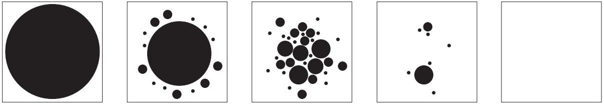
Random k-SAT with n variables, $m \sim \text{Poisson}(n\alpha)$ clauses:



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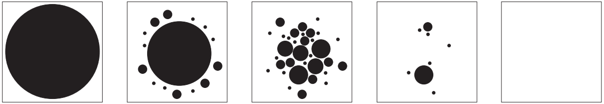

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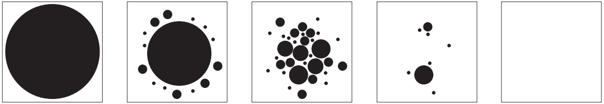

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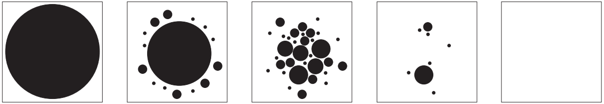

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What are the implications for the rigorous approaches to α_{sat}? For example, how does all this relate back to $\mathbb{E}Z$?

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Under an additional set of assumptions (1RSB) (more later)

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$\mathbb{E}Z \doteq \exp\{n[s_\star + \Sigma(s_\star)]\}$ where $\Sigma'(s_\star) = -1$. As α increases:

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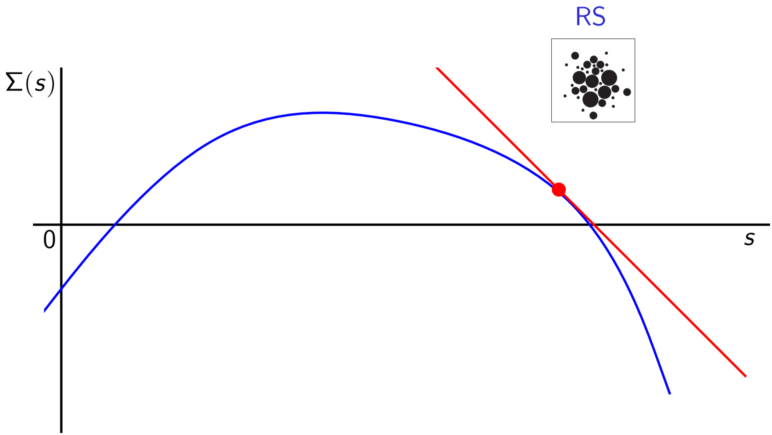

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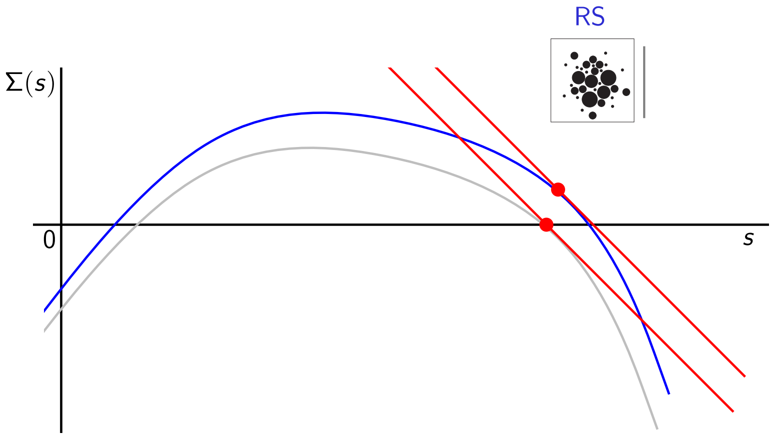

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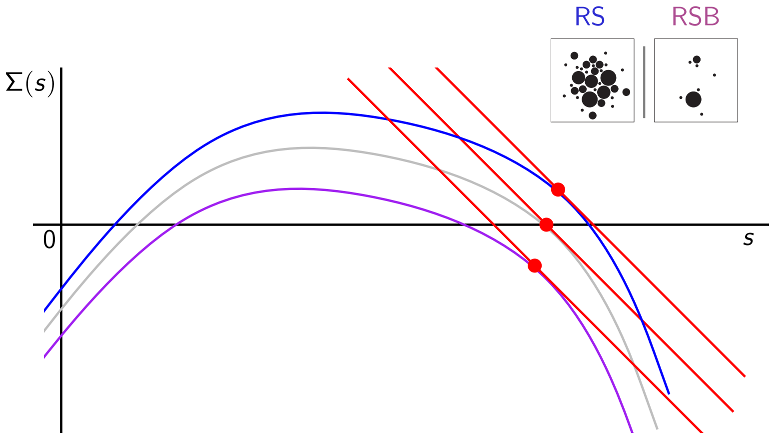

