The Classification Program II: Tractable Classes and Hardness Proof

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Previously ...
By a Pfaffian orientation, one can compute $\mathrm{PerfMatch}(G)$ in polynomial time.

Definition

A matchgate is an undirected weighted plane graph G with a subset of distinguished nodes on its outer face, called the external nodes, ordered in a clockwise order.

Let G be a matchgate with k external nodes. For each $\alpha \in \{0,1\}^k$, G defines a subgraph G^α obtained from G by moving all external nodes i (and incident edges) such that $\alpha_i = 1$.

Definition

We define the signature of a matchgate G as the vector $\Gamma_G = (\Gamma_G^\alpha)$, indexed by $\alpha \in \{0,1\}^k$ in lexicographic order, as follows:

$$\Gamma_G^\alpha = \mathrm{PerfMatch}(G^\alpha) = \sum_{M \in \mathcal{M}(G^\alpha)} \prod_{e \in M} w(e). \quad (1)$$

Perfect Matchings as a Holant Sum

Counting the number of Perfect Matchings can be viewed as follows:

$$\mathrm{Holant}(G) = \sum_{\sigma : E \to \{0,1\}} \prod_{v \in V} f_v(\sigma \mid_{E(v)}).$$

where every vertex v is labeled by an $\textsc{Exact-One}$ function f_v of arity $\deg(v)$.
We then consider

$$\mathrm{Holant}(G) = \sum_{\sigma : E \to \{0,1\}} \prod_{v \in V} f_v(\sigma \mid_{E(v)}).$$

Each product term gives a one if $\sigma^{-1}(1)$ is a Perfect Matching, and zero otherwise.

Holant Sum

Definition

Let $\mathcal{F}$ be a set of constraint functions (signatures). A signature grid is a tuple $\Omega = (G, \pi)$ where π assigns a function $f \in \mathcal{F}$ to each vertex of G.

Definition

For a set of signatures $\mathcal{F}$, $\mathrm{Holant}(\mathcal{F})$ is the following class of problems:
Input: A *signature grid* $\Omega = (G, \pi)$ over $\mathcal{F}$;
Output:

$$\mathrm{Holant}(\Omega; \mathcal{F}) = \sum_{\sigma : E \to \{0,1\}} \prod_{v \in V} f_v(\sigma \mid_{E(v)}),$$

where

- $E(v)$ denotes the incident edges of v and
- $\sigma \mid_{E(v)}$ denotes the restriction of σ to $E(v)$, and $f_v(\sigma \mid_{E(v)})$ is the evaluation of f_v on the ordered input tuple $\sigma \mid_{E(v)}$.

#PL-3-NAE-ICE
INPUT: A planar graph $G = (V, E)$ of maximum degree 3.
OUTPUT: The number of orientations such that no node has all incident edges directed toward it or all incident edges directed away from it.

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If f is a symmetric function on $\{x_1, x_2, \ldots, x_n\}$, we can denote it as $[f_0, f_1, \ldots, f_n]$, where f_w is the value of f on input of Hamming weight w. Thus the ternary NOT-ALL-EQUAL function f is $[0, 1, 1, 0]$.

For every edge, we can replace it by a path of length 2, and assign the binary DISEQUALITY function $[0, 1, 0]$ at the new vertex.

#PL-3-NAE-ICE continued

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The Holant sum on the bipartite Edge-Vertex incidence graph of G is a sum over $2^{2|E|}$ terms.

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Each vertex function $[0, 1, 1, 0]$ evaluates to 1 if the no-sink-no-source condition is satisfied, and it evaluates to 0 otherwise.

Holant Sum as a Dot Product

This Holant Sum can be viewed as a (long) dot product of the following two vectors:
On LHS: we take the tensor product of all $[0, 1, 0]$, one per each edge.
On RHS: we take the tensor product of all $[0, 1, 1, 0]$, one per each vertex.
The indices of the two (long) vectors (each of dimension $2^{2|E|}$) are matched up by the connection of the graph.

Holographic Transformation

We can perform a local transformation by $H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

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$$\begin{aligned} [0,1,1,0] &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}^{\otimes 3} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\otimes 3} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}^{\otimes 3} \\ \mapsto H^{\otimes 3}[0,1,1,0] &= \begin{bmatrix} 2 \\ 0 \end{bmatrix}^{\otimes 3} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}^{\otimes 3} - \begin{bmatrix} 1 \\ -1 \end{bmatrix}^{\otimes 3} = [6,0,-2,0], \end{aligned}$$

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and

$$\leftarrow\!\shortmid \quad [0,1,0] = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\otimes 2} - \begin{bmatrix} 1 & 0 \end{bmatrix}^{\otimes 2} - \begin{bmatrix} 0 & 1 \end{bmatrix}^{\otimes 2}$$
$$[0,1,0](H^{-1})^{\otimes 2} = [\frac{1}{2}, 0, \frac{-1}{2}] = \frac{1}{2}[1,0,-1].$$

Valiant's Holant Theorem

Theorem

If there is a holographic transformation mapping signature grid Ω to Ω', then $Holant_{\Omega} = Holant_{\Omega'}$.

