Markov Chain Mixing Times And Applications III:

Conductance and Canonical Paths

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Simons Institute for the Theory of Computing
Counting Complexity and Phase Transitions Bootcamp

January 27th, 2016

Outline

Other techniques for bounding the mixing time:

- conductance
- canonical paths
- canonical flows

Thanks to:

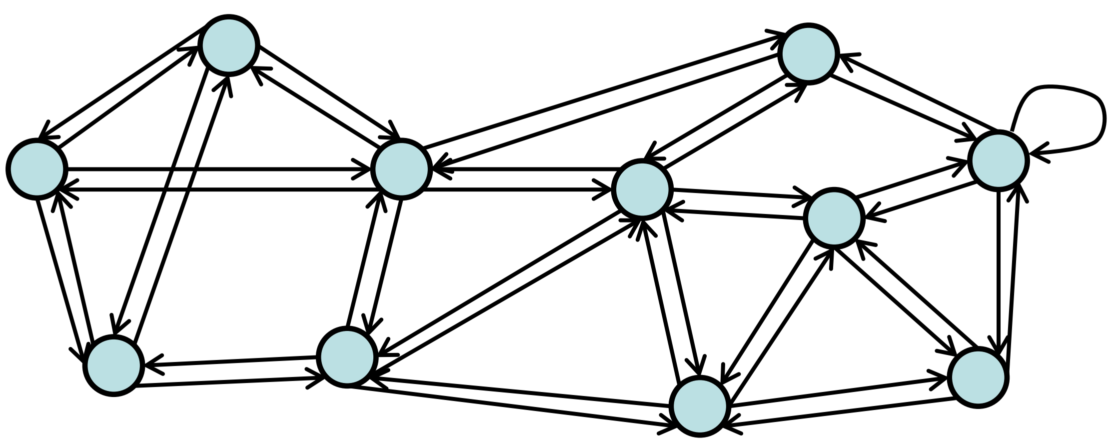
Bhatnagar, Diaconis, Dyer, Jerrum, Lawler, Müller, Randall, Sinclair, Sokal, Štefankovič, Stroock, Vazirani, Vigoda, …

Outline

Recall:

- Ergodic MC (Ω,P) => unique stationary distribution π
- Mixing time: $t_{mix}(\epsilon)$ = minimum t such that for every start state x, after t steps within ϵ of π

An ergodic reversible Markov chain (Ω,P):



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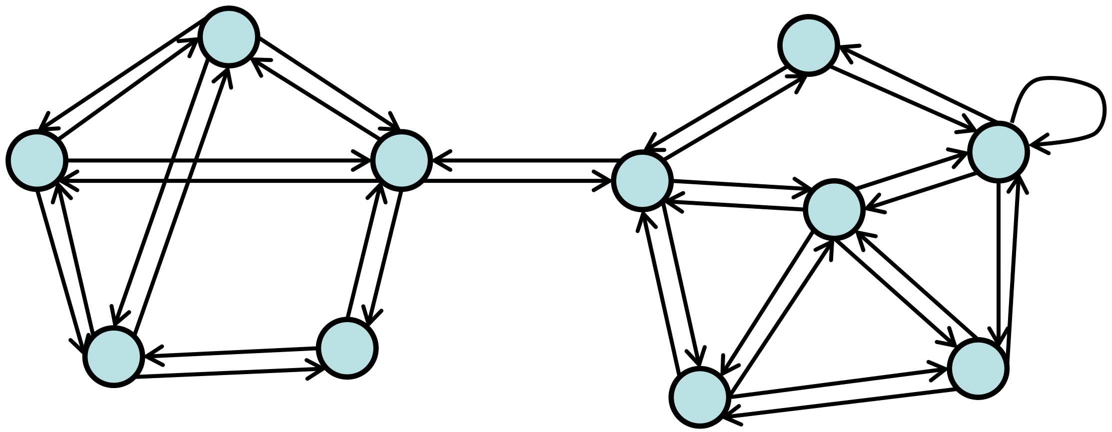
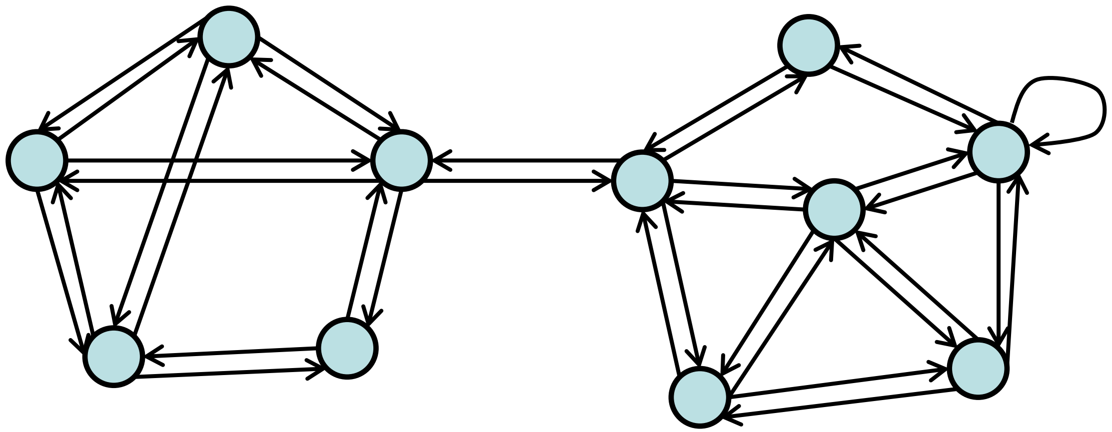


Conductance

Def: For an ergodic reversible MC (Ω,P), its conductance is defined as:

$$\Phi = \min_{S: S \subseteq \Omega, \pi(S) \le 1/2} \frac{\sum_{x \in S, y \notin S} \pi(x) P(x, y)}{\pi(S)}$$



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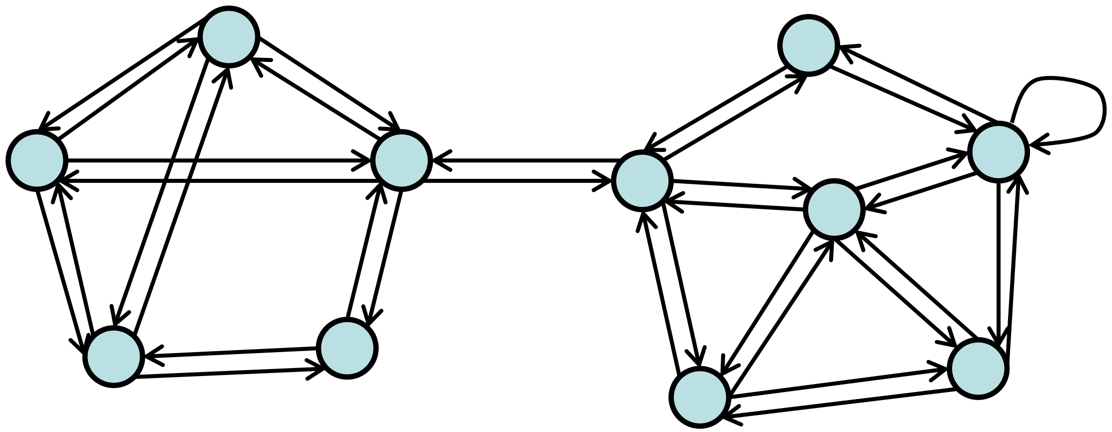
Example: suppose π is uniform:



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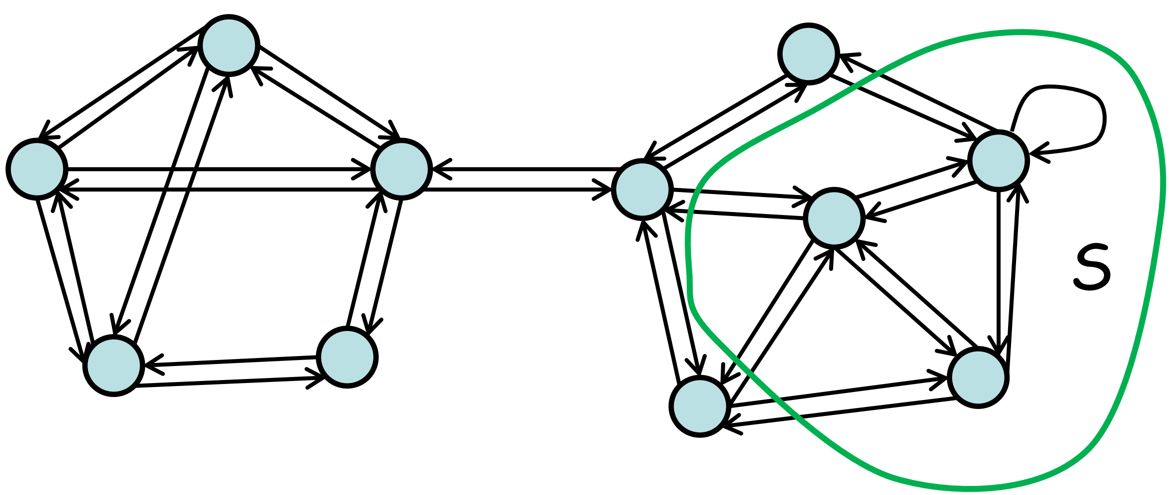

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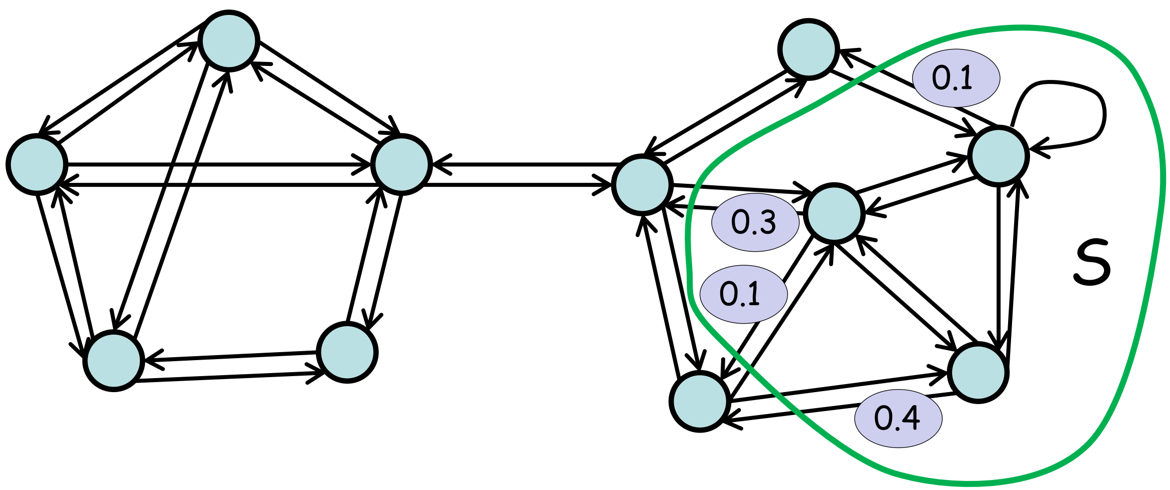

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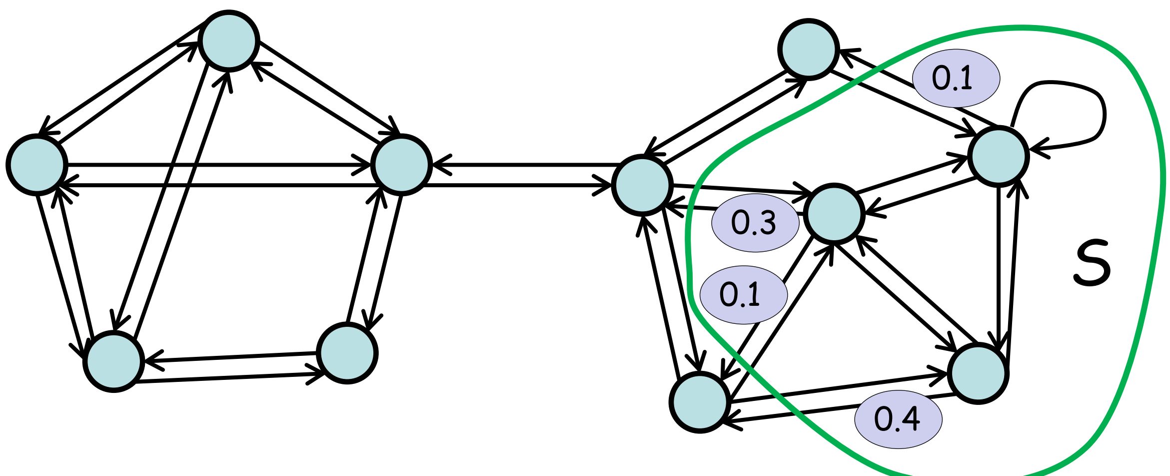
Example: suppose π is uniform: $\Phi_S = \dfrac{\frac{1}{11}0.1+\frac{1}{11}0.3+\frac{1}{11}0.1+\frac{1}{11}0.4}{\frac{3}{11}} = 0.3$



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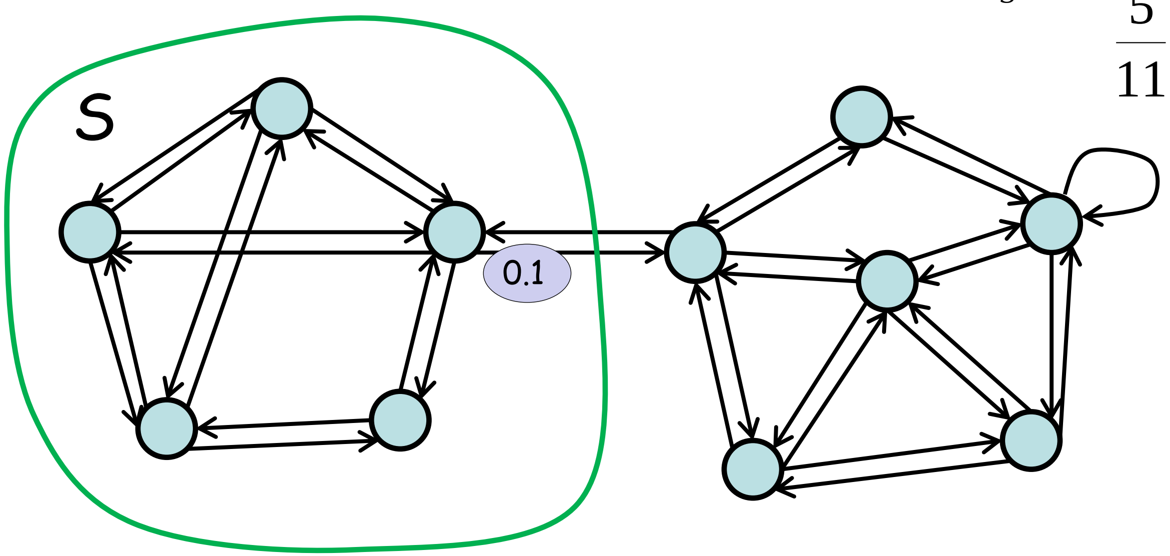
Example: suppose π is uniform: $\Phi_S = \dfrac{\frac{1}{11}0.1}{\frac{5}{11}} = 0.02$



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Thm: $\boxed{\Phi^2/2 \leq \textit{spectral gap} \leq 2\Phi}$

Recall:

Thm: For an ergodic MC, let λ_2 be the 2nd largest eigenvalue of P and $\pi_{\min} := \min_x \pi(x)$. Then

$$\frac{|\lambda_2|}{\textit{spectral gap}} \log\left(\frac{1}{2\varepsilon}\right) \leq t_{mix}(\varepsilon) \leq \frac{1}{\textit{spectral gap}} \log\left(\frac{1}{\varepsilon\pi_{\min}}\right)$$

[Jerrum-Sinclair, Diaconis-Stroock, Lawler-Sokal]

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Thm: For a lazy ergodic MC, where $\pi_{\min} := \min_x \pi(x)$:

$$\boxed{\frac{1}{2}\left(\frac{1}{2\Phi}-1\right)\log\left(\frac{1}{2\varepsilon}\right) \le t_{mix}(\varepsilon) \le \frac{2}{\Phi^2}\log\left(\frac{1}{\varepsilon\pi_{\min}}\right)}$$

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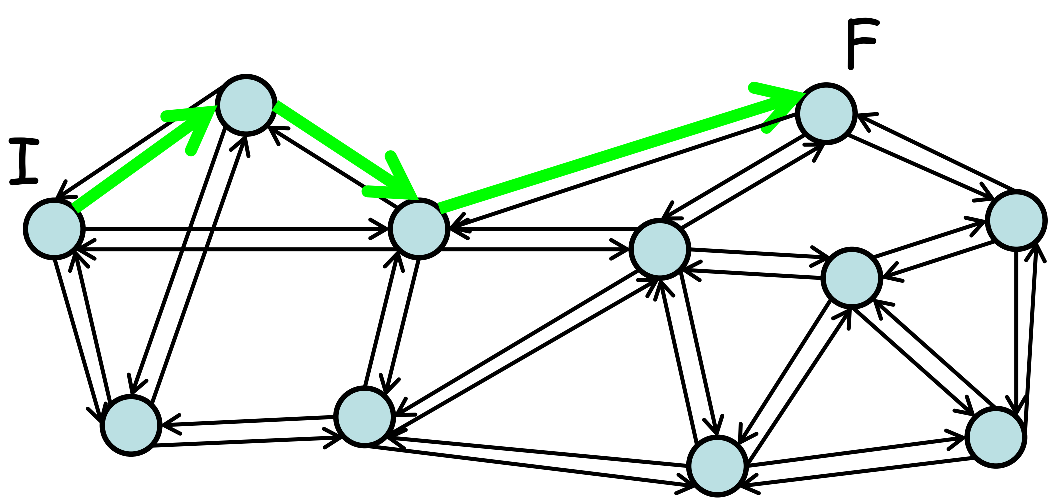
Canonical Paths

Bounding the conductance:

- Find a path in the transition graph from every state I to every other state F:



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Bounding the conductance:

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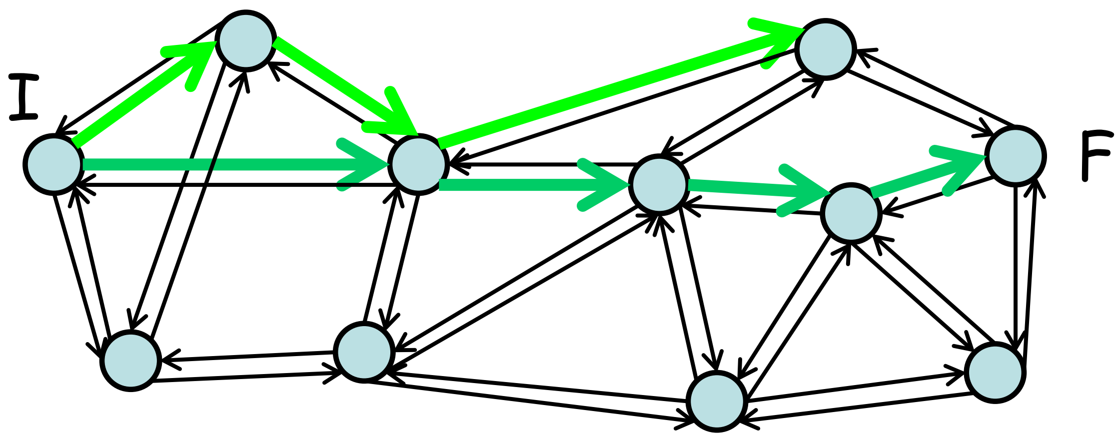