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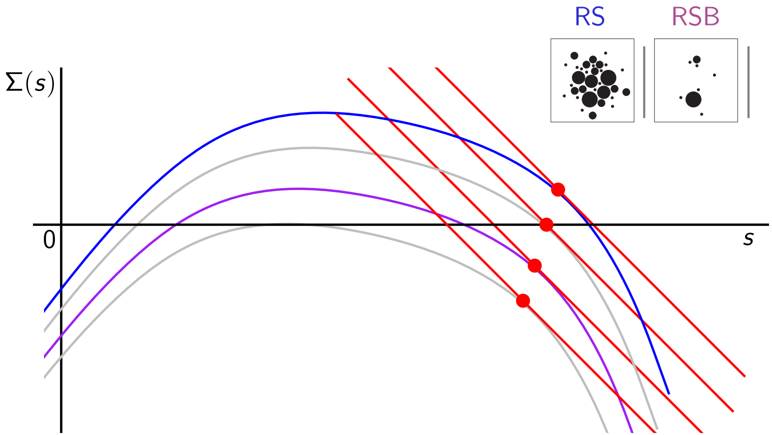

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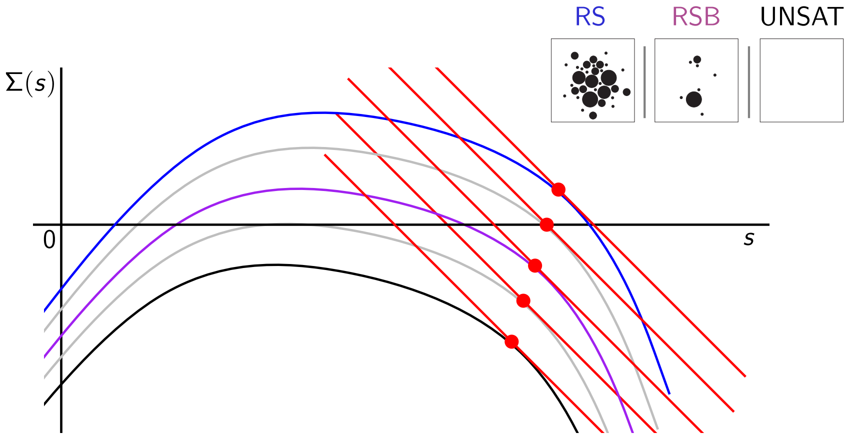

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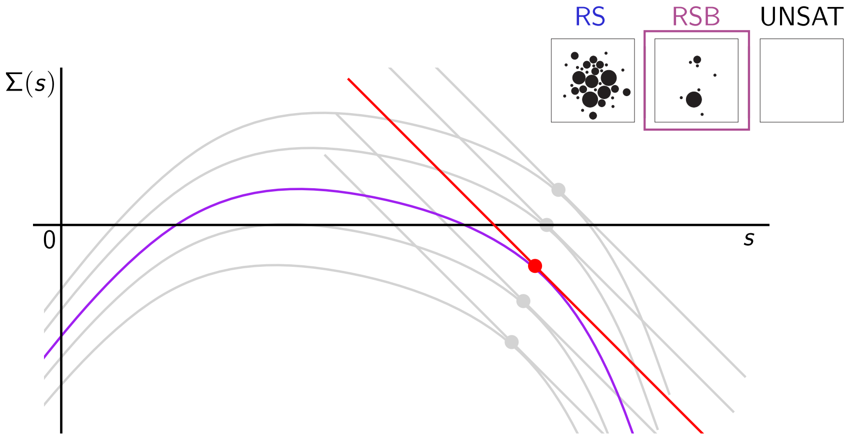

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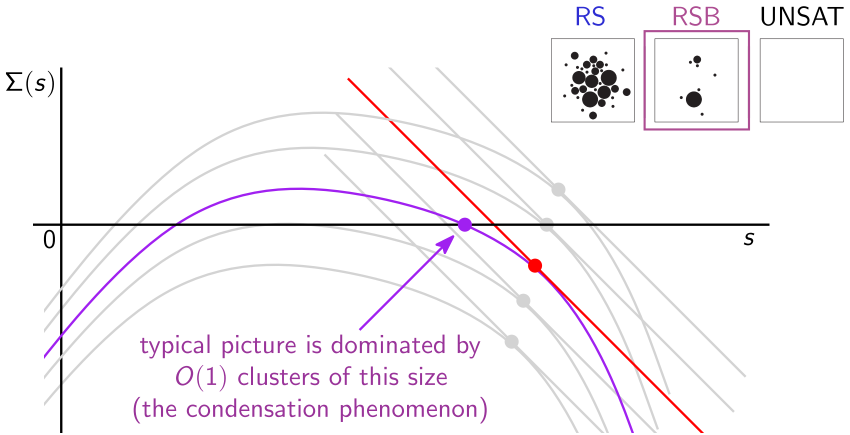