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If there is a holographic transformation mapping signature grid Ω to Ω', then $\text{Holant}_\Omega = \text{Holant}_{\Omega'}$.

Hence the same quantity is obtained for #PL-3-NAE-ICE if we use the signature $[6, 0, -2, 0] = H^{\otimes 3}[0, 1, 1, 0]$ for each vertex,
And the signature $\frac{1}{2}[1, 0, -1] = [0, 1, 0](H^{-1})^{\otimes 2}$ for each edge.

Both $[6, 0, -2, 0]$ and $\frac{1}{2}[1, 0, -1]$ are matchgate signatures.

Holographic Algorithms by Matchgates

Both $[6, 0, -2, 0]$ and $\frac{1}{2}[1, 0, -1]$ are matchgate signatures.

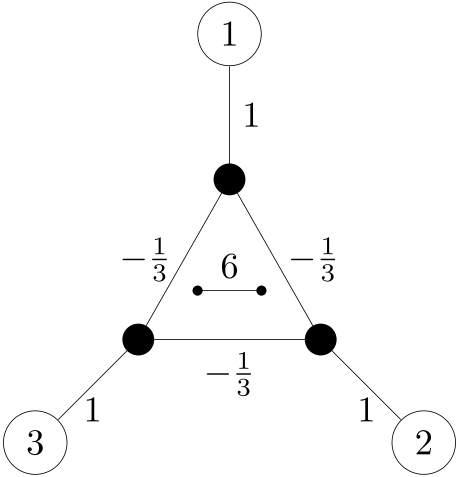

Figure: A matchgate with signature $[6, 0, -2, 0]$

Another Matchgate

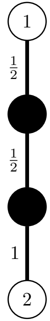

Figure: A matchgate with signature $\frac{1}{2}[1, 0, -1]$

Thus #PL-3-NAE-ICE is computable in P.

A list of symmetric matchgate signatures

Theorem

A symmetric signature is the signature of a matchgate iff it has the following form, for some $a, b \in \mathbb{C}$ *and integer* k *(we take the convention that* $0^0 = 1$*):*

1. $[a^k b^0, 0, a^{k-1} b, 0, a^{k-2} b^2, 0, \ldots, a^0 b^k]$ *(arity* $2k \geq 2$*)*
2. $[a^k b^0, 0, a^{k-1} b, 0, a^{k-2} b^2, 0, \ldots, a^0 b^k, 0]$ *(arity* $2k + 1 \geq 1$*)*
3. $[0, a^k b^0, 0, a^{k-1} b, 0, a^{k-2} b^2, 0, \ldots, a^0 b^k]$ *(arity* $2k + 1 \geq 1$*)*
4. $[0, a^k b^0, 0, a^{k-1} b, 0, a^{k-2} b^2, 0, \ldots, a^0 b^k, 0]$ *(arity* $2k + 2 \geq 2$*).*

Fibonacci Gates

Recall symmetric signatures are denoted as $[f_0, f_1, \ldots, f_n]$.

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Definition

For any $n \geq 1$, a signature $f = [f_0, f_1, \ldots, f_n]$ is a Fibonacci gate if

$$f_{k+2} = f_{k+1} + f_k, \quad 0 \leq k \leq n - 2.$$

A set of signatures $\mathcal{F}$ is called Fibonacci if every signature in $\mathcal{F}$ is a Fibonacci gate.

Fibonacci are Tractable

Recall

$$\text{Holant}(\Omega; \mathcal{F}) = \sum_{\sigma:E\to\{0,1\}} \prod_{v\in V} f_v(\sigma\mid_{E(v)}),$$

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Theorem

For any finite set of Fibonacci gates $\mathcal{F}$, the Holant problem Holant($\mathcal{F}$) is computable in polynomial time.

Fibonacci Gates

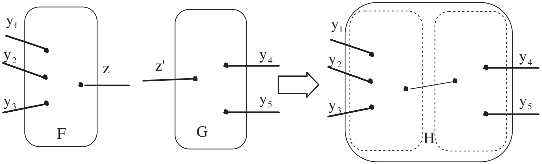

Figure: First operation.

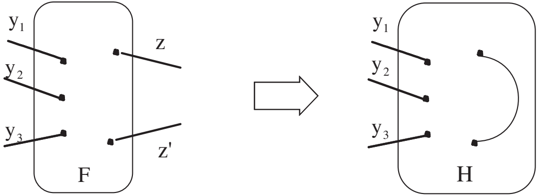

Figure: Second operation.

Definition

For any $n \geq 1$, and a parameter $\lambda \in \mathbb{C}$, a signature $f = [f_0, f_1, \ldots, f_n]$ is a generalized Fibonacci gate (with parameter λ) if

$$f_{k+2} = \lambda f_{k+1} + f_k, \quad 0 \leq k \leq n-2. \tag{2}$$

A set of signatures $\mathcal{F}$ is called generalized Fibonacci if for some $\lambda \in \mathbb{C}$, every signature in $\mathcal{F}$ is a generalized Fibonacci gate with parameter λ.

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Gen-Eq are Generalized Equalities: $[*, 0, \ldots, 0, *]$.

Fibonacci Gates Under Holographic Transformation

Define

$$\mathscr{F} = \{f \mid f \text{ satisfies (2) for some } \lambda \neq \pm 2\mathfrak{i}\} \cup \textsc{Gen-Eq}. \tag{3}$$

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Theorem

For any $f \in \mathscr{F}$ in (3),

1. *There exists an orthogonal T such that Tf is a $\textsc{Gen-Eq}$.*
2. *There exists an orthogonal T such that Tf is a Fibonacci gate satisfying Definition 7.*
3. *For all orthogonal T, $Tf \in \mathscr{F}$.*

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Remark: In (3), when $\lambda = \pm 2\mathfrak{i}$, f is a vanishing signature.

Fibonacci Gates Under Holographic Transformation

Theorem

A symmetric signature $[f_0, f_1, \ldots, f_n]$ *can be transformed by some invertible holographic transformation to a Fibonacci gate according to Definition 7 (equivalently to a signature in* $\mathscr{F}$ *defined in (3)) iff there exist three constants* a, b *and* c*, such that* $b^2 - 4ac \neq 0$*, and for all* $0 \leq k \leq n-2$*,*

$$af_k + bf_{k+1} + cf_{k+2} = 0. \tag{4}$$

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$\mathrm{Holant}^*(\mathcal{F})$ is the problem $\mathrm{Holant}(\mathcal{F} \cup \mathcal{U})$, where $\mathcal{U}$ is the set of all unary signatures.

Degeneracy

A signature is degenrate if it is a tensor product of unary signatures.

This includes all unary signatures.

If $\mathcal{F}$ consists of degenrate signatures, then $\mathrm{Holant}(\mathcal{F})$ is tractable.

The Signatures of Product Type

Definition

A function has product type if it can be expressed as a product of unary functions, binary Equality functions $((=_2) = [1, 0, 1])$ and binary Disequality functions $((\neq_2) = [0, 1, 0])$, on not necessarily disjoint subsets of variables.

We denote by $\mathscr{P}$ the set of all functions of *product type*.

Theorem

Let $\mathcal{F}$ be a set of non-degenerate symmetric signatures over $\mathbb{C}$. Then $\operatorname{Holant}^*(\mathcal{F})$ is #P-hard, unless $\mathcal{F}$ satisfies the following conditions, in which case it is computable in polynomial time.

1. *All signatures in $\mathcal{F}$ have arity at most 2.*
2. *There exists some $M \in \mathbf{GL}_2(\mathbb{C})$ such that $(=_2)M^{\otimes 2} \in \mathscr{P}$ and $\mathcal{F} \subseteq M\mathscr{P}$.*
3. *There exists $\lambda \in \{2\mathfrak{i}, -2\mathfrak{i}\}$, such that every signature $f \in \mathcal{F}$ of arity n satisfies the recurrence*

$$f_{k+2} = \lambda f_{k+1} + f_k, \qquad \text{for} \quad 0 \leq k \leq n-2.$$

The counting constraint satisfaction problem $\#\mathrm{CSP}(\mathcal{F})$ is defined as follows: The input I is a finite sequence of constraints on variables $x_1, x_2, \ldots, x_n$ of the form $F(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$, where $F \in \mathcal{F}$. The output is called the partition function

$$Z(I) = \sum_{x_1, x_2, \ldots, x_n \in \{0,1\}} \prod F(x_{i_1}, x_{i_2}, \ldots, x_{i_k}),$$

where the product is over all constraints occurring in I.
For now we will restrict to the Boolean domain.

Definition

A function is of affine type if it can be expressed as

$$\lambda \cdot \chi_{AX} \cdot \mathfrak{i}^{L_1(X)+L_2(X)+\cdots+L_n(X)},$$

where $X = (x_1, x_2, \ldots, x_k, 1)$ $\lambda \in \mathbb{C}$, $\mathfrak{i} = \sqrt{-1}$, each L_j is an integer 0-1 indicator function of the form $\langle \alpha_j, X \rangle$, where α_j is a $k+1$ dimensional vector over $\mathbb{Z}_2$ and the dot product $\langle \cdot, \cdot \rangle$ is computed over $\mathbb{Z}_2$.
The set of all functions of *affine type* is denoted by $\mathscr{A}$.

Another Definition of Affine Signatures

Theorem

A function f belongs to $\mathscr{A}$ iff it can be expressed as $\lambda \chi_{AX} \mathfrak{i}^{Q(x_1, \ldots, x_k)}$ where Q is a homogeneous quadratic polynomial over $\mathbb{Z}$ with the additional requirement that every cross term $x_s x_t$ has an even coefficient, where $s \neq t$. We may also use all, not necessarily homogeneous, polynomials over $\mathbb{Z}$ of degree at most 2, with the same requirement on cross terms.

Symmetric Affine Signatures

$$\begin{aligned}
\mathscr{F}_1 &= \{\lambda([1,0]^{\otimes k} + \mathfrak{i}^r[0,1]^{\otimes k}) \mid \lambda \in \mathbb{C}, k = 1,2,\ldots, \text{ and } r = 0,1,2,3\}, \\
\mathscr{F}_2 &= \{\lambda([1,1]^{\otimes k} + \mathfrak{i}^r[1,-1]^{\otimes k}) \mid \lambda \in \mathbb{C}, k = 1,2,\ldots, \text{ and } r = 0,1,2,3\}, \\
\mathscr{F}_3 &= \{\lambda([1,\mathfrak{i}]^{\otimes k} + \mathfrak{i}^r[1,-\mathfrak{i}]^{\otimes k}) \mid \lambda \in \mathbb{C}, k = 1,2,\ldots, \text{ and } r = 0,1,2,3\}.
\end{aligned}$$

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We note that expressions in complex numbers appear naturally, even for real-valued functions. The special case where $r = 1$, $k = 2$ and $\lambda = (1 + \mathfrak{i})^{-1}$ in $\mathscr{F}_3$ is noteworthy. In this case we get a real-valued binary symmetric function $H = [1, 1, -1]$. In other words, $H(0,0) = H(0,1) = H(1,0) = 1$ and $H(1,1) = -1$. The matrix form of this function is the Hadamard matrix $H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Explicit list of $\mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$

1. $[1, 0, \ldots, 0, \pm 1]$; $(\mathscr{F}_1, r = 0, 2)$
2. $[1, 0, \ldots, 0, \pm \mathfrak{i}]$; $(\mathscr{F}_1, r = 1, 3)$
3. $[1, 0, 1, 0, \ldots, 0 \text{ or } 1]$; $(\mathscr{F}_2, r = 0)$
4. $[1, -\mathfrak{i}, 1, -\mathfrak{i}, \ldots, (-\mathfrak{i}) \text{ or } 1]$; $(\mathscr{F}_2, r = 1)$
5. $[0, 1, 0, 1, \ldots, 0 \text{ or } 1]$; $(\mathscr{F}_2, r = 2)$
6. $[1, \mathfrak{i}, 1, \mathfrak{i}, \ldots, \mathfrak{i} \text{ or } 1]$; $(\mathscr{F}_2, r = 3)$
7. $[1, 0, -1, 0, 1, 0, -1, 0, \ldots, 0 \text{ or } 1 \text{ or } (-1)]$; $(\mathscr{F}_3, r = 0)$
8. $[1, 1, -1, -1, 1, 1, -1, -1, \ldots, 1 \text{ or } (-1)]$; $(\mathscr{F}_3, r = 1)$
9. $[0, 1, 0, -1, 0, 1, 0, -1, \ldots, 0 \text{ or } 1 \text{ or } (-1)]$; $(\mathscr{F}_3, r = 2)$
10. $[1, -1, -1, 1, 1, -1, -1, 1, \ldots, 1 \text{ or } (-1)]$. $(\mathscr{F}_3, r = 3)$

A Dichotomy Theorem of #CSP over Boolean Domain

Theorem

Suppose $\mathscr{F}$ is a set of functions mapping Boolean inputs to complex numbers. If $\mathscr{F} \subseteq \mathscr{A}$ or $\mathscr{F} \subseteq \mathscr{P}$, then #CSP($\mathscr{F}$) is computable in polynomial time. Otherwise, #CSP($\mathscr{F}$) is $\#\mathrm{P}$-hard.

Three Frameworks for Counting Problems

1. Graph Homomorphisms
2. Constraint Satisfaction Problems (#CSP)
3. Holant Problems

In each framework, there has been remarkable progress in the classification program of the complexity of counting problems.

L. Lovász:
Operations with structures, Acta Math. Hung. 18 (1967), 321-328.

http://www.cs.elte.hu/~lovasz/hom-paper.html

Let $\mathbf{A} = (A_{i,j}) \in \mathbb{C}^{\kappa \times \kappa}$ be a symmetric complex matrix.

The Graph Homomorphism problem is:
INPUT: An undirected graph $G = (V, E)$.
OUTPUT:

$$Z_{\mathbf{A}}(G) = \sum_{\xi : V \to [\kappa]} \prod_{(u,v) \in E} A_{\xi(u),\xi(v)}.$$

Theorem

[C., Xi Chen and Pinyan Lu] For any symmetric complex valued matrix $\mathbf{A} \in \mathbb{C}^{\kappa \times \kappa}$, the problem of computing $Z_{\mathbf{A}}(G)$, for any input G, is either in P or #P-hard.
Given $\mathbf{A}$, whether $Z_{\mathbf{A}}(\cdot)$ is in P or #P-hard can be decided in polynomial time in the size of $\mathbf{A}$.

SIAM J. Comput. 42(3): 924-1029 (2013) (106 pages)

Many partial results: Dyer, Greenhill, Bulatov, Grohe, Goldberg, Jerrum, Thurley, . . .

[C., Xi Chen]

Theorem

Every finite set $\mathcal{F}$ of complex valued constraint functions on any finite domain set $[\kappa]$ defines a counting CSP problem $\#$CSP($\mathcal{F}$) that is either computable in P or $\#$P-hard.

[C., Xi Chen]

Theorem

Every finite set $\mathcal{F}$ of complex valued constraint functions on any finite domain set $[\kappa]$ defines a counting CSP problem $\#$CSP($\mathcal{F}$) that is either computable in P or $\#$P-hard.

The decision version of this is open.
The decidability of this #CSP Dichotomy is open.

Creignou, Hermann, ..., Bulatov, Dalmau, Dyer, Richerby, Lu ...
Creignou, Khanna, Sudan: Complexity Classifications of Boolean Constraint Satisfaction Problems, SIAM.

A Holant problem is parametrized by a set of signatures.

Definition

Given a set of signatures $\mathcal{F}$, we define the counting problem Holant($\mathcal{F}$) as:
Input: A *signature grid* $\Omega = (G, \pi)$;
Output: Holant($\Omega; \mathcal{F}$).

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The problem Pl-Holant($\mathcal{F}$) is defined similarly using a planar signature grid.

Definition

We say a signature set $\mathcal{F}$ is $\mathscr{C}$-transformable for Holant($\mathcal{F}$), if there exists $T \in \mathbf{GL}_2(\mathbb{C})$ such that $(=_2)T^{\otimes 2} \in \mathscr{C}$ and $T^{-1}f \in \mathscr{C}$ for all $f \in \mathcal{F}$.

[C., Heng Guo, Tyson Williams]

Theorem

Let $\mathcal{F}$ be any set of symmetric, complex-valued signatures in Boolean variables. Then $\text{Holant}(\mathcal{F})$ *is #P-hard unless $\mathcal{F}$ satisfies one of the following conditions, in which case the problem is in P:*

1. *All non-degenerate signatures in $\mathcal{F}$ have arity ≤ 2;*
2. *$\mathcal{F}$ is $\mathcal{A}$-transformable;*
3. *$\mathcal{F}$ is $\mathcal{P}$-transformable;*
4. $\mathcal{F} \subseteq \mathcal{V}^\sigma \cup \{f \in \mathcal{R}_2^\sigma \mid \text{arity}(f) = 2\}$ *for* $\sigma \in \{+, -\}$;
5. *All non-degenerate signatures in $\mathcal{F}$ are in $\mathcal{R}_2^\sigma$ for $\sigma \in \{+, -\}$.*

A planar matchgate $\Gamma = (G, X)$ is a weighted graph $G = (V, E, W)$ with a planar embedding, having external nodes, placed on the outer face.

Define $\mathrm{PerfMatch}(G) = \sum_M \prod_{(i,j) \in M} w_{ij}$, where the sum is over all perfect matchings M.
A matchgate Γ is assigned a Matchgate Signature

$$G = (G^S),$$

where

$$G^S = \mathrm{PerfMatch}(G - S).$$

The matchgate signatures are characterized by: (1) Parity Condition: either all even entries are 0 or all odd entries are 0.
(2) Matchgate Identities (MGI): For any patterns $\alpha, \beta \in \{0,1\}^n$, let bitwise XOR $\alpha \oplus \beta$ have bit 1 at $1 \leq p_1 < p_2 < \ldots < p_\ell \leq n$. Then

$$\sum_{i=1}^{\ell} (-1)^i f_{\alpha \oplus e_{p_i}} f_{\beta \oplus e_{p_i}} = 0. \tag{5}$$

Matchgate-Identities

Valiant first proved MGI for arity at most 4. General proofs are given in [C., Choudhary, Lu][C., Lu]. See also
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Planarity $\Longrightarrow$ j has the opposite parity as i.

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Planarity $\Longrightarrow$ j has the opposite parity as i.
Now flipping edges along the alternating path, we get

$$M \Longrightarrow \widehat{M} \in \mathcal{M}^{\alpha \oplus e_{p_j}} \quad M' \Longrightarrow \widehat{M'} \in \mathcal{M}^{\beta \oplus e_{p_j}}$$

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This sets up a bijective mapping

$$\bigcup_{i even} \left[\mathcal{M}^{\alpha + e_{p_i}} \times \mathcal{M}^{\beta + e_{p_i}} \right] \leftrightarrow \bigcup_{j odd} \left[\mathcal{M}^{\alpha + e_{p_j}} \times \mathcal{M}^{\beta + e_{p_j}} \right]$$

maintaining weights.

Matchgate-Transformable

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Let $H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

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$$(=_k)H^{\otimes k} = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\otimes k} + \begin{bmatrix} 1 & -1 \end{bmatrix}^{\otimes k} = 2[1, 0, 1, 0, \ldots] \in \mathscr{M}.$$

Let $\widehat{\mathscr{M}} = H\mathscr{M}$.
Then for any $\mathcal{F} \subseteq \widehat{\mathscr{M}}$, PI-#CSP($\mathcal{F}$) is tractable.

Dichotomy Theorem for Planar #CSP

Heng Guo, Tyson Williams

Theorem

Let $\mathcal{F}$ be any set of symmetric, complex-valued signatures in Boolean variables. Then Pl-#CSP($\mathcal{F}$) *is #P-hard unless $\mathcal{F} \subseteq \mathscr{A}$, $\mathcal{F} \subseteq \mathscr{P}$, or $\mathcal{F} \subseteq \widehat{\mathscr{M}}$, in which case the problem is computable in polynomial time.*

Vanishing Signatures

Definition

A set of signatures $\mathcal{F}$ is called vanishing if the value $\text{Holant}_{\Omega}(\mathcal{F})$ is zero for every signature grid Ω. A signature f is called vanishing if the singleton set $\{f\}$ is vanishing.

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Definition

Let S_n be the symmetric group of degree n. Then for positive integers t and n with $t \leq n$ and unary signatures $v, v_1, \ldots, v_{n-t}$, we define

$$\text{Sym}_n^t(v; v_1, \ldots, v_{n-t}) = \sum_{\pi \in S_n} u_{\pi(1)} \otimes u_{\pi(2)} \cdots \otimes u_{\pi(k)}, \tag{6}$$

where the ordered sequence $(u_1, u_2, \ldots, u_n) = (\underbrace{v, \ldots, v}_{t \text{ copies}}, v_1, \ldots, v_{n-t})$.

Definition

A nonzero symmetric signature f of arity n has positive vanishing degree $k \geq 1$, denoted by $\text{vd}^+(f) = k$, if $k \leq n$ is the largest positive integer such that there exists $n - k$ unary signatures $v_1, \ldots, v_{n-k}$ satisfying

$$f = \text{Sym}_n^k([1, \mathfrak{i}]; v_1, \ldots, v_{n-k}).$$

If f cannot be expressed as such a symmetrization form, we define $\text{vd}^+(f) = 0$. If f is the all zero signature, define $\text{vd}^+(f) = n + 1$.

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Definition

For $\sigma \in \{+, -\}$, we define $\mathscr{V}^\sigma = \{f \mid 2\,\text{vd}^\sigma(f) > \text{arity}(f)\}$.

Definition

An arity n symmetric signature of the form $f = [f_0, f_1, \ldots, f_n]$ is in $\mathscr{R}_t^+$ for a nonnegative integer $t \geq 0$ if $t > n$; or for any $0 \leq k \leq n-t$, $f_k, \ldots, f_{k+t}$ satisfy the recurrence relation of order t

$$\binom{t}{t} \mathfrak{i}^t f_{k+t} + \binom{t}{t-1} \mathfrak{i}^{t-1} f_{k+t-1} + \cdots + \binom{t}{0} \mathfrak{i}^0 f_k = 0. \tag{7}$$

We define $\mathscr{R}_t^-$ similarly but with $-\mathfrak{i}$ in place of $\mathfrak{i}$ in (7).

Theorem

Let $\mathcal{F}$ be a set of symmetric signatures. Then $\mathcal{F}$ is vanishing if and only if $\mathcal{F} \subseteq \mathscr{V}^+$ or $\mathcal{F} \subseteq \mathscr{V}^-$.

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Theorem

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Let $Z = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]$,

Theorem

Suppose f is a symmetric signature of arity n. Let $\hat{f} = (Z^{-1})^{\otimes n} f$. If $\operatorname{vd}^+(f) = n - d$, then $\hat{f} = [\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_d, 0, \ldots, 0]$ and $\hat{f}_d \neq 0$.

Note that $[1, 0, 1] Z^{\otimes 2} = [0, 1, 0]$.

Definition

Let $G = (V, E)$ be an undirected graph, the Tutte polynomial of G is defined as

$$\mathrm{T}(G; x, y) = \sum_{A \subseteq E} (x-1)^{k(A)-k(E)} (y-1)^{k(A)+|A|-|V|}, \tag{8}$$

where $k(A)$ denotes the number of connected components of the graph (V, A).

Jaeger, Vertigan and Welsh

Theorem

For $x, y \in \mathbb{C}$, evaluating the Tutte polynomial at (x, y) is #P-hard over graphs unless

$$(x-1)(y-1) = 1$$

or
$(x, y) \in \{(1,1), (-1,-1), (0,-1), (-1,0), (\mathrm{i}, -\mathrm{i}), (-\mathrm{i}, \mathrm{i}), (\omega, \omega^2), (\omega^2, \omega)\}$, *where* $\omega = e^{2\pi \mathrm{i}/3}$. *In each exceptional case, the problem is in polynomial time.*

Theorem

For $x, y \in \mathbb{C}$, evaluating the Tutte polynomial at (x, y) is #P-hard over planar graphs unless

$$(x-1)(y-1) \in \{1, 2\} \quad \text{or} \quad (x, y) \in \{(1,1), (-1,-1), (\omega, \omega^2), (\omega^2, \omega)\},$$

where $\omega = e^{2\pi i/3}$. In each exceptional case, the problem is in polynomial time.

Medial Graph

Definition

Given a connected plane graph G, its *medial graph* G_m has a vertex e' for each edge e of G, and vertices e'_1 and e'_2 in G_m are joined by an edge for each face of G in which their corresponding edges e_1 and e_2 in G occur consecutively.

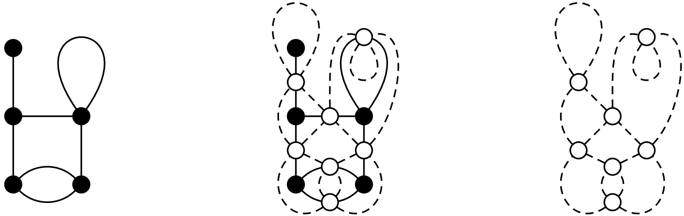

Figure: A plane graph, its medial graph, and the two graphs superimposed.

Tutte Polynomial and Eulerian Orientation

Definition

Given a graph G, an orientation is an *Eulerian orientation* if for each vertex v of G, the number of incoming edges of v equals the number of outgoing edges of v.

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Michel Las Vergnas

Theorem

Let G be a connected plane graph and let $\mathscr{O}(G_m)$ be the set of all Eulerian orientations in the medial graph G_m of G. Then

$$2 \cdot \mathrm{T}(G; 3, 3) = \sum_{O \in \mathscr{O}(G_m)} 2^{\beta(O)}, \tag{9}$$

where $\beta(O)$ is the number of saddle vertices in the orientation O, i.e. the number of vertices in which the edges are oriented "in, out, in, out" in cyclic order.

Theorem

*$\#$*EULERIAN-ORIENTATIONS *is $\#$P-hard for planar 4-regular graphs.*

Proof: 1. The Tutte Polynomial problem (right-hand side of (9)) is the bipartite planar Holant problem Pl-Holant $(\neq_2 \mid f)$, where the signature matrix of f is

$$M_f = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

2. By $Z = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{smallmatrix}\right]$, the Tutte Polynomial problem becomes

$$\begin{aligned} \text{Pl-Holant}\,(\neq_2 \mid f) &\equiv_T \text{Pl-Holant}\left([0,1,0](Z^{-1})^{\otimes 2} \mid Z^{\otimes 4} f\right) \\ &\equiv_T \text{Pl-Holant}\left([1,0,1] \mid \hat{f}\right) \\ &\equiv_T \text{Pl-Holant}(\hat{f}), \end{aligned}$$

where the signature matrix of $\hat{f}$ is

$$M_{\hat{f}} = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

3. On the other side, the Eulerian Orientation problem is

$$
\begin{aligned}
&\text{Pl-Holant}\,(\neq_2 \mid [0,0,1,0,0]) \\
&\equiv_T \text{Pl-Holant}\,\big([0,1,0](Z^{-1})^{\otimes 2} \mid Z^{\otimes 4}[0,0,1,0,0]\big) \\
&\equiv_T \text{Pl-Holant}\,\big([1,0,1] \mid \tfrac{1}{2}[3,0,1,0,3]\big) \\
&\equiv_T \text{Pl-Holant}([3,0,1,0,3]).
\end{aligned}
$$

4. Moreover, by assigning the transformed Eulerian Orientation signature $[3, 0, 1, 0, 3]$ at every vertex

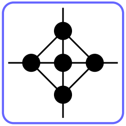

Figure: The planar tetrahedron gadget. Each vertex is assigned $[3, 0, 1, 0, 3]$.

We have

$$\text{Pl-Holant}(\hat{g}) \leq_T \text{Pl-Holant}([3, 0, 1, 0, 3])$$

with

$$M_{\hat{g}} = \frac{1}{2} \begin{bmatrix} 19 & 0 & 0 & 7 \\ 0 & 7 & 5 & 0 \\ 0 & 5 & 7 & 0 \\ 7 & 0 & 0 & 19 \end{bmatrix}.$$

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5. Finally, we finish the proof by reducing the Tutte Polynomial problem $\hat{f}$ to the Eulerian Orientation problem via $\hat{g}$:

Interpolate $\hat{f}$ using $\hat{g}$.

$$M_{\hat{f}} = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \qquad M_{\hat{g}} = \frac{1}{2}\begin{bmatrix} 19 & 0 & 0 & 7 \\ 0 & 7 & 5 & 0 \\ 0 & 5 & 7 & 0 \\ 7 & 0 & 0 & 19 \end{bmatrix}.$$

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Interpolation

Now we show how to reduce Pl-Holant($\hat{f}$) ($\textsc{Tutte}$) to Pl-Holant($\hat{g}$) ($\#\mathrm{EO}$) by interpolation.

Let Ω be an instance of Pl-Holant($\hat{f}$), $\hat{f}$ appears n times.

We construct from Ω a sequence of instances Ω_s of Holant($\hat{g}$) indexed by $s \geq 1$.

We obtain Ω_s from Ω by replacing each occurrence of $\hat{f}$ with the gadget N_s with $\hat{g}$ assigned to all vertices. .

Notice that $\hat{f}$ and $\hat{g}$ are *rotationally symmetric*.

Interpolation

To obtain Ω_s from Ω, we effectively replace $M_{\hat{f}}$ with $M_{N_s} = (M_{\hat{g}})^s$.

To obtain Ω_s from Ω, we effectively replace $M_{\hat{f}}$ with $M_{N_s} = (M_{\hat{g}})^s$.
Let

$$T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

$$\Lambda_{\hat{f}} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \text{and} \qquad \Lambda_{\hat{g}} = \begin{bmatrix} 13 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

Then

$$M_{\hat{f}} = T\Lambda_{\hat{f}}T^{-1} \qquad \text{and} \qquad M_{\hat{g}} = T\Lambda_{\hat{g}}T^{-1}$$

Interpolation

We can view our construction of Ω_s as first replacing each $M_{\hat{f}}$ by $T \Lambda_{\hat{f}} T^{-1}$ to obtain a signature grid Ω', which does not change the Holant value, We can view our construction of Ω_s as first replacing each $M_{\hat{f}}$ by $T\Lambda_{\hat{f}}T^{-1}$ to obtain a signature grid Ω', which does not change the Holant value, and then replacing each $\Lambda_{\hat{f}}$ with $\Lambda_{\hat{g}}^s$.

We can view our construction of Ω_s as first replacing each $M_{\hat{f}}$ by $T\Lambda_{\hat{f}}T^{-1}$ to obtain a signature grid Ω', which does not change the Holant value, and then replacing each $\Lambda_{\hat{f}}$ with $\Lambda_{\hat{g}}^s$.
We stratify the assignments in Ω' based on the assignment to $\Lambda_{\hat{f}}$. Recall that the rows of $\Lambda_{\hat{f}}$ and $\Lambda_{\hat{g}}$ are indexed by $00, 01, 10, 11$ and the columns are indexed by $00, 10, 01, 11$, in their respective orders.

$$\Lambda_{\hat{f}} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \text{and} \qquad \Lambda_{\hat{g}} = \begin{bmatrix} 13 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$$\Lambda_{\hat{f}} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \text{and} \qquad \Lambda_{\hat{g}} = \begin{bmatrix} 13 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

We only need to consider the assignments to $\Lambda_{\hat{f}}$ that assign

- $(00, 00)$ j many times,
- $(01, 10)$ or $(11, 11)$ k many times, and
- $(10, 01)$ ℓ many times,

where $j + k + \ell = n$, the total number of occurrences of $\Lambda_{\hat{f}}$ in Ω'.

Let $c_{jk\ell}$ be the sum over all such assignments of the products of evaluations from T and T^{-1} but excluding $\Lambda_{\hat{f}}$ on Ω'. Then

$$\text{PI-Holant}_\Omega = \sum_{j+k+\ell=n} 3^j c_{jk\ell}$$

Let $c_{jk\ell}$ be the sum over all such assignments of the products of evaluations from T and T^{-1} but excluding $\Lambda_{\hat{f}}$ on Ω'. Then

$$\text{Pl-Holant}_{\Omega} = \sum_{j+k+\ell=n} 3^j c_{jk\ell}$$

and the value of the Holant on Ω_s, for $s \geq 1$, is

$$\text{Pl-Holant}_{\Omega_s} = \sum_{j+k+\ell=n} (13^j 6^k)^s c_{jk\ell}. \tag{10}$$

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This is a linear equation system with unknowns $c_{jk\ell}$, and a coefficient matrix whose rows are indexed by s and columns are indexed by (j, k), where $0 \leq j, k$ and $j + k \leq n$.

Interpolation

$$\text{Pl-Holant}_{\Omega_s} = \sum_{j+k+\ell=n} (13^j 6^k)^s c_{jk\ell}. \tag{11}$$

We take $1 \leq s \leq \binom{n+2}{2}$. Then the coefficient matrix in the linear system is Vandermonde

Interpolation

$$\text{PI-Holant}_{\Omega_s} = \sum_{j+k+\ell=n} (13^j 6^k)^s c_{jk\ell}. \tag{11}$$

We take $1 \leq s \leq \binom{n+2}{2}$. Then the coefficient matrix in the linear system is Vandermonde and has full rank since for any $j, k, j', k' \geq 0$, if $(j, k) \neq (j', k')$ then $13^j 6^k \neq 13^{j'} 6^{k'}$.

Therefore, after obtaining the values of $\text{PI-Holant}_{\Omega_s}$ by oracle calls to $\#\mathrm{EO}$, for $1 \leq s \leq \binom{n+2}{2}$, we can solve the linear system for the unknown $c_{jk\ell}$'s and obtain the value of $\text{PI-Holant}_{\Omega}$ ($\textsc{Tutte}$).

http://www.cs.wisc.edu/~jyc/dichotomy-book.pdf

Some papers can be found on my web site
http://www.cs.wisc.edu/~jyc

THANK YOU!