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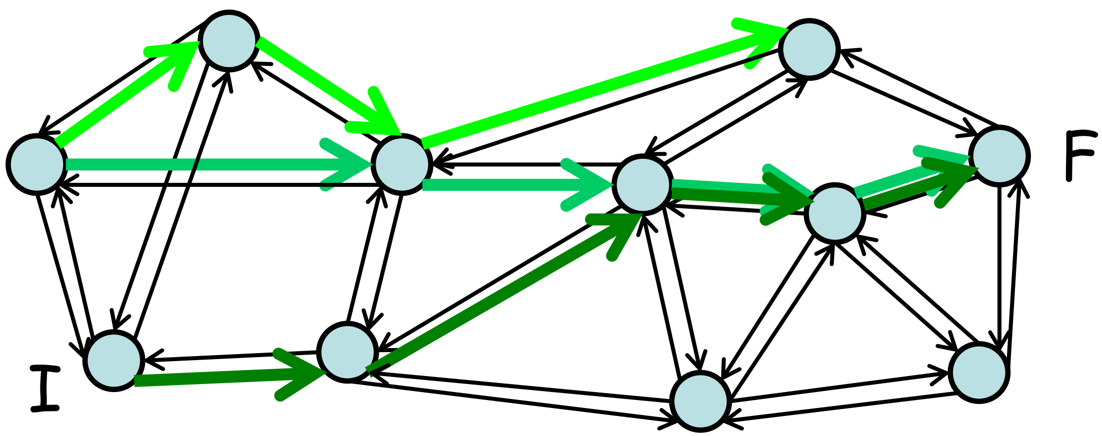

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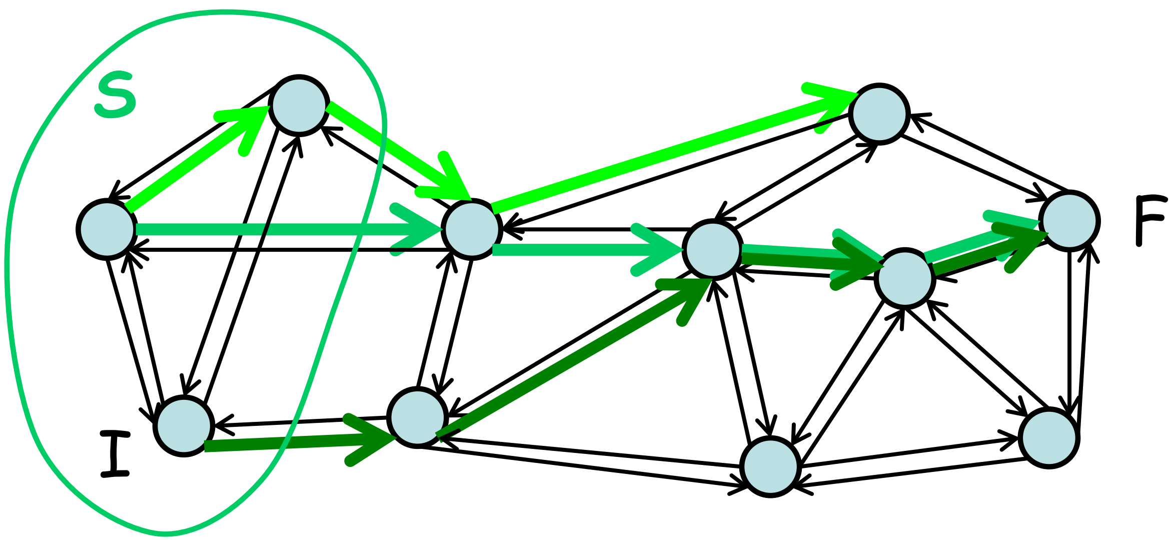
- Then, take any S:

$$\pi(S)\pi(\bar{S}) \quad = \sum_{I \in S,\, F \notin S} \pi(I)\pi(F)$$



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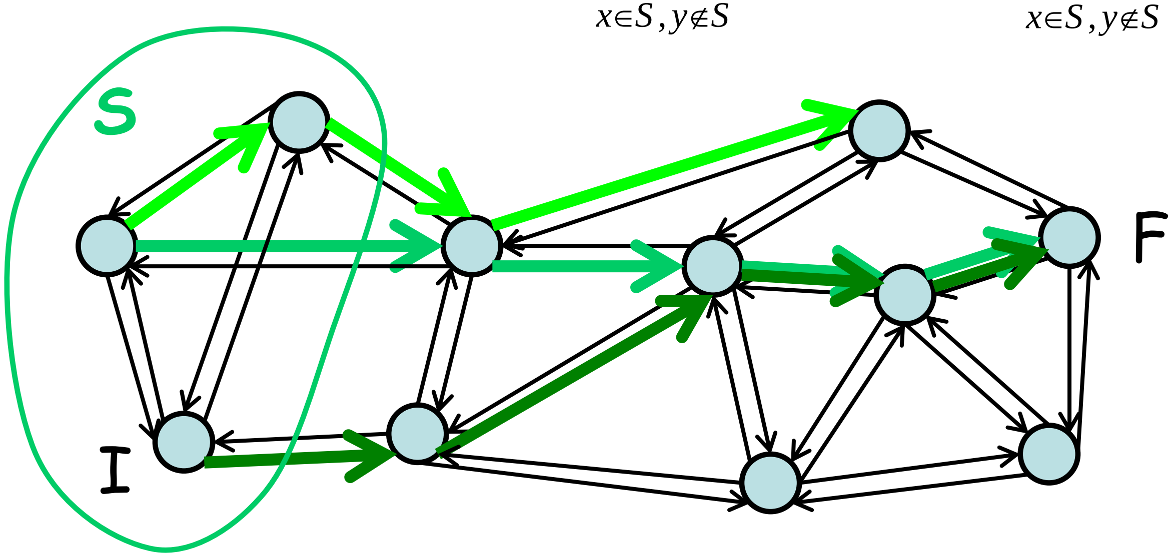

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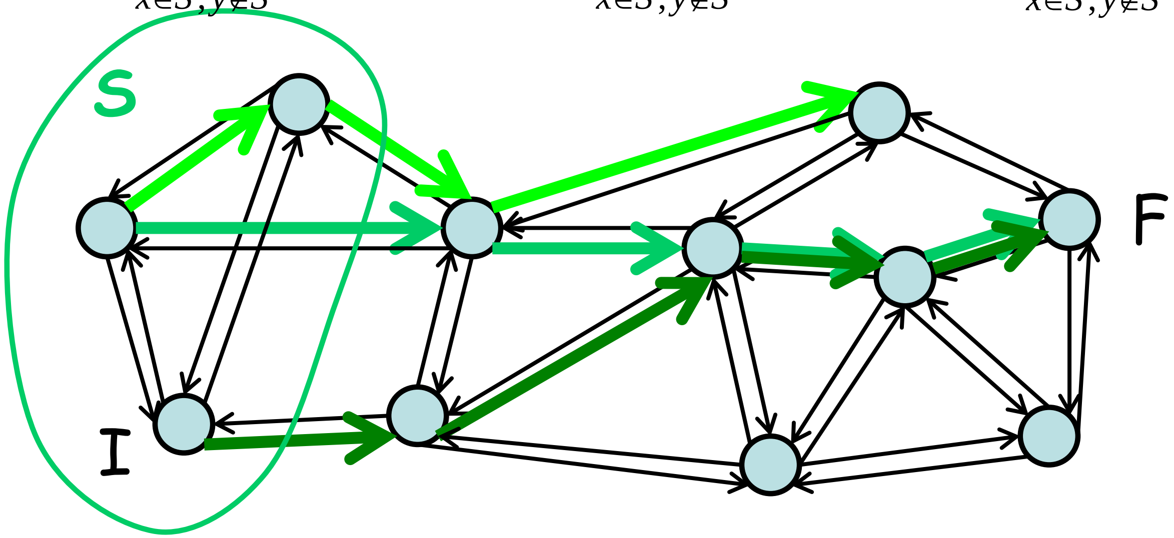

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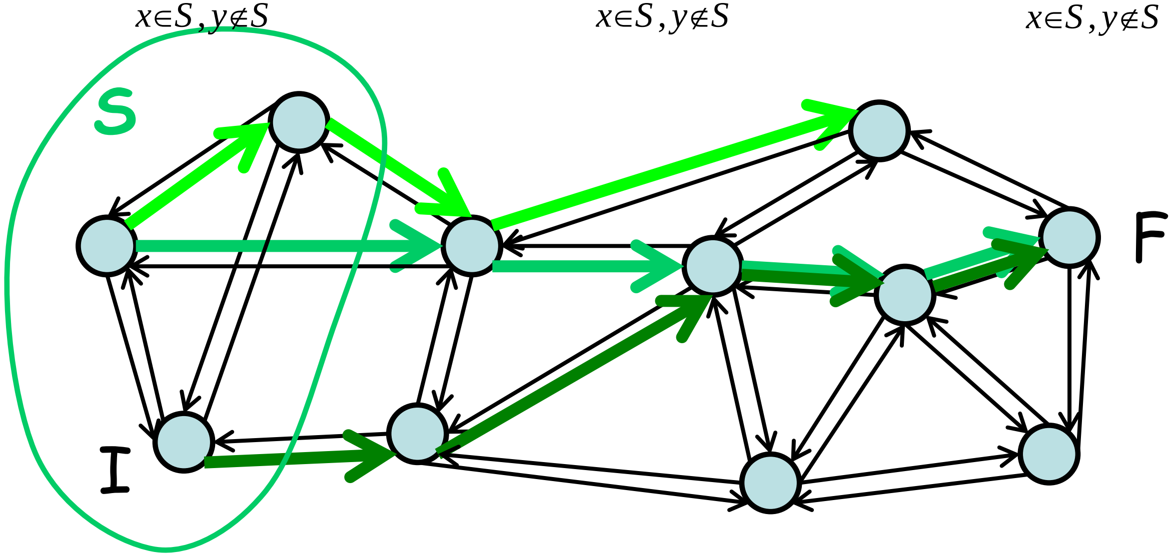

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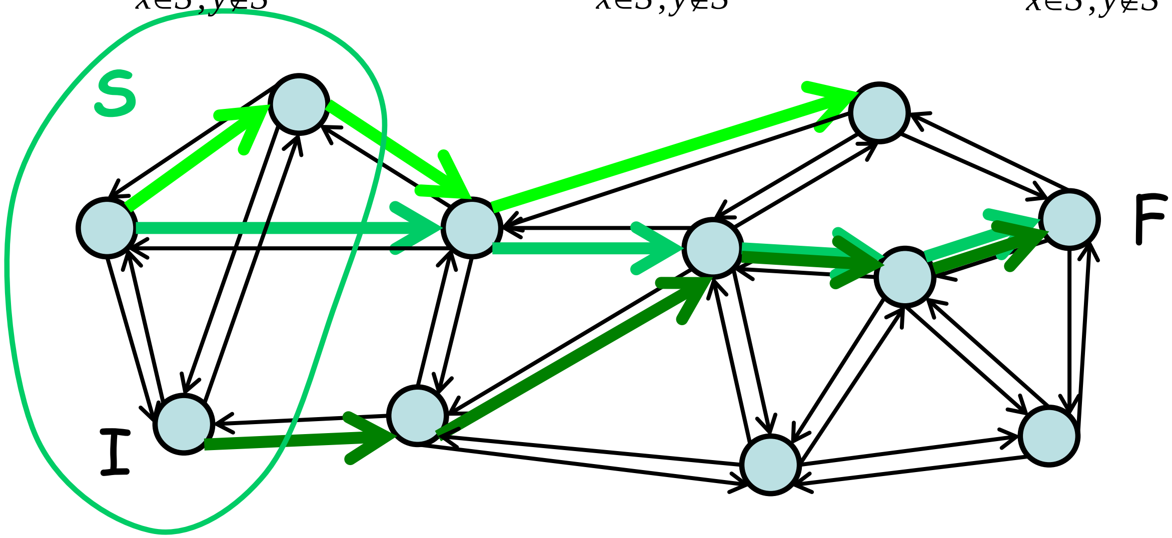

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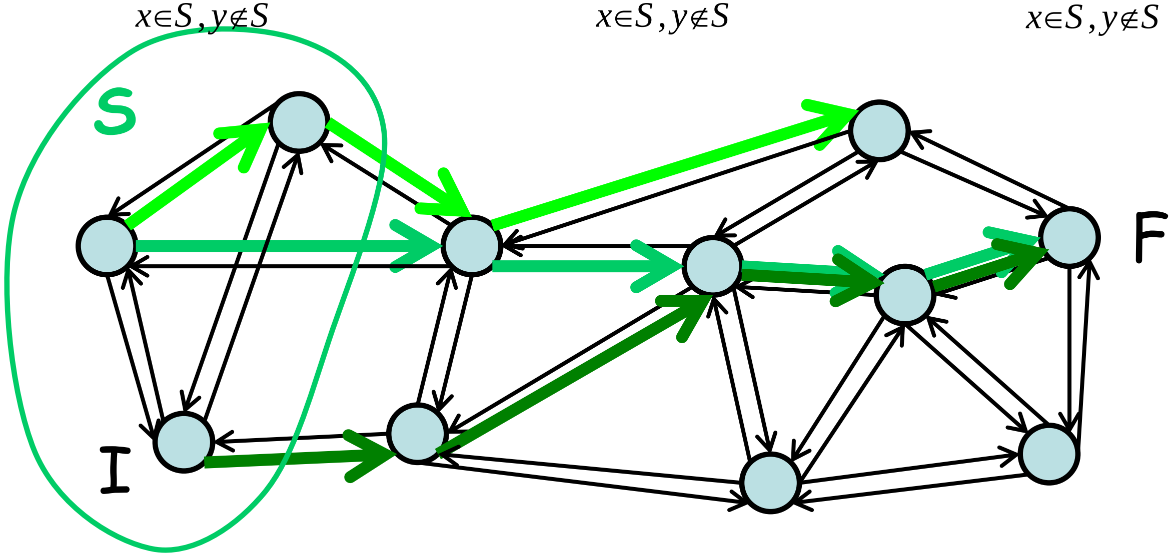

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for some u in S, v not in S.

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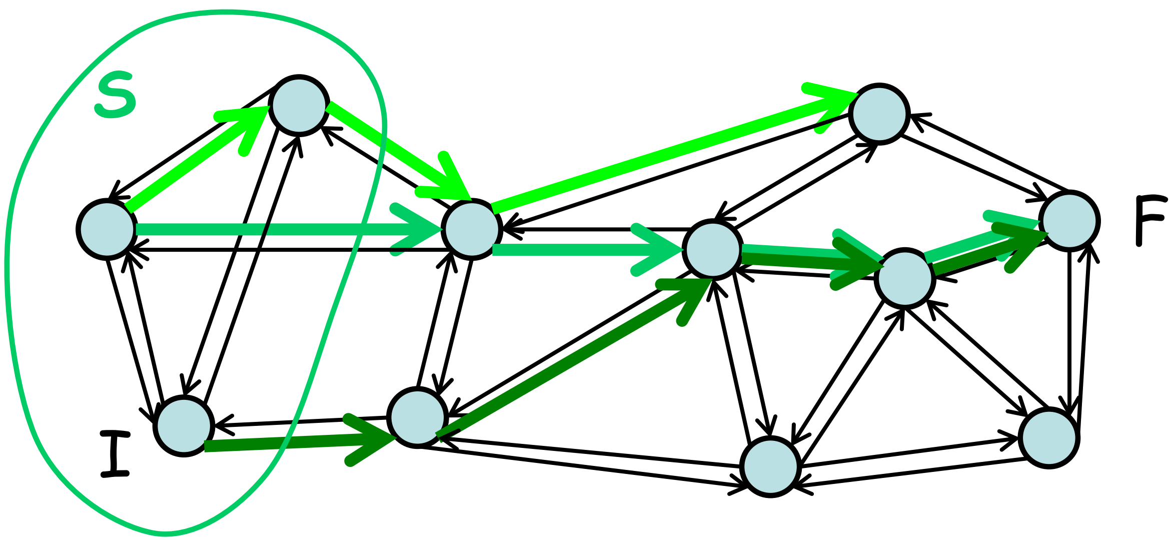

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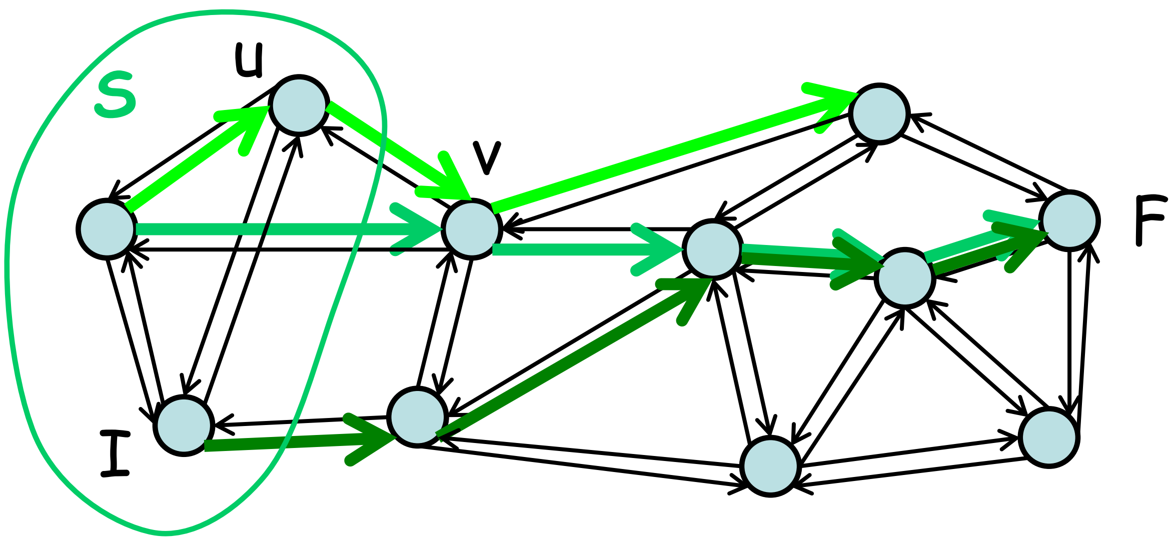
Congestion through transition (u,v)



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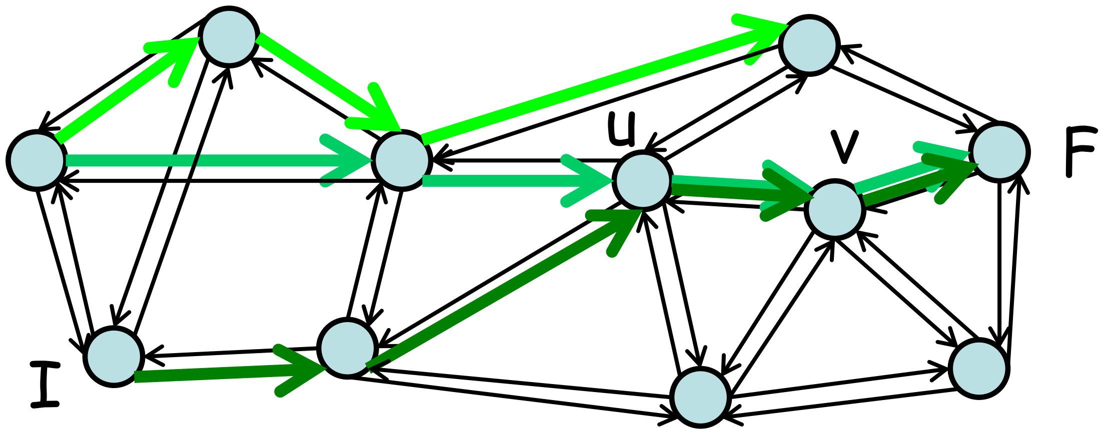
Def: Congestion

$$\rho := \max_{u,v} \frac{1}{\pi(u)P(u,v)} \sum_{\substack{I \to F \text{ path} \\ \text{through } (u,v)}} \pi(I)\pi(F)(\text{length of } I \to F \text{ path})$$



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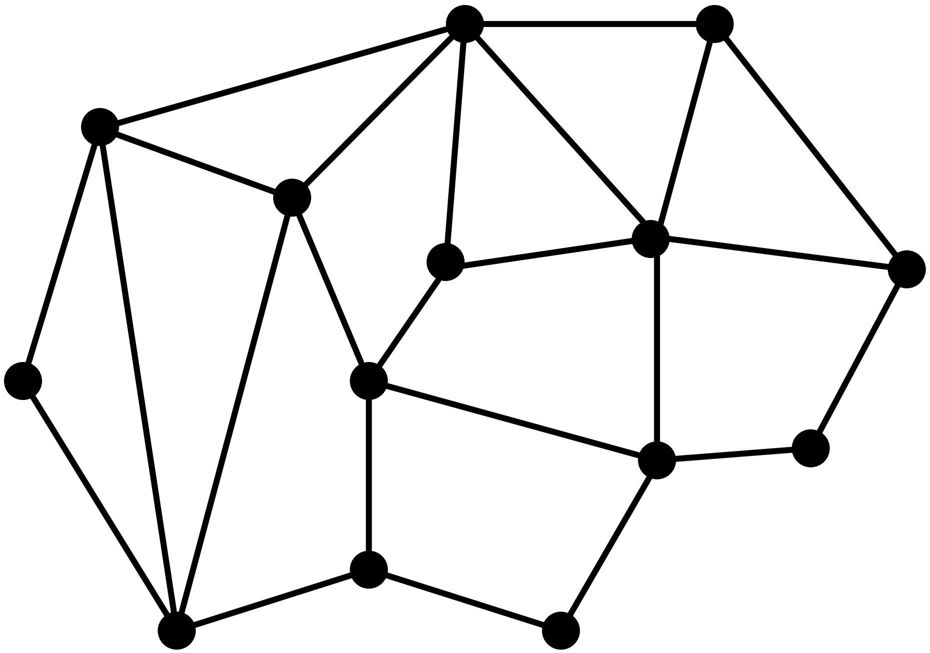
Thm [Sinclair]: For a lazy ergodic reversible MC:

$$t_{mix}(\varepsilon) \le 4\rho \ln\left(\frac{1}{\varepsilon\, \pi_{\min}}\right)$$

Matchings Revisited

Given an undirected graph $G=(V,E)$, a **matching** $M \subseteq E$ is a set of vertex disjoint edges. A matching is **perfect** if $|M|=n/2$, where n = # vertices (and m = # edges).

Example:



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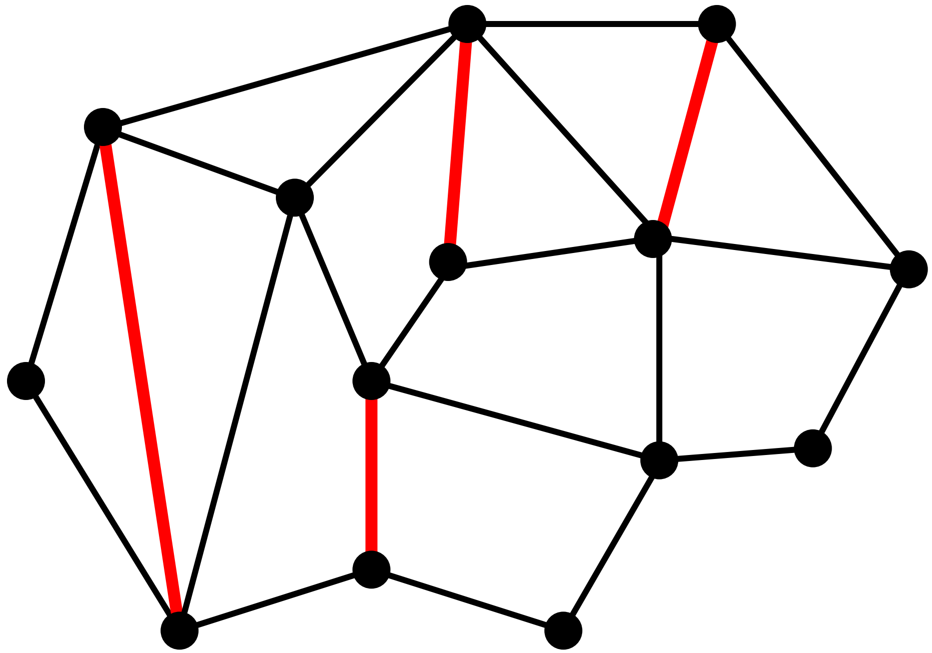

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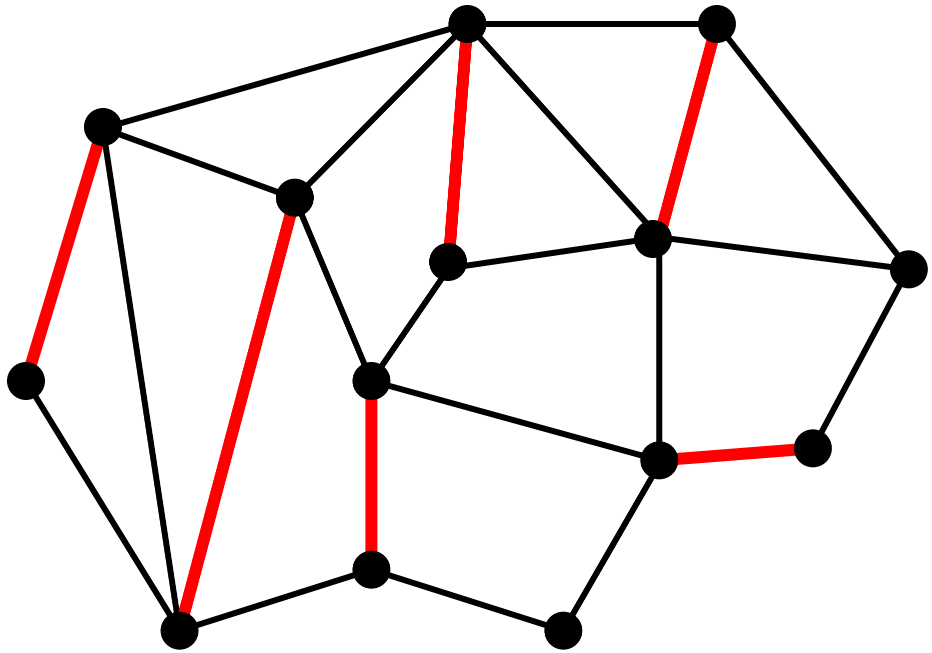

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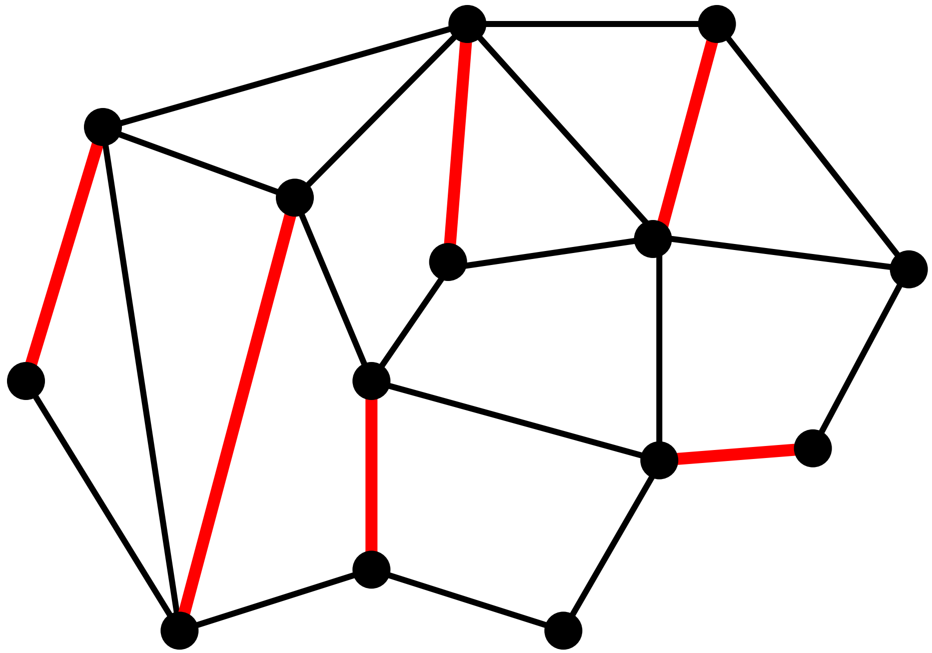

Goal:

An FPRAS for

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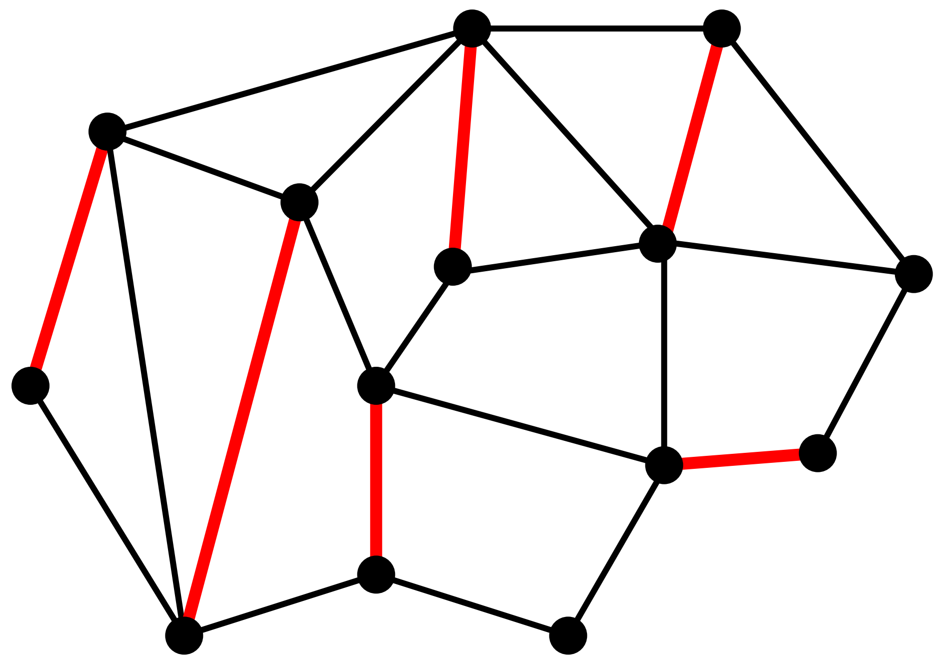

Goal:

FPAUS (sampler)

An FPRAS for

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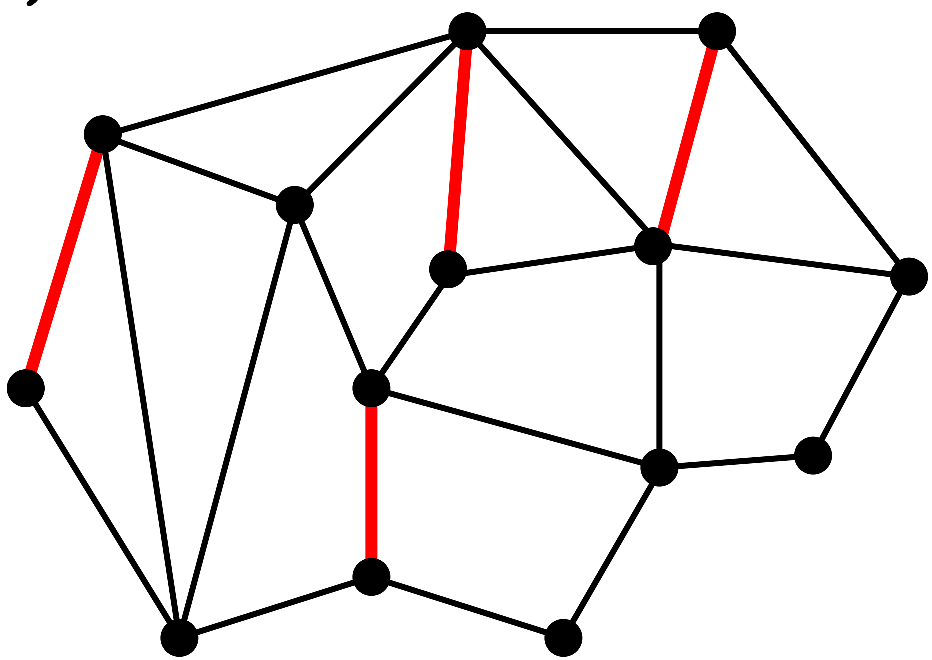
A Markov Chain for Matchings

Input: a graph G

State space Ω: all matchings of G

Markov chain (slide chain):
Let M be the current matching, we get the next state by choosing a random edge $e=(u,v) \in E$ and:



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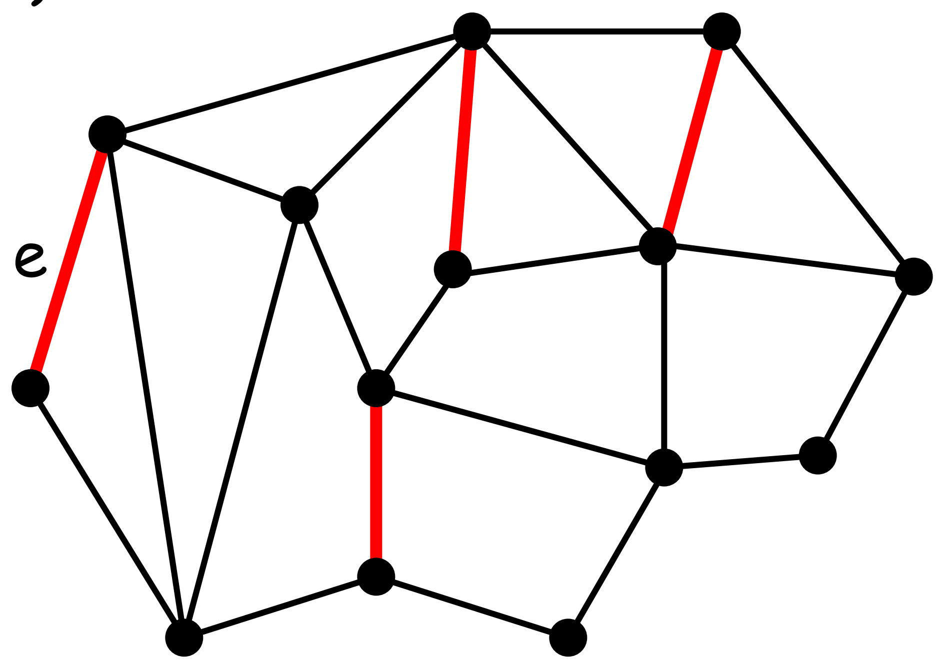

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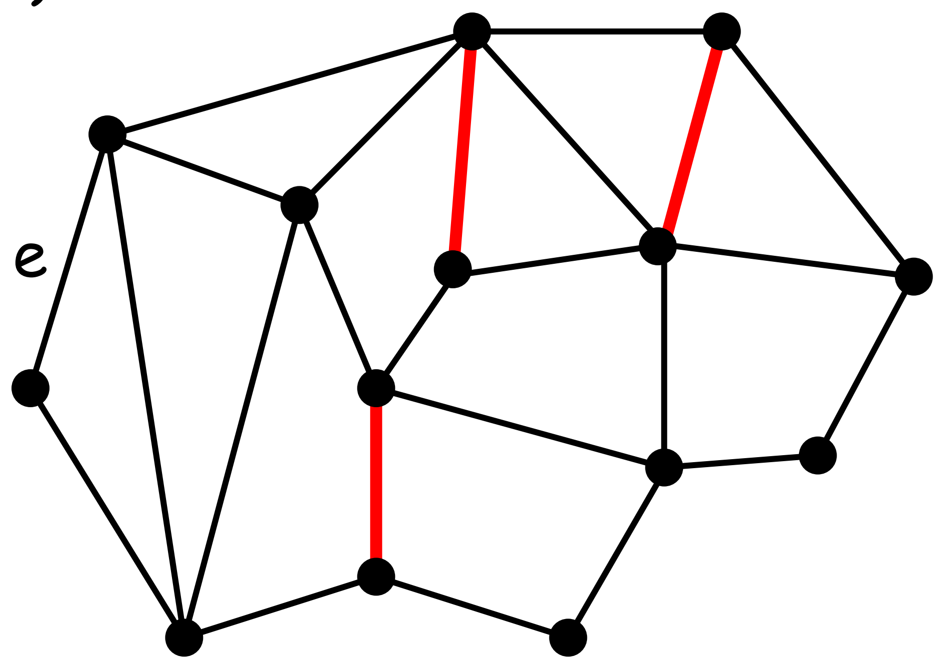

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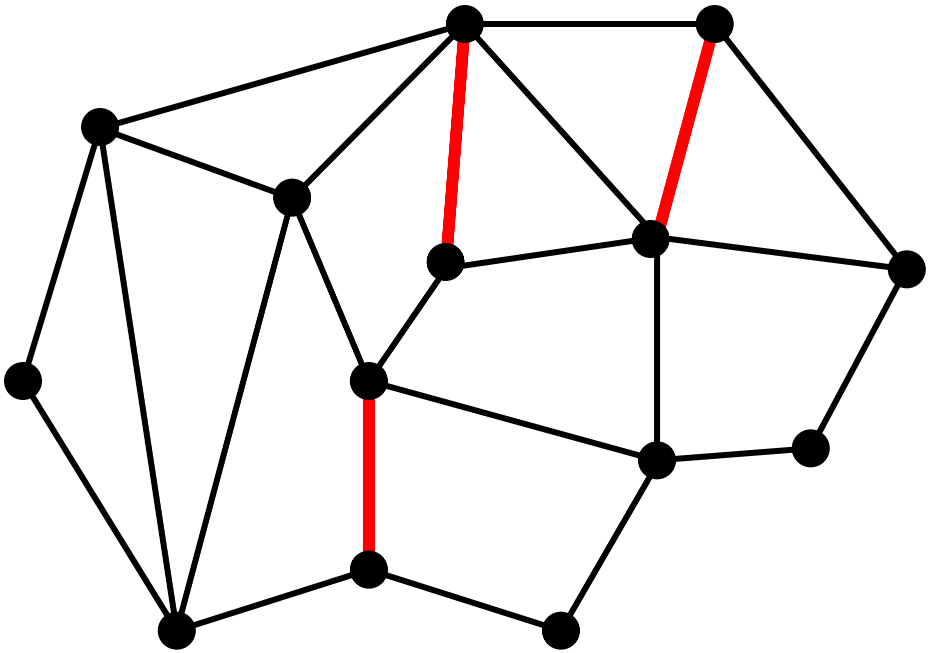

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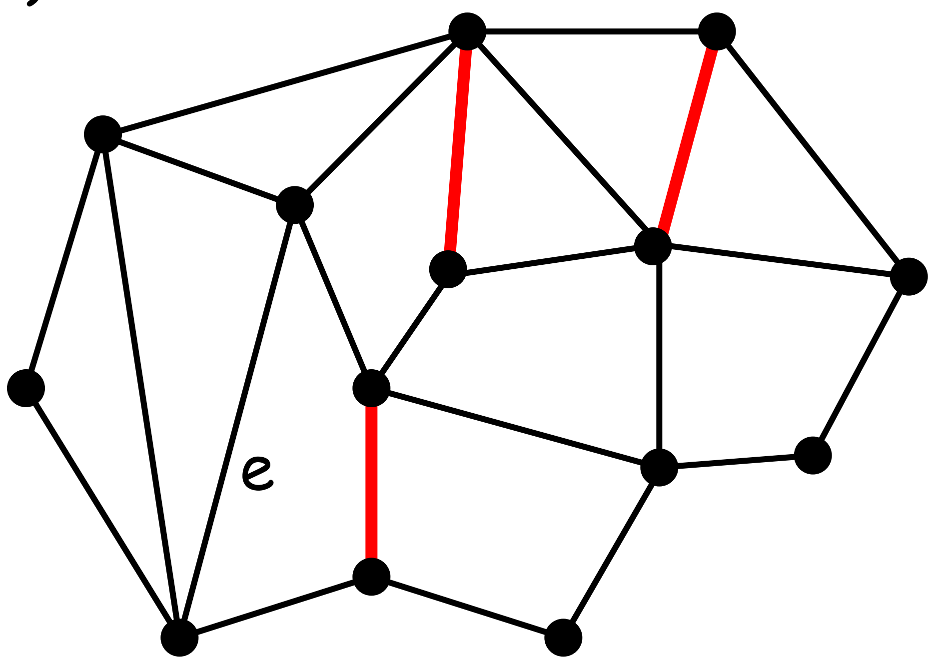

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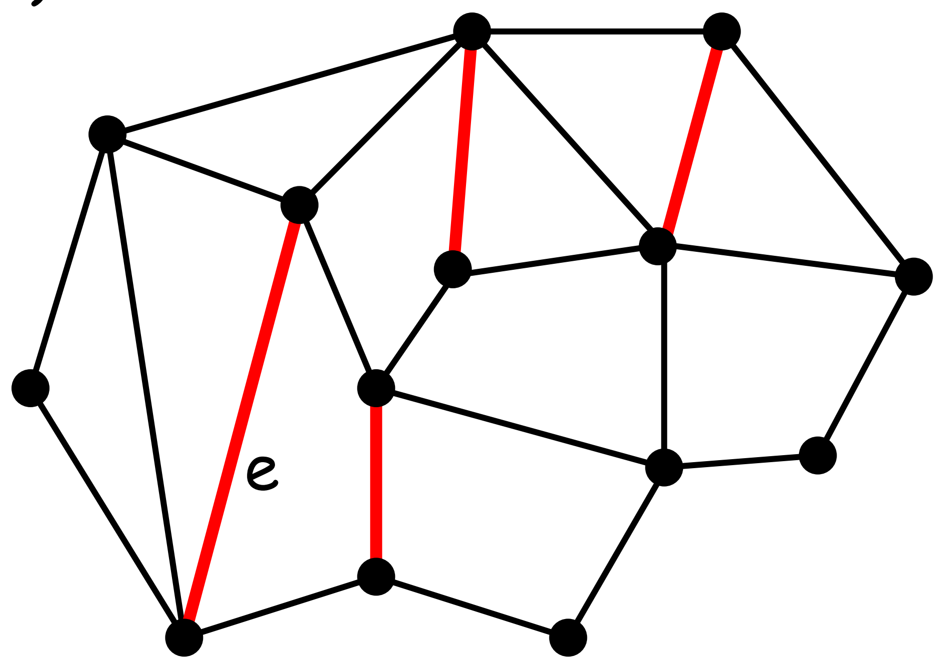

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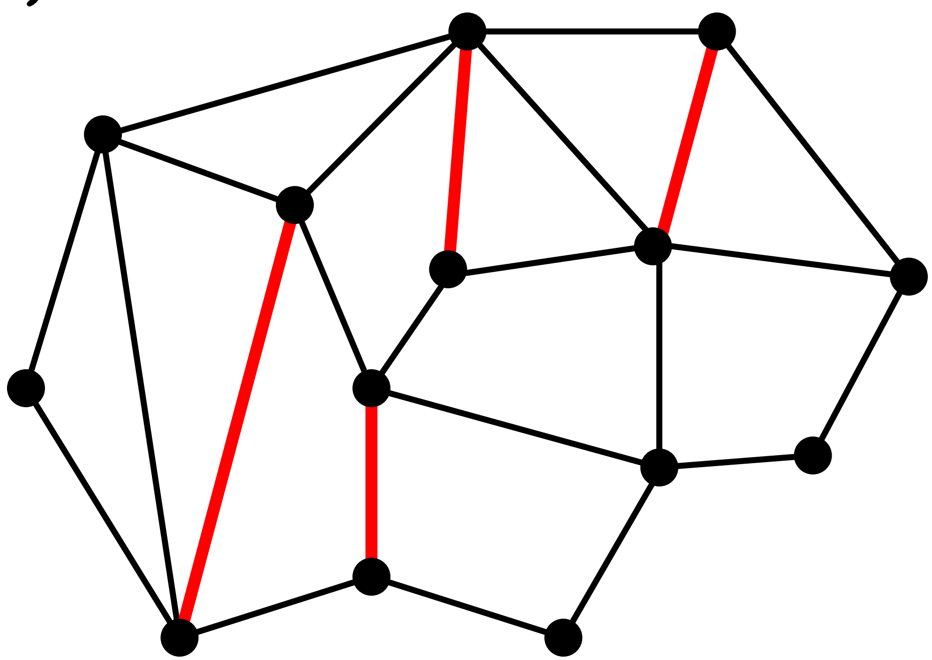

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A Markov Chain for Matchings

Input: a graph G

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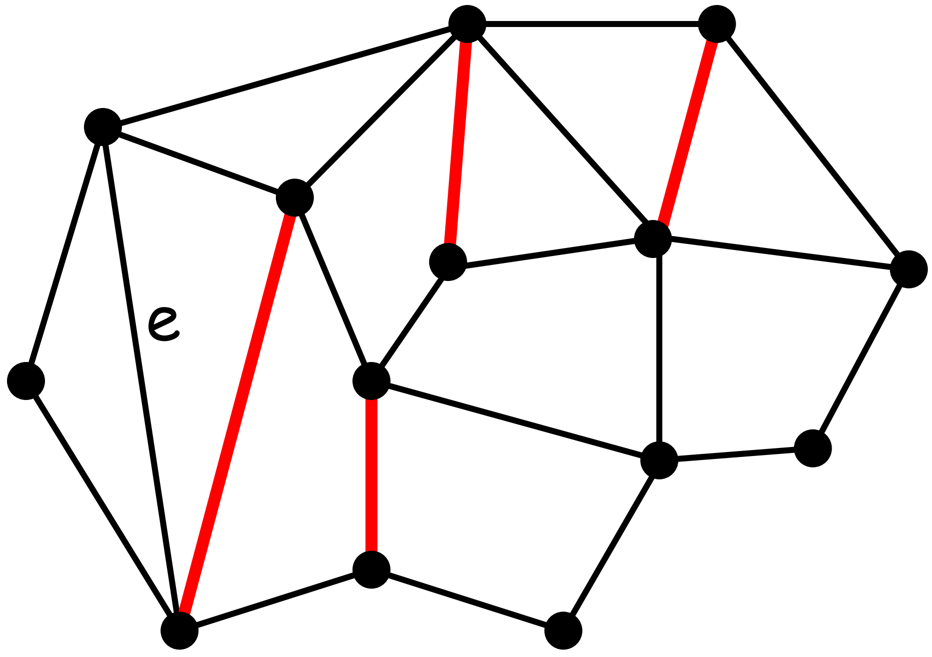

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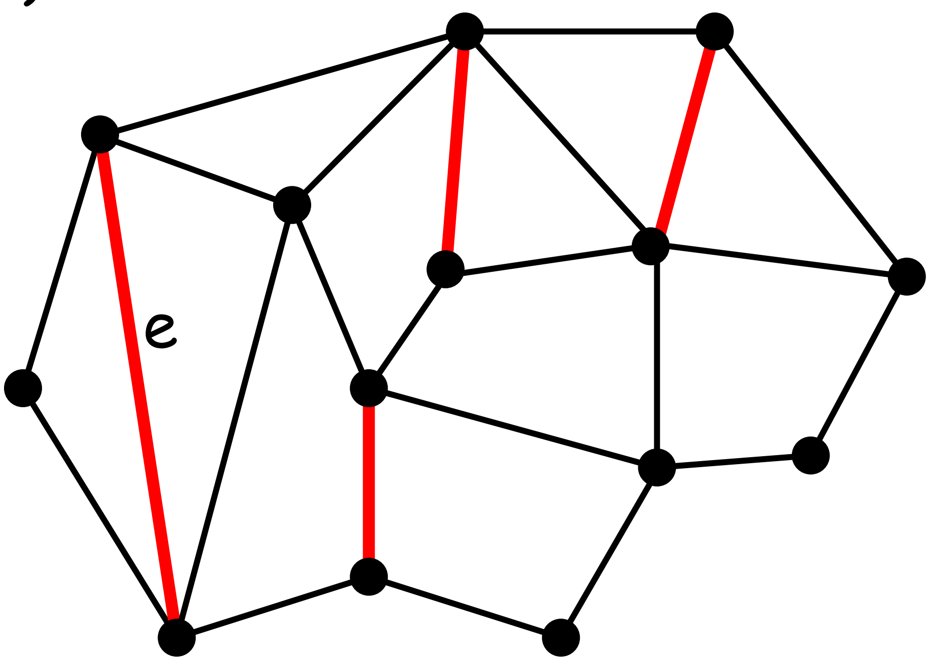

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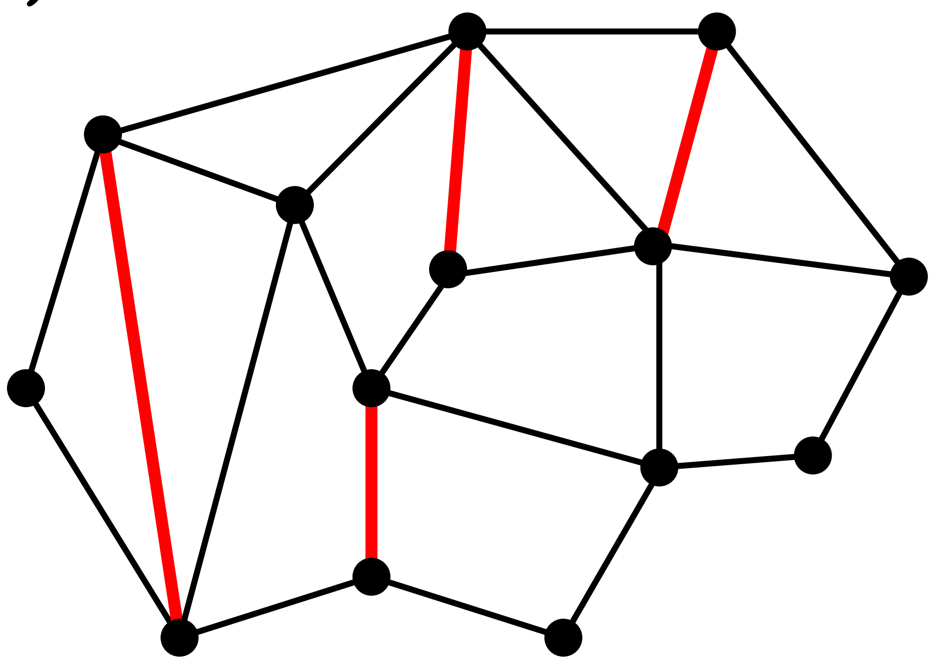

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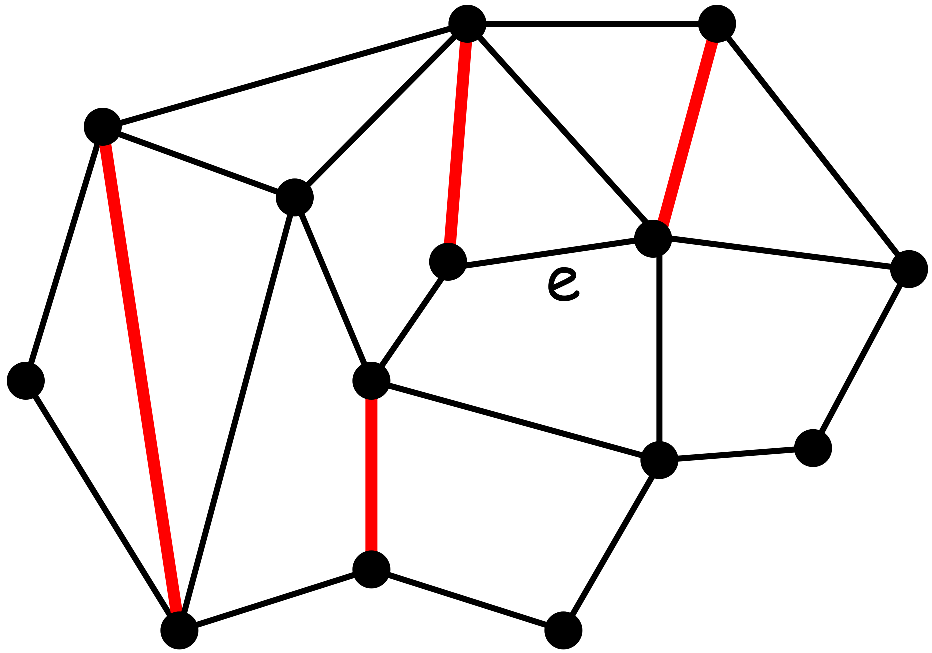

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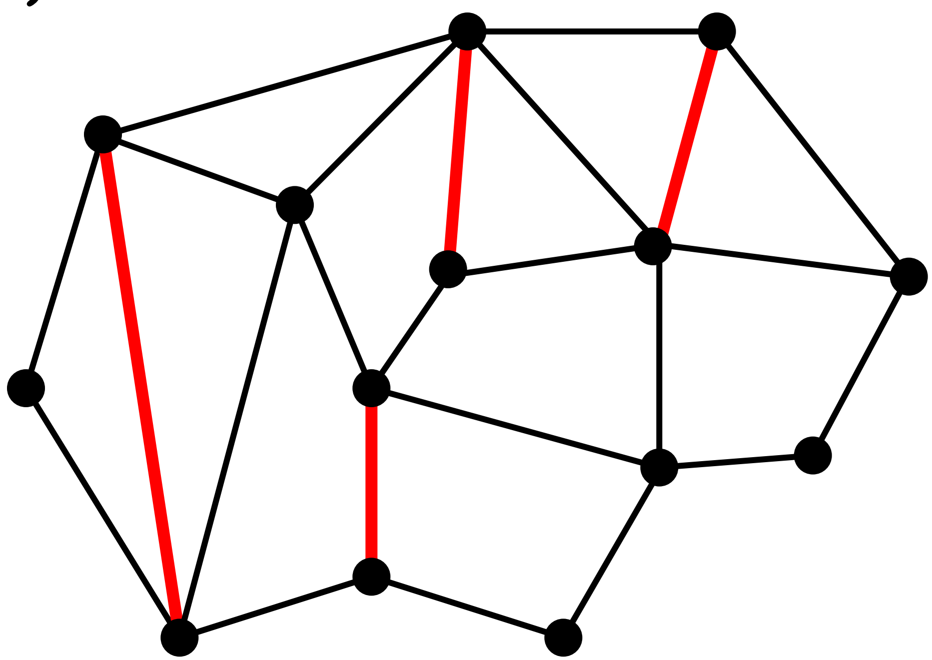

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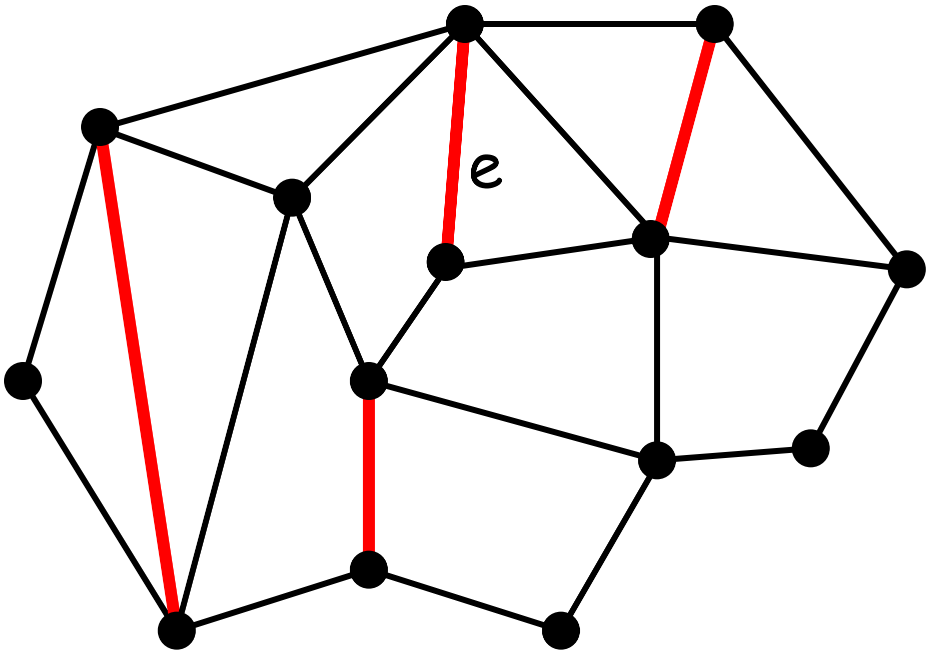

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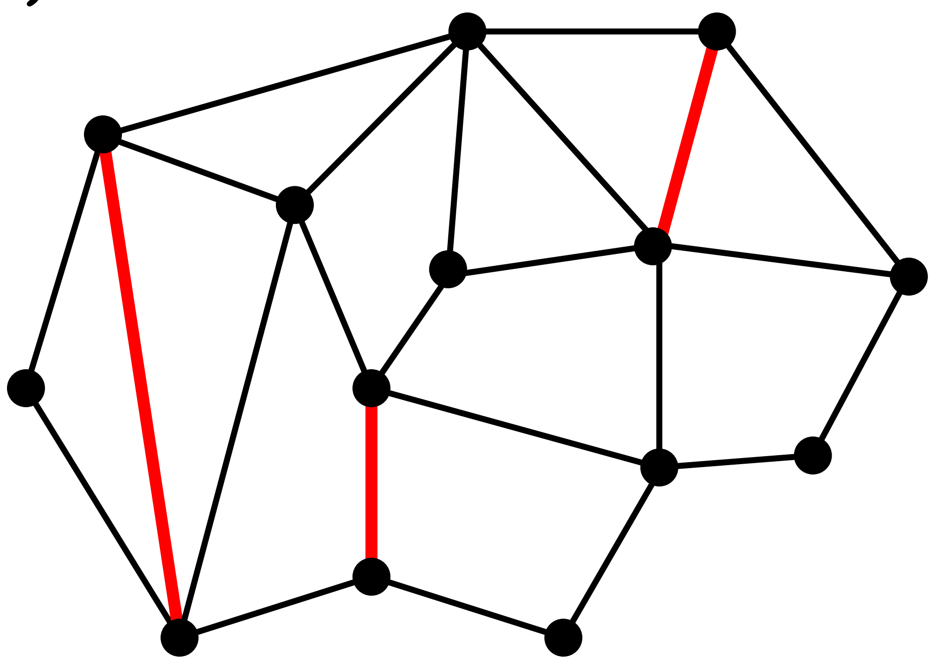

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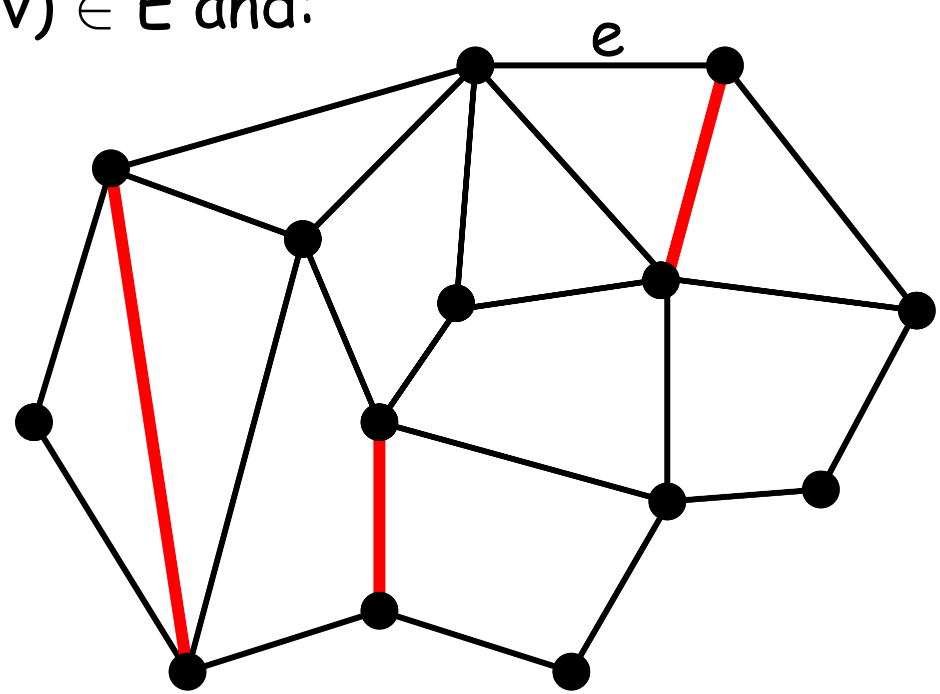

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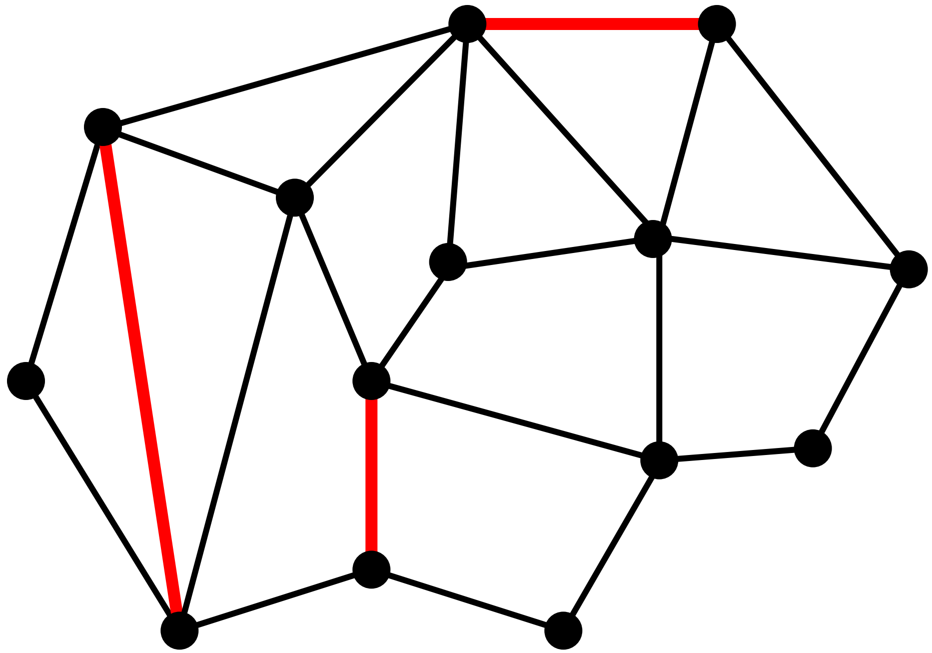

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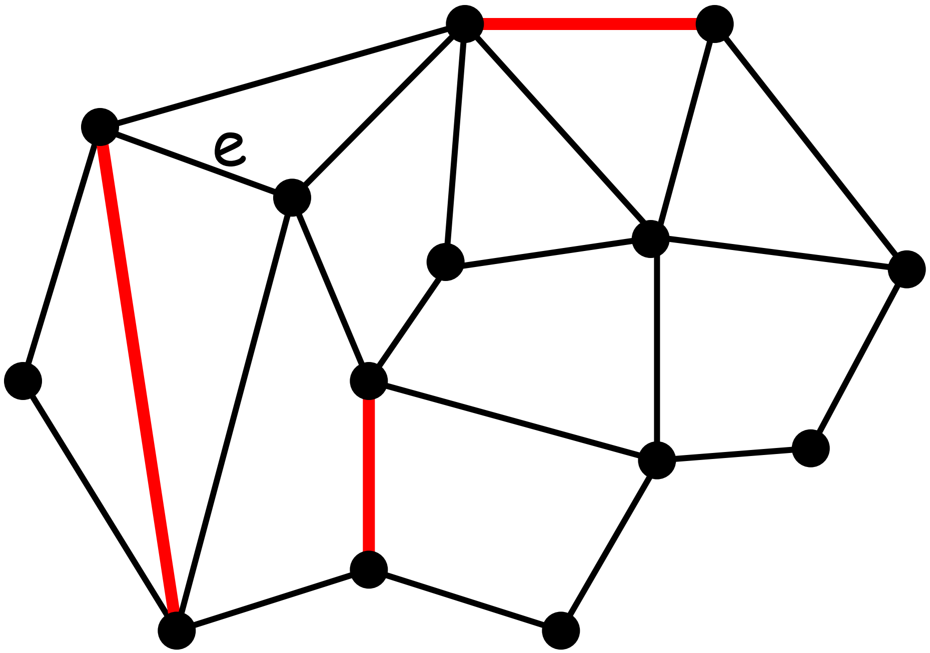

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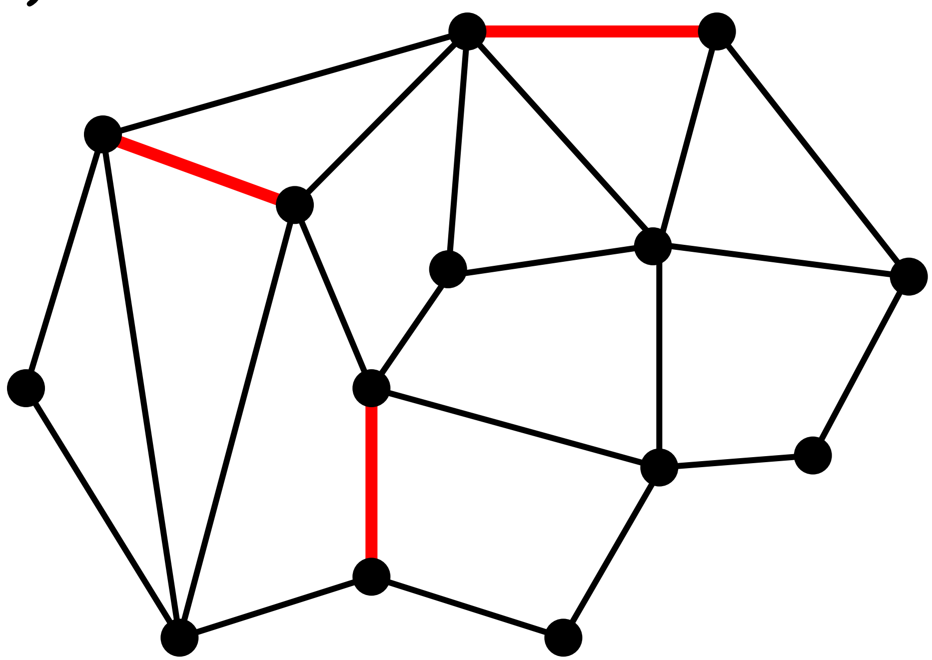

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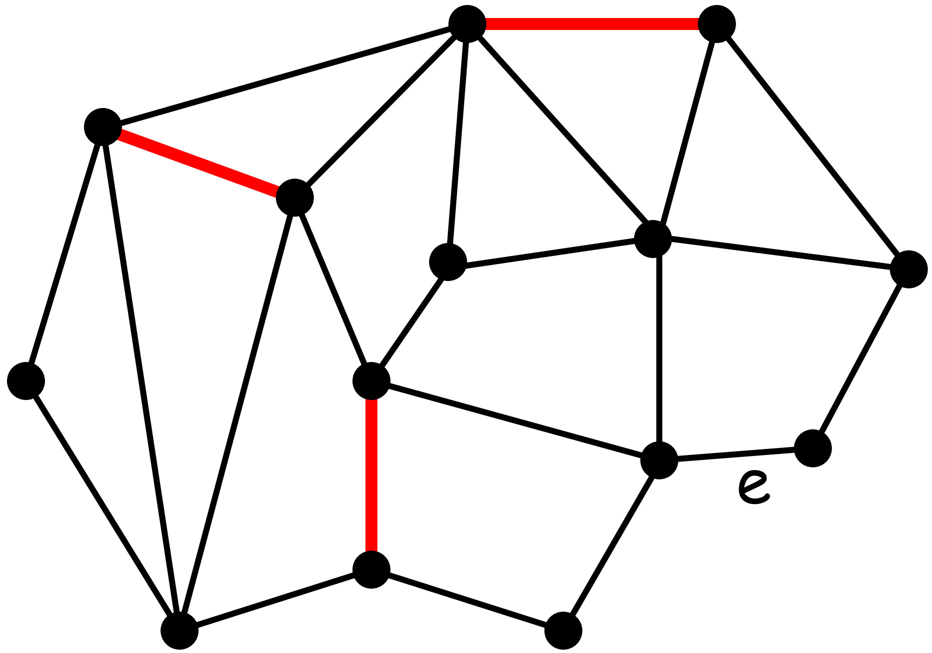

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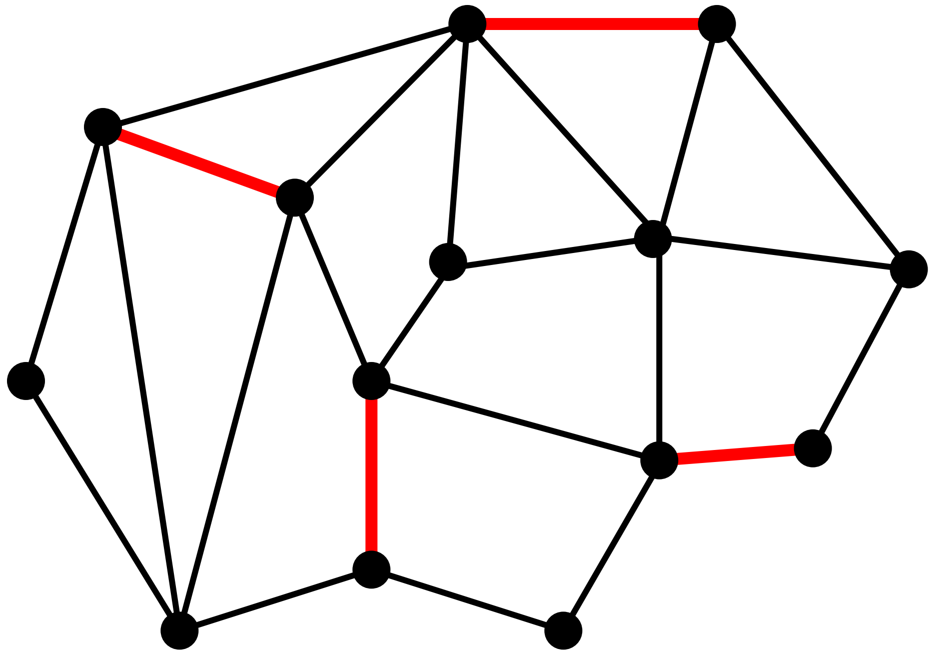

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A Markov Chain for Matchings

Technicality:
A **lazy** chain: with probability 1/2 stay in M, otherwise

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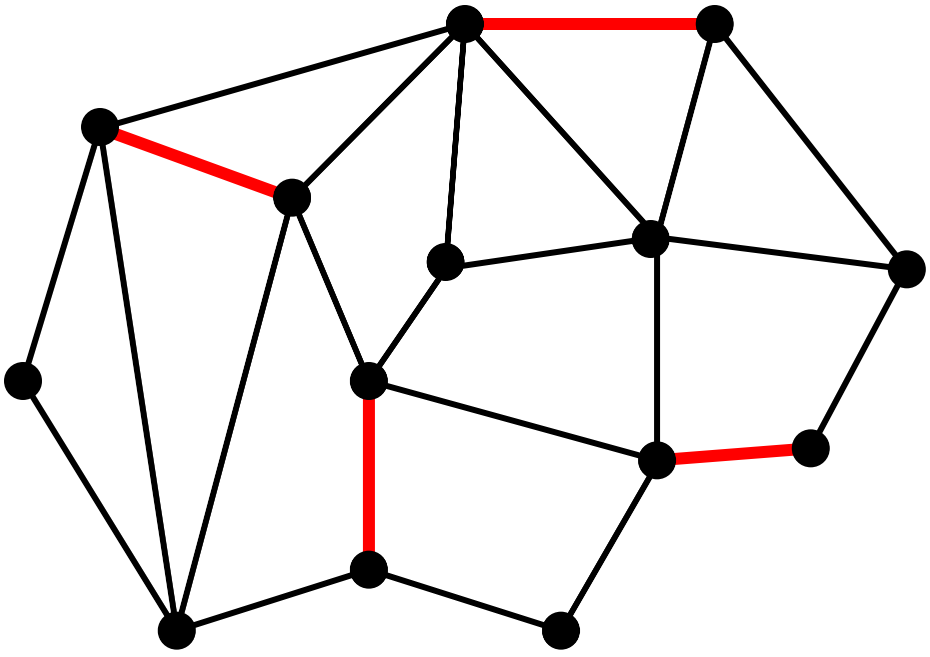
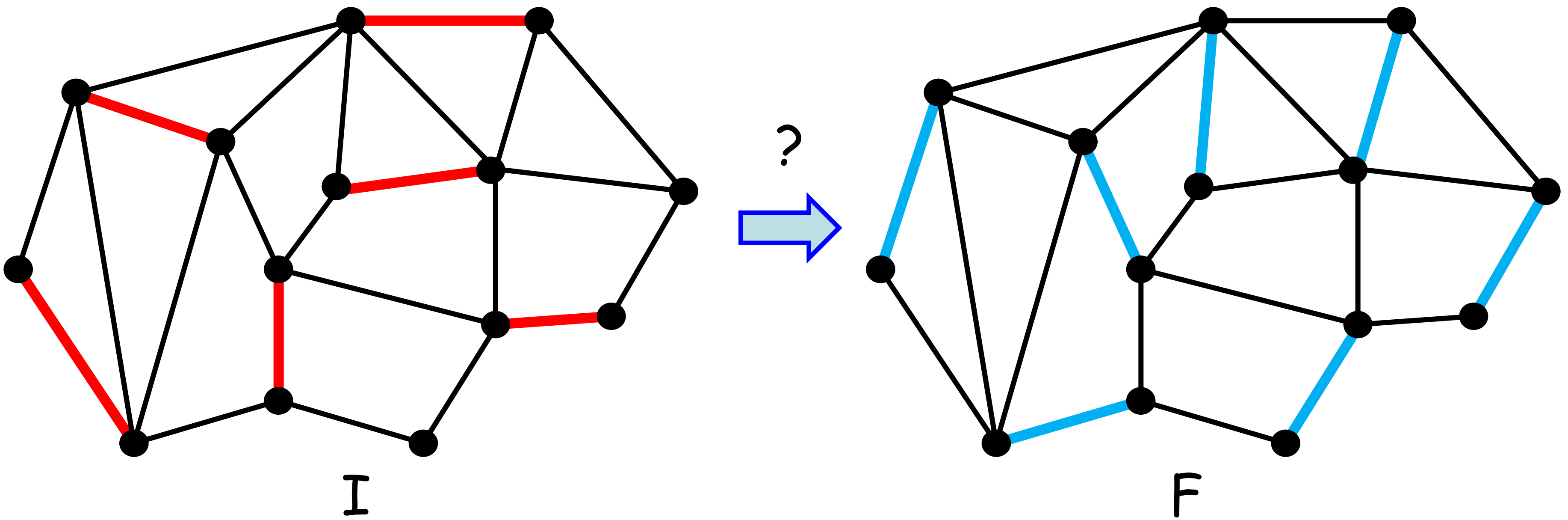


Canonical Paths

For any pair of matchings I,F, define a path from I to F in the transition graph:



[Jerrum-Sinclair]

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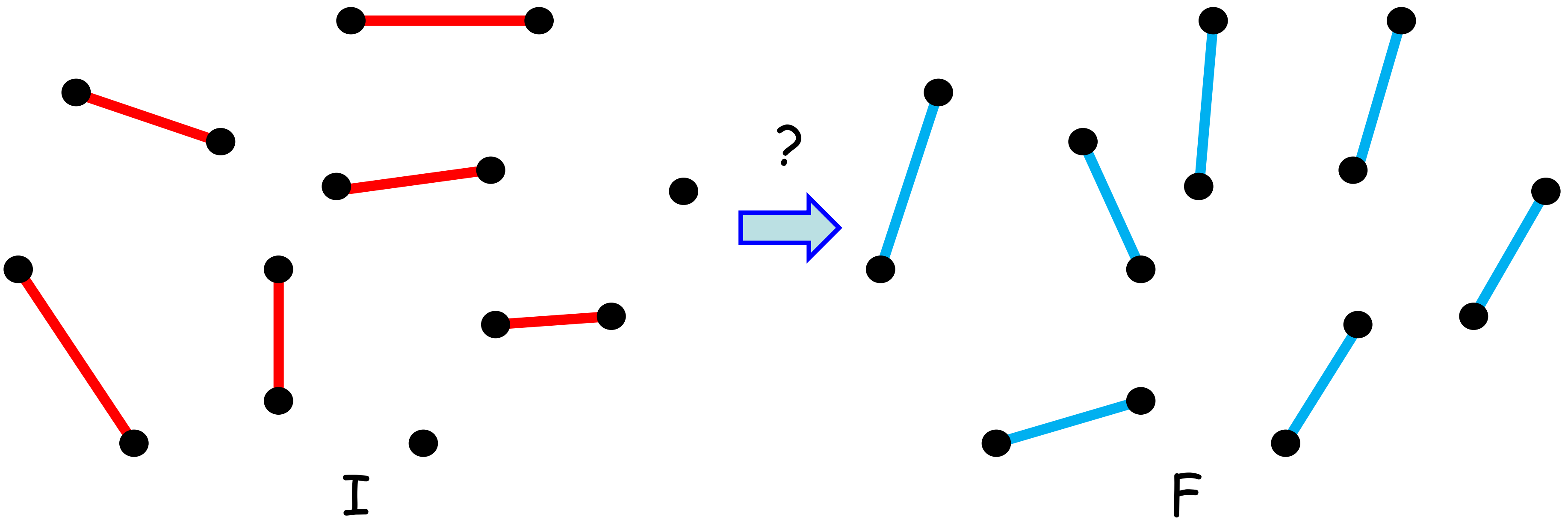

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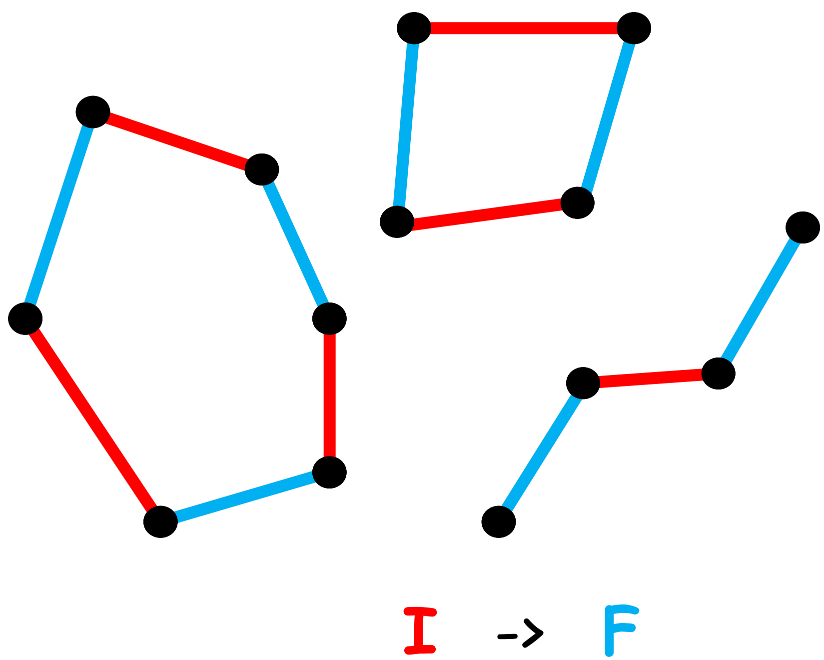

Going from red to blue:

- take I $\oplus$ F (sym. difference)

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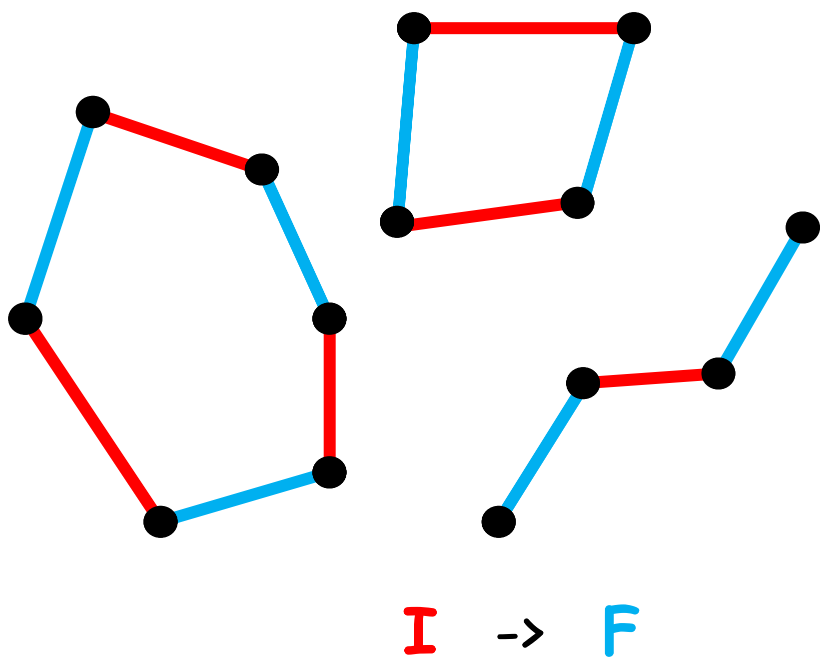

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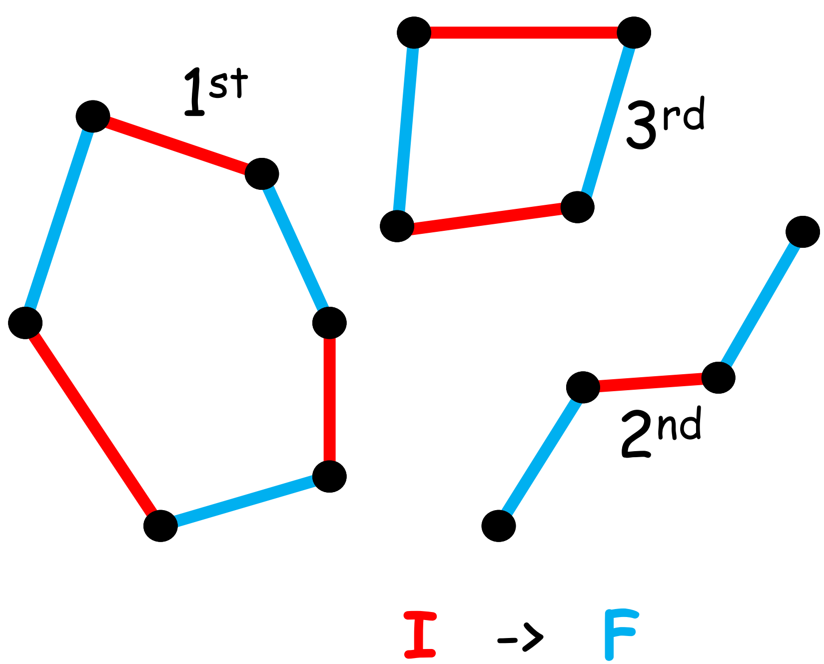

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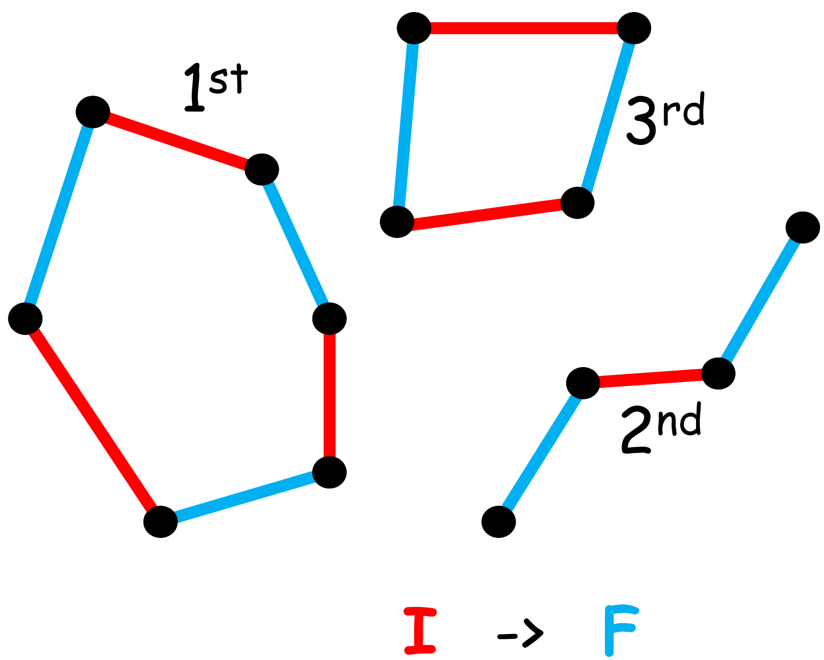

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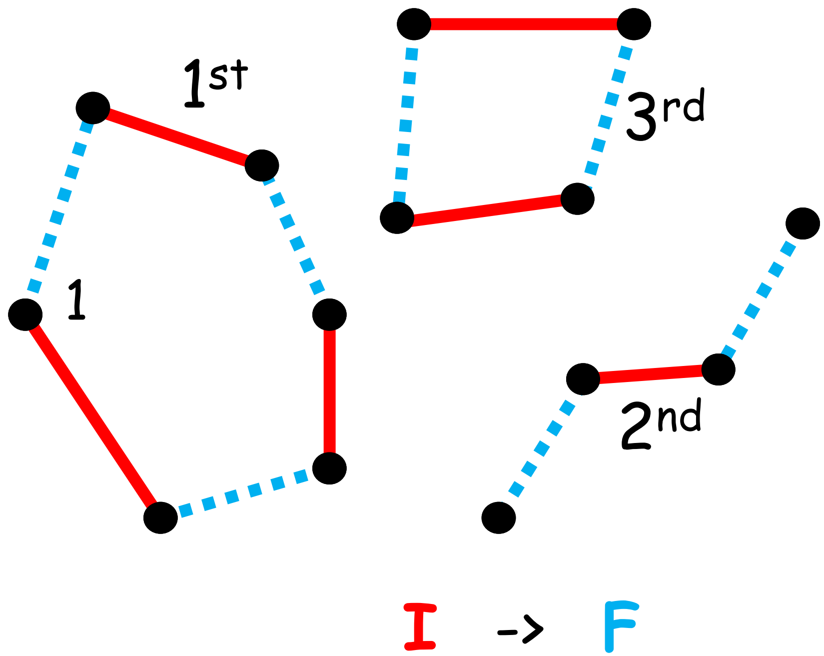

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(dashed edges:
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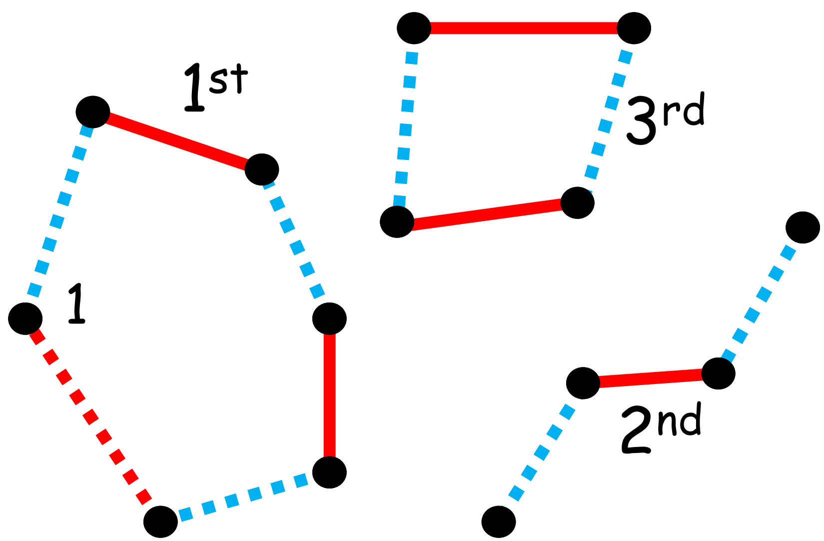

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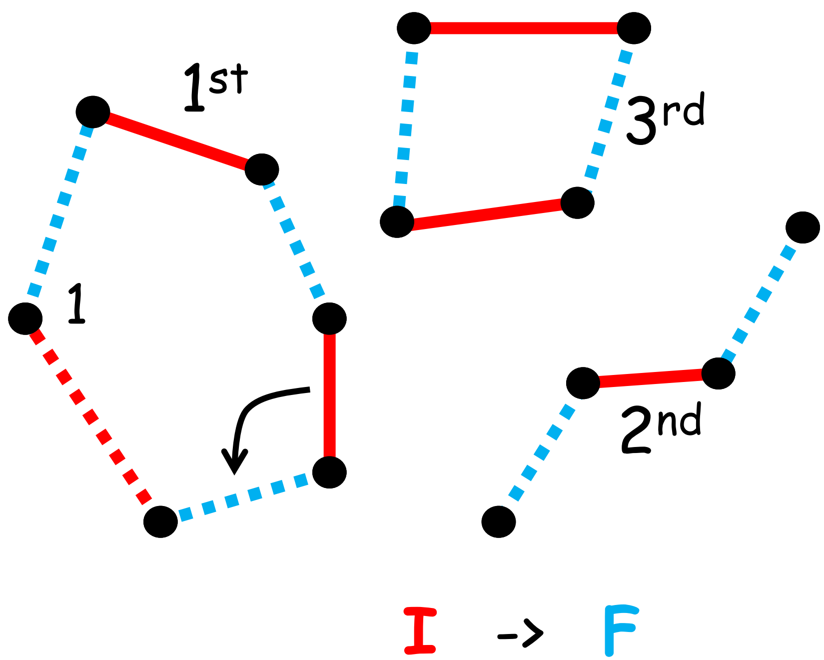

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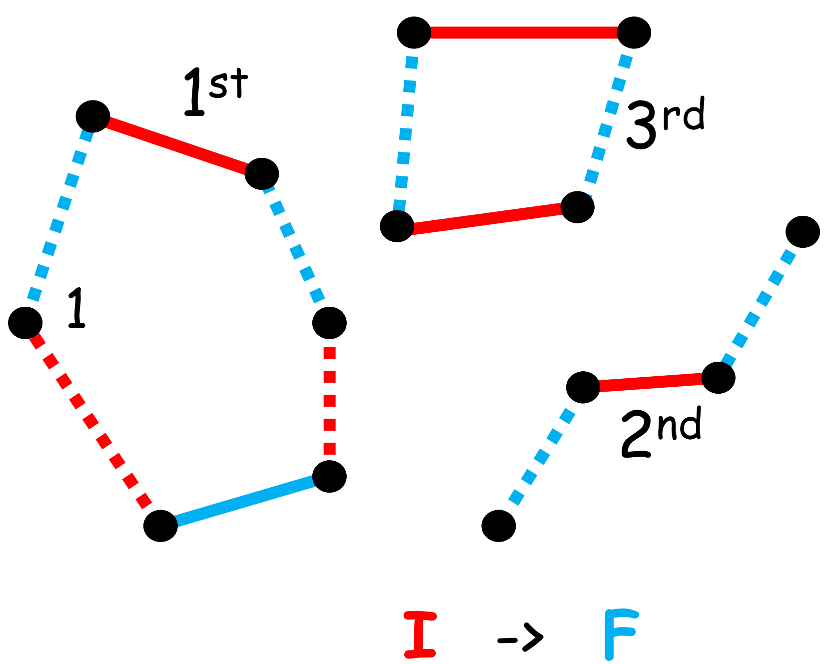

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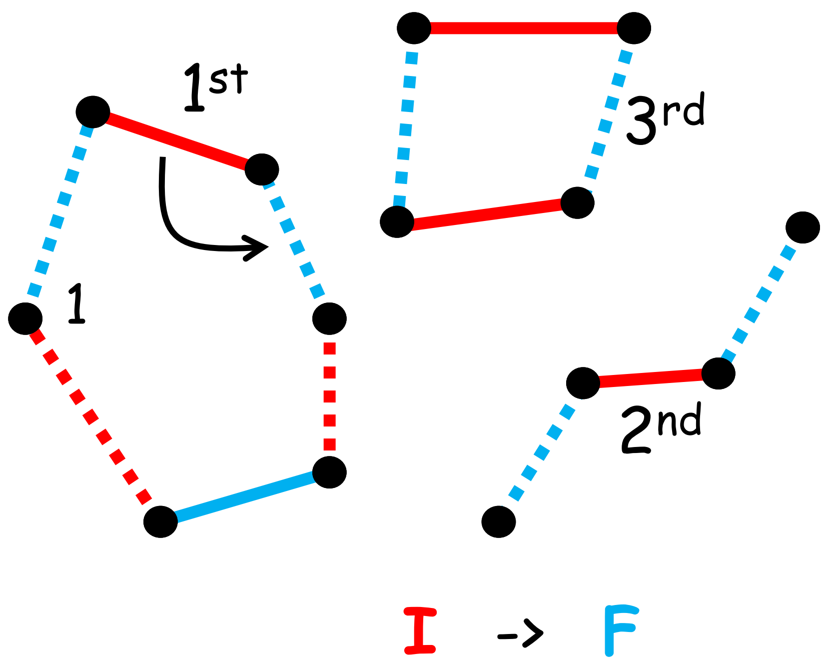

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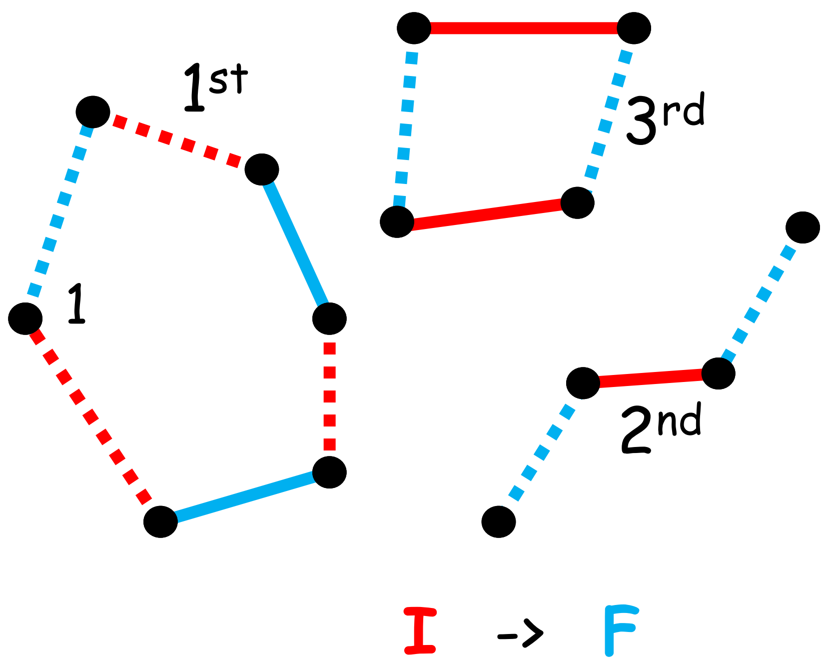

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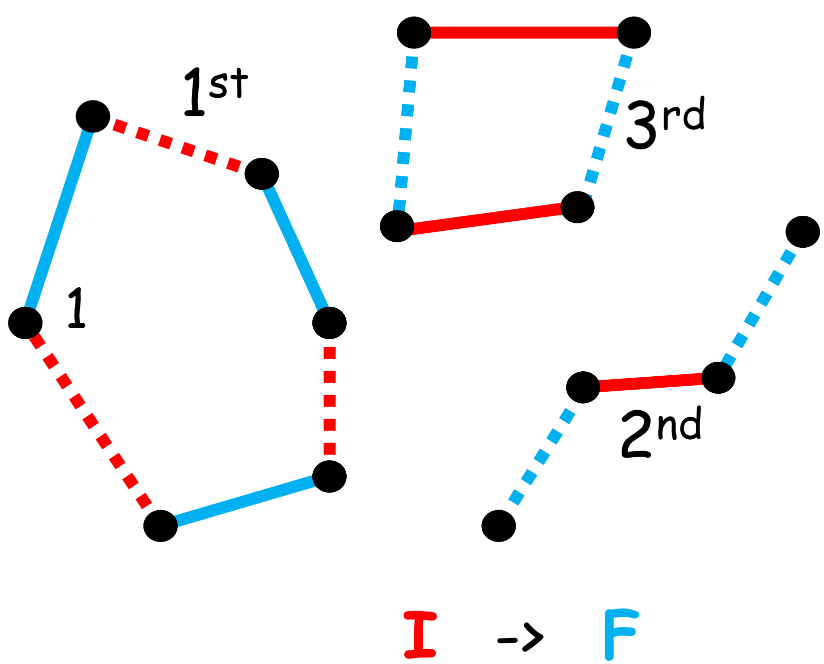

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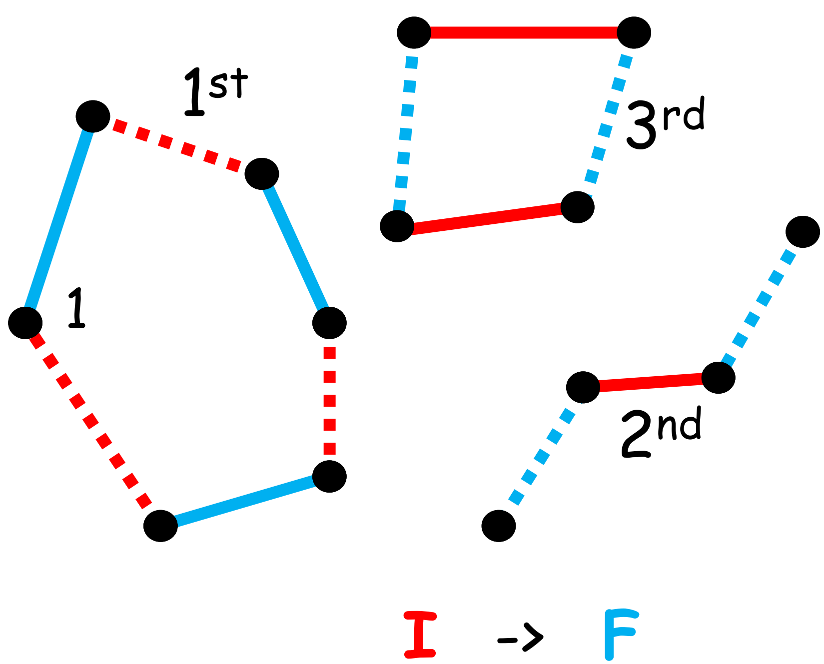

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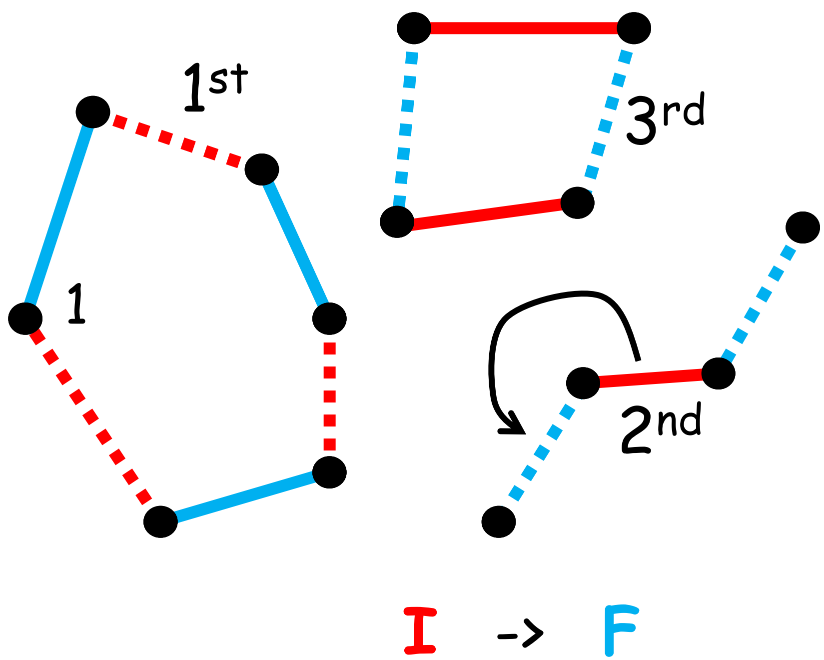

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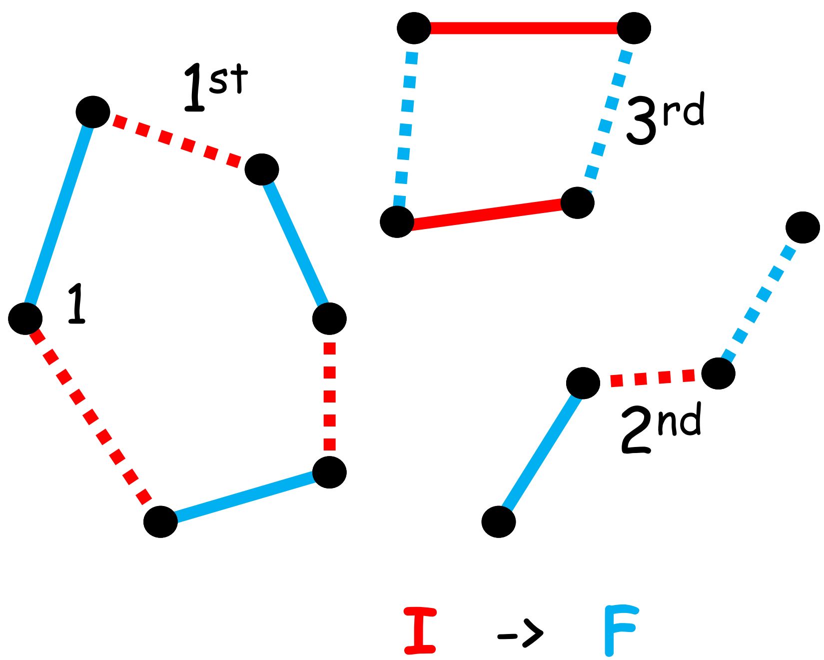

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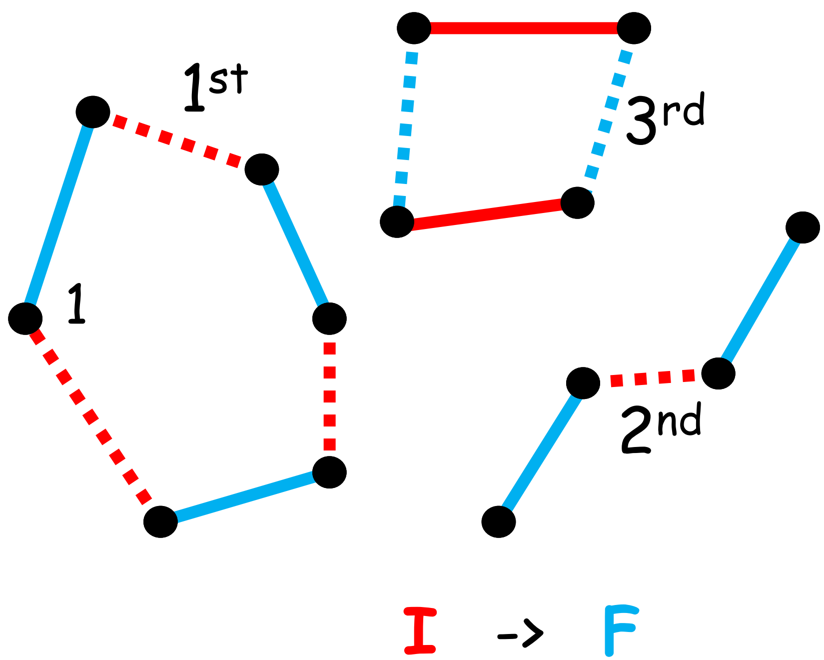

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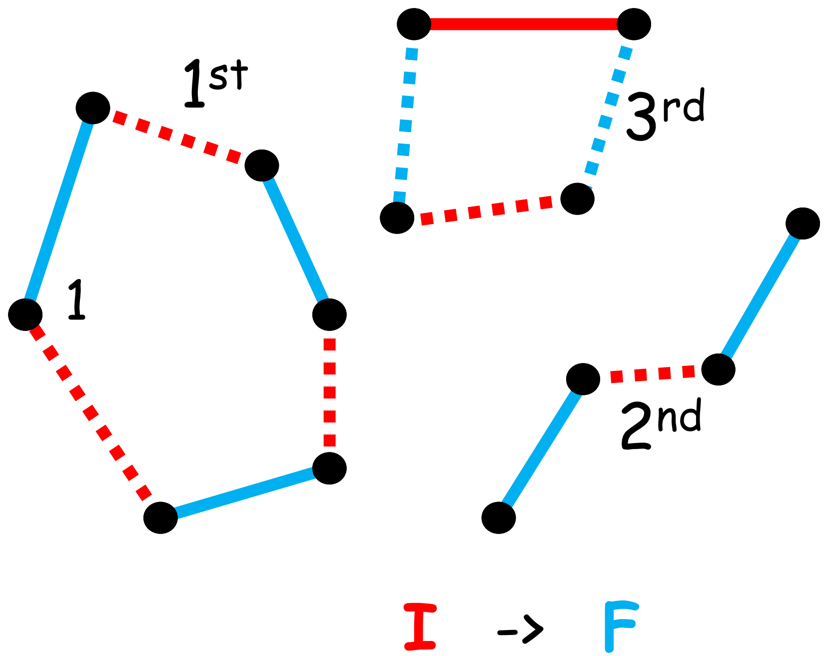

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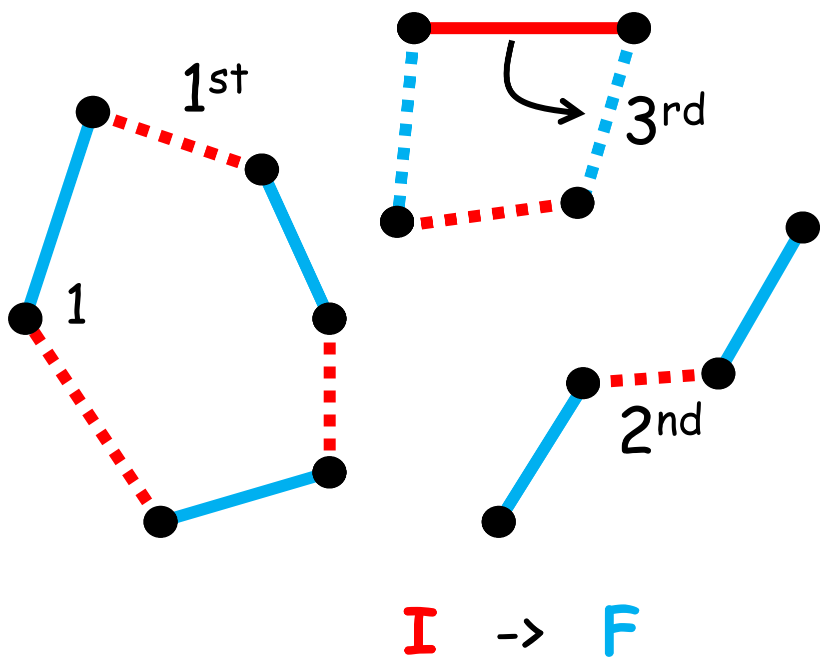

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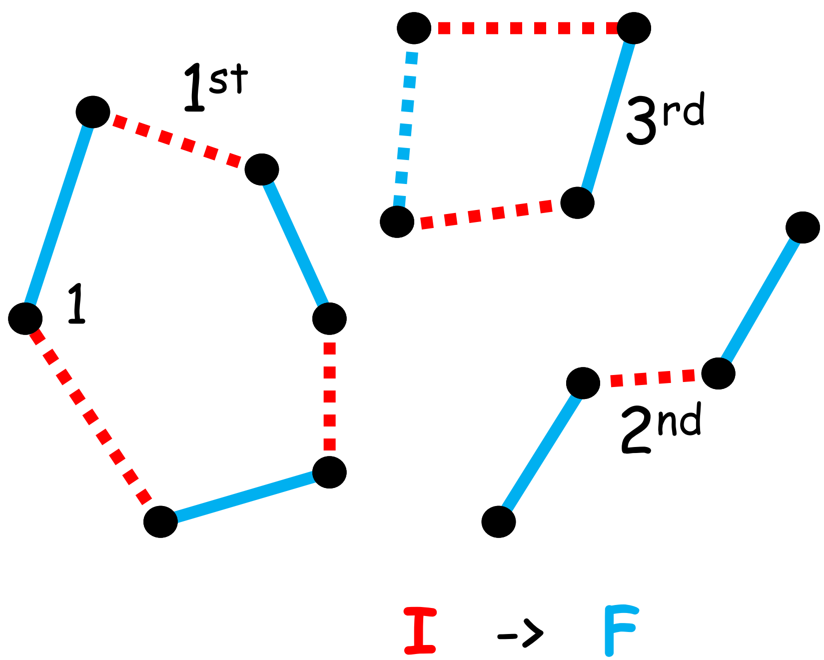

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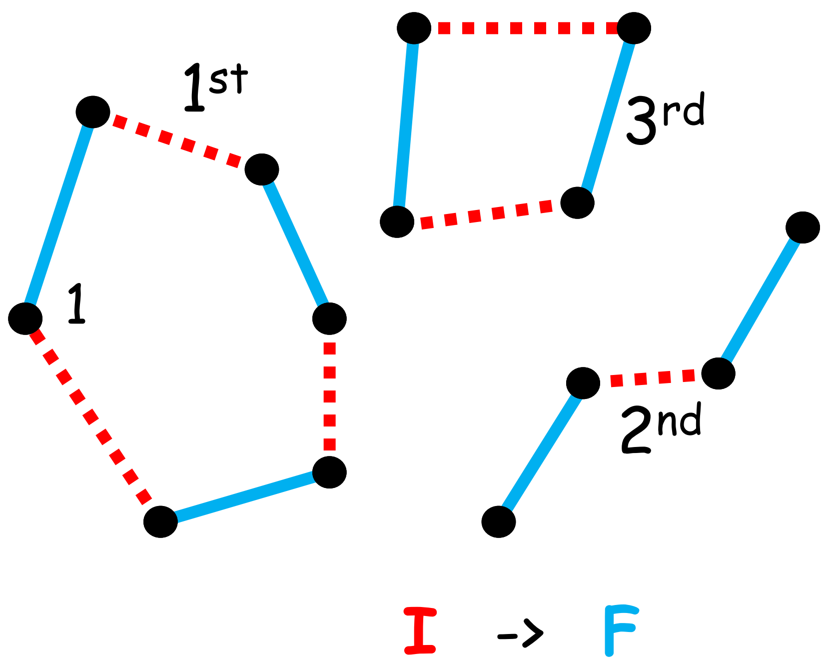

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Bounding the Congestion

Congestion through transition M->M':

$$\frac{1}{\pi(M)P(M,M')} \sum_{I\text{->}F \text{ path through } M\text{->}M'} \pi(I)\pi(F)\ \text{length}(I\text{->}F)$$

Since $\pi(M)=\pi(I)=\pi(F)=1/|\Omega|$ and $P(M,M')=1/(2m)$, and length(I->F)$\leq$n:

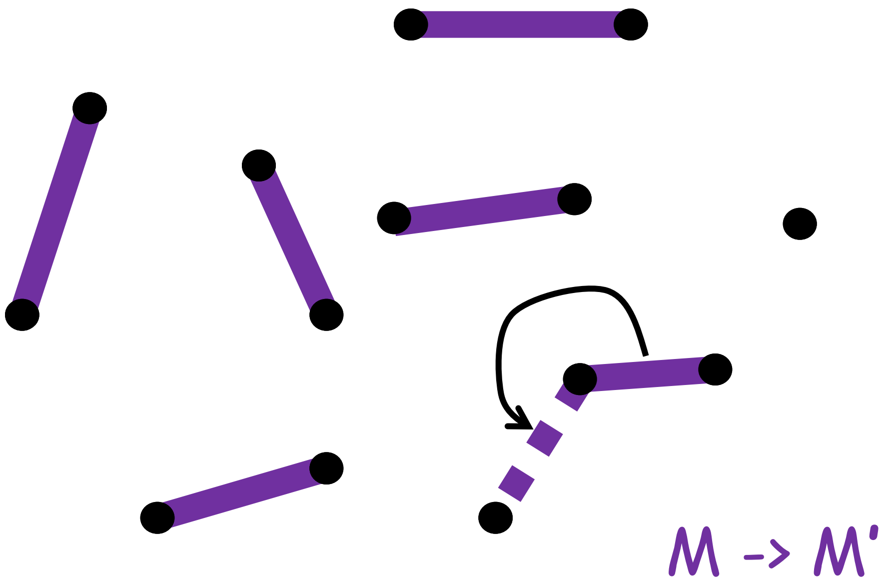
$$\leq \frac{2m}{|\Omega|} \sum_{I\text{->}F \text{ path through } M\text{->}M'} n$$

$$= \frac{2mn}{|\Omega|} (\#\ \text{canonical paths through } M\text{->}M')$$

Bounding the Congestion: Encoding

Let M->M' be a transition.

How many canonical paths go through it ? [Want $\leq |\Omega|\text{poly}(n)$]



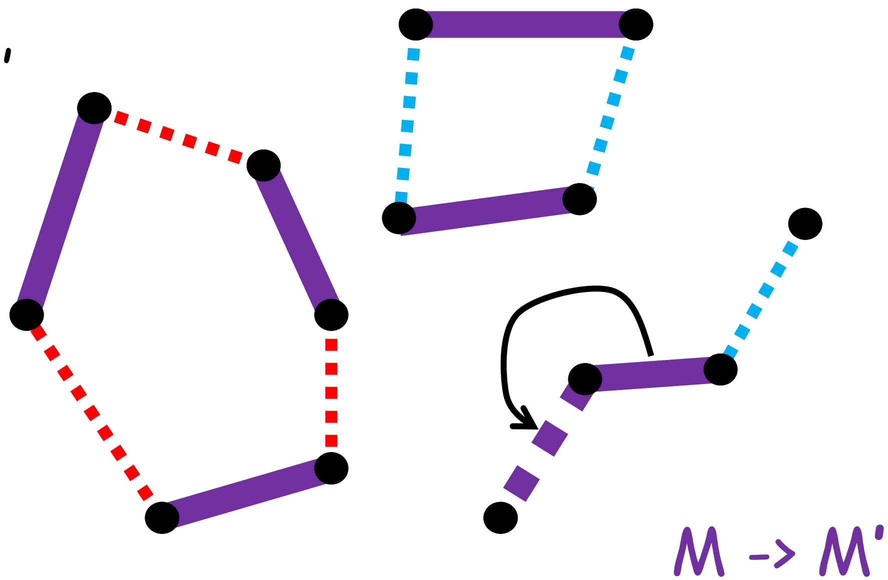
<u>Legend:</u>

- **purple**: transition

Bounding the Congestion: Encoding

Let M->M' be a transition.
How many canonical paths go through it ? [Want $\leq |\Omega|$poly(n)]

An I -> F path
through M->M'



Legend:
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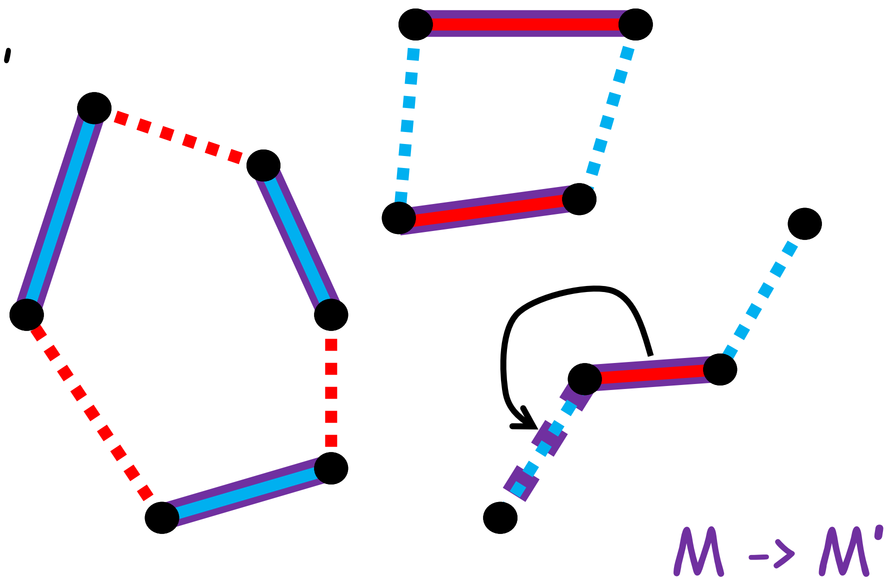

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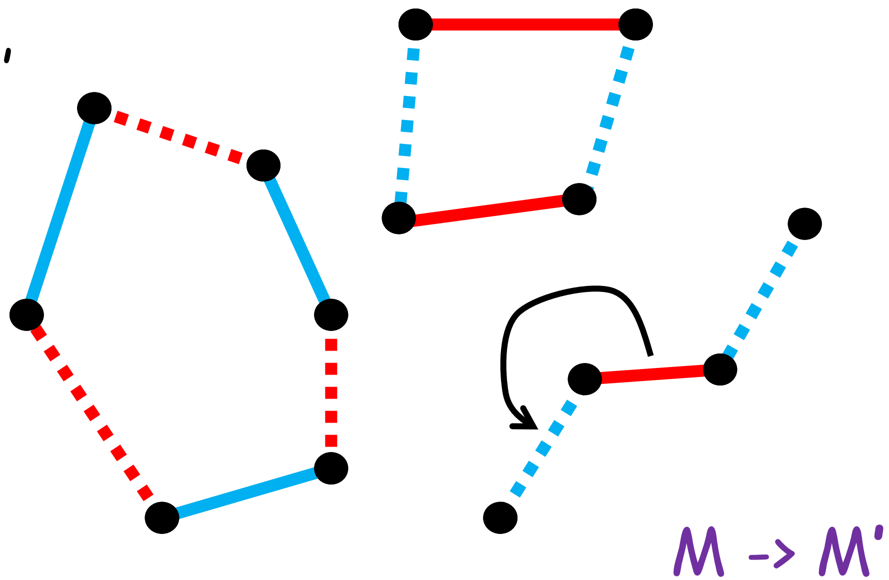

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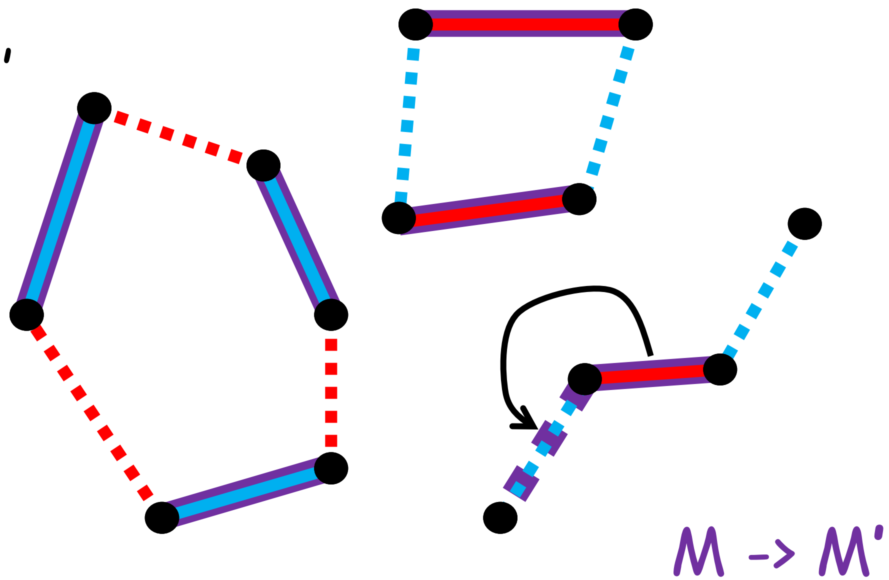

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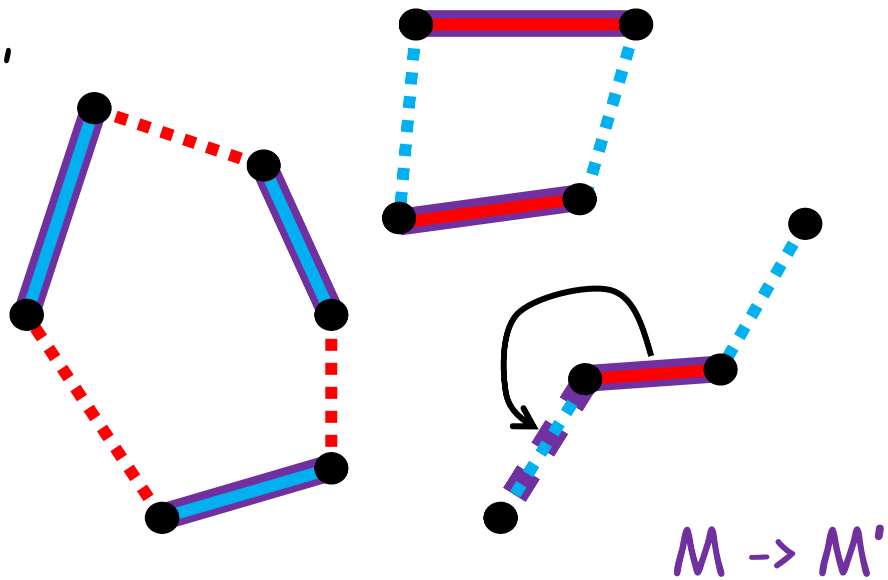

<u>Observation:</u>

$I \oplus F - (M \cup M')$
is a matching.

=> encoding E
(for I,F given M)

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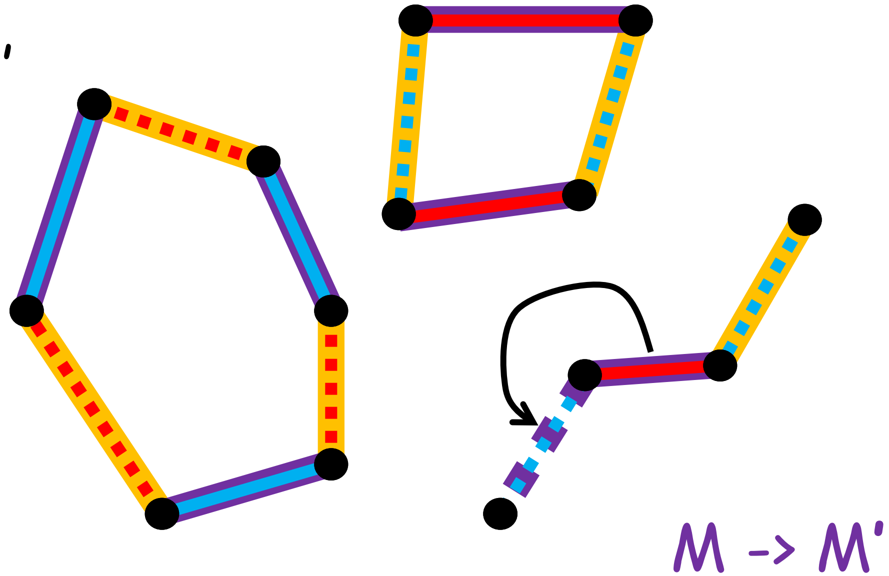

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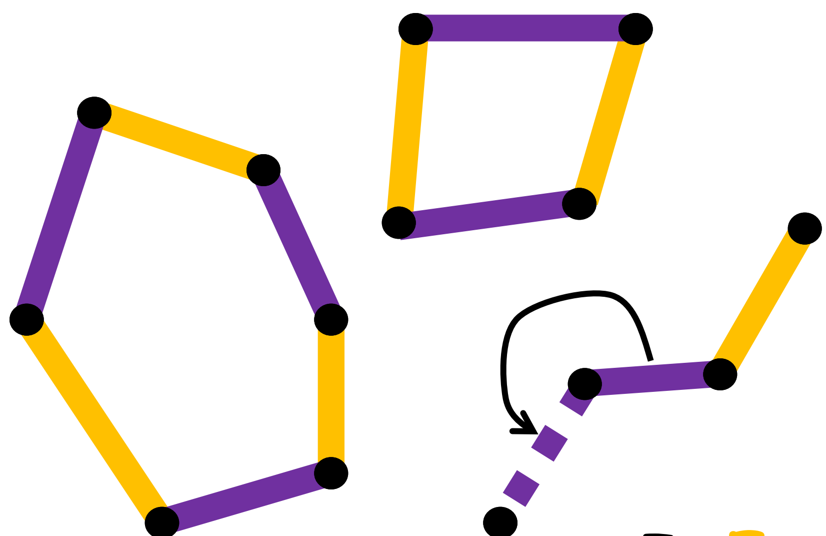

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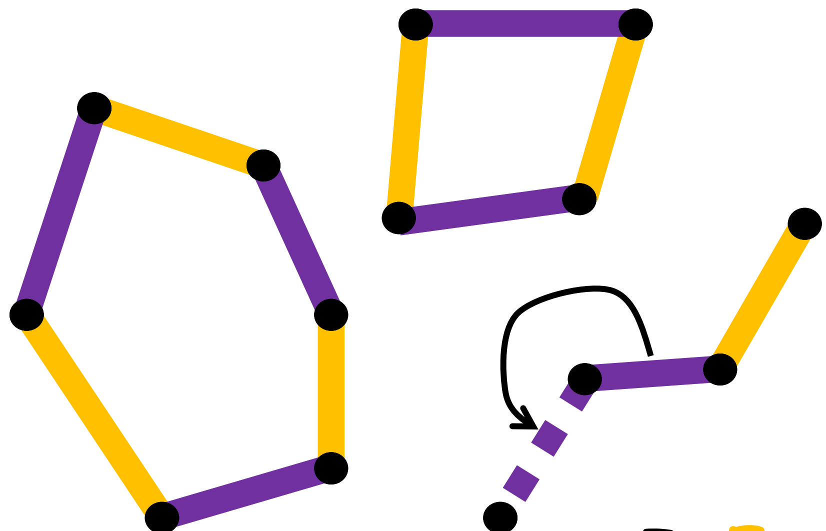
Is E an "encoding"?
(Given M->M' and E, can reconstruct I,F?)
If yes, then # can.paths through M->M' is $\leq |\Omega|$

Bounding the Congestion: Encoding

Reconstructing I,F from E,M->M':



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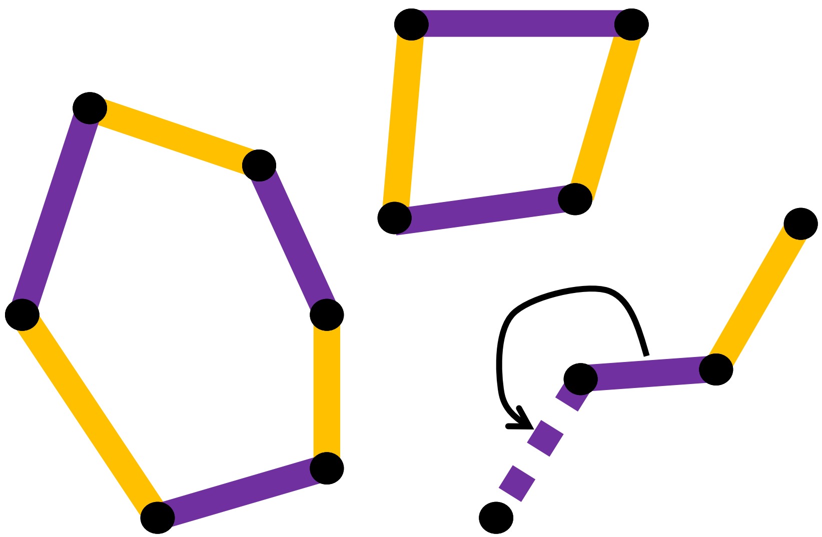

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Know:

- order of components
- currently working on 2nd
- 1st done, 3rd not yet
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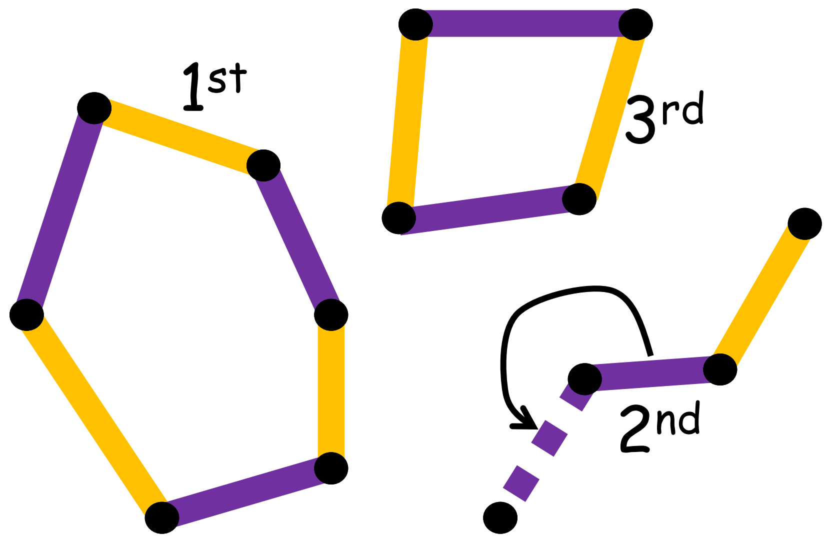

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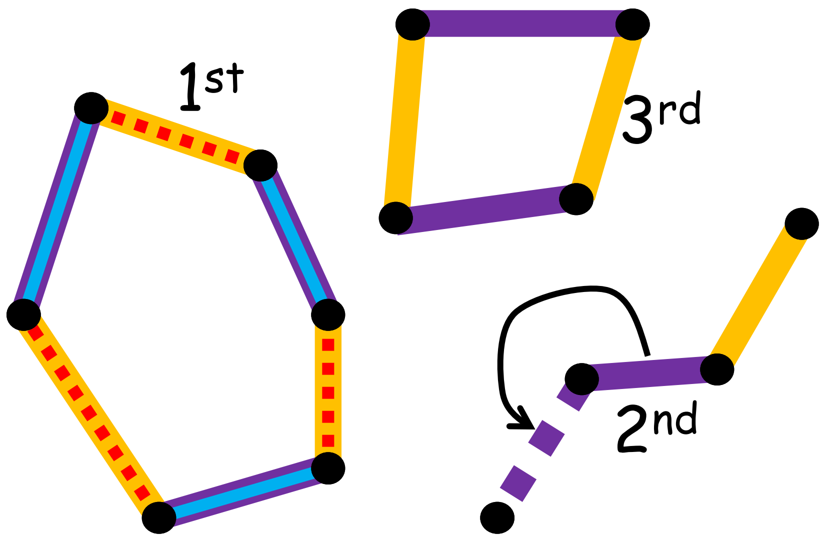

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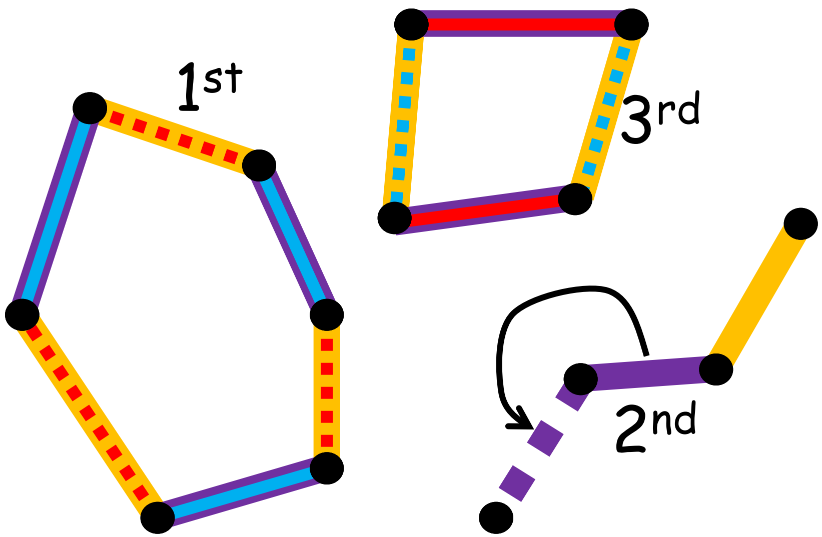

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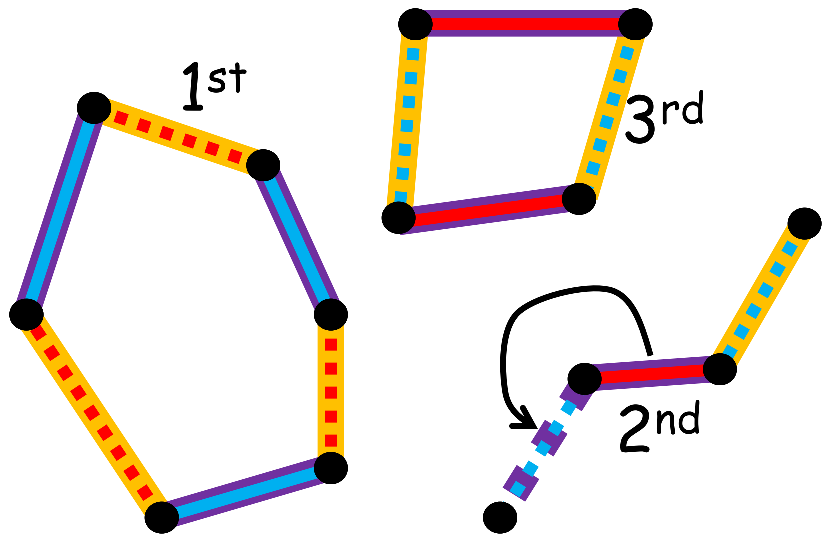

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