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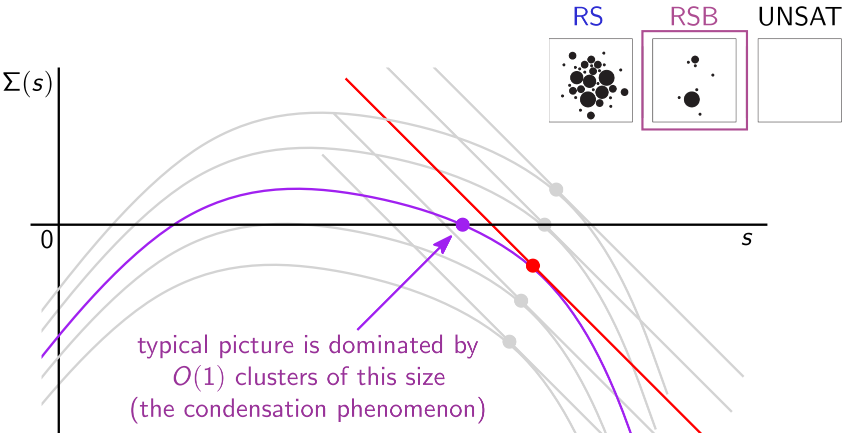

Upon onset of RSB (condensation/Kauzmann transition), $\mathbb{E}Z$ becomes dominated by atypically large clusters. $Z \ll \mathbb{E}Z$ whp.

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The detailed phase diagram is derived with the assumption

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This assumption underlies the **explicit derivation of** $\Sigma(s)$*, and yields* $\alpha_\star = \max\{\alpha : \Sigma_{\max}(\alpha) \equiv \max_s \Sigma(s;\alpha) > 0\}$.

Exact formulas

1RSB formulas

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no 'clusters within clusters'

How to get from this to formulas? What does it really mean for clusters to be RS? Need *graphical model of clusters*.

here, graphical model = (weighted) CSP

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Cavity measure and fixed points

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so $(\star)$ yields $Z_n \doteq \phi^n$ for **explicit** ϕ!

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Lift G to new CSP G^{BP} whose constraints are the BP eqns.: $\{\text{clusters } \gamma\} \leftrightarrow \{\text{BP fixed points } \underline{q}^{\gamma}\} \leftrightarrow \{\text{solutions of } G^{\mathrm{BP}}\}$.

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With more work, can predict full curve $\Sigma(s;\alpha)$.

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If only interested in $\max_s \Sigma(s)$, can further reduce BP to WP: $q_{x\to y}$ (measure on $\{+,-\}$) projects to $\pi_{x\to y} \in \{+,-,\texttt{free}\}$.

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So far, in models where α_{sat} was rigorously determined, lower bounds go through WP configurations $\underline{\pi}$. Informal idea is to show that the $\underline{\pi}$'s 'do not cluster' — partially confirms 1RSB.

Some open questions

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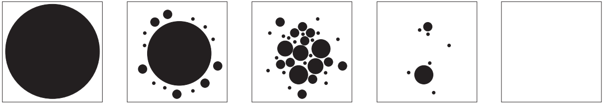
What is the typical value of Z?

Other aspects of phase diagram (structural properties of **SOL**)?



How does the picture change at positive temperature?

Models with higher levels of RSB (MAX-CUT)?

Explicit k-SAT threshold & thanks!

Let $\mathscr{P} \equiv$ space of probability measures on $[0,1]$. Define the distributional recursion $\boldsymbol{R}_\alpha : \mathscr{P} \to \mathscr{P}$,

$$(\boldsymbol{R}_\alpha \mu)(B) \equiv \sum_{\underline{d} \equiv (d^+, d^-)} \pi_\alpha(\underline{d}) \int \mathbf{1}\left\{ \frac{(1-\Pi^-)\Pi^+}{\Pi^+ + \Pi^- - \Pi^+\Pi^-} \in B \right\} \prod_{i,j} d\mu(\eta_{ij}^{\pm})$$

$$\text{with } \pi_\alpha(\underline{d}) \equiv \frac{e^{-k\alpha}(k\alpha/2)^{d^+ + d^-}}{(d^+)!(d^-)!}, \; \Pi^{\pm} \equiv \Pi^{\pm}(\underline{d}, \underline{\eta}) \equiv \prod_{i=1}^{d^{\pm}} \left(1 - \prod_{j=1}^{k-1} \eta_{ij}^{\pm} \right).$$

We show $(\boldsymbol{R}_\alpha)^\ell \mathbf{1}_{1/2} \xrightarrow{\ell \to \infty} \mu_\alpha$, and use μ_α to define

$$\Phi(\alpha) = \sum_{\underline{d}} \pi_\alpha(\underline{d}) \int \ln \left(\frac{\Pi^+ + \Pi^- - \Pi^+\Pi^-}{(1 - \prod_{j=1}^{k} \eta_j)^{\alpha(k-1)}} \right) \prod_j d\mu_\alpha(\eta_j) \prod_{i,j} d\mu_\alpha(\eta_{ij}^{\pm}).$$

For $k \geqslant k_0$, the random k-SAT threshold $\alpha_{\text{sat}} = \alpha_\star$ is the unique solution of $\Phi(\alpha) = 0$ in the interval $2^k \ln 2 - 2 \leqslant \alpha \leqslant 2^k \ln 2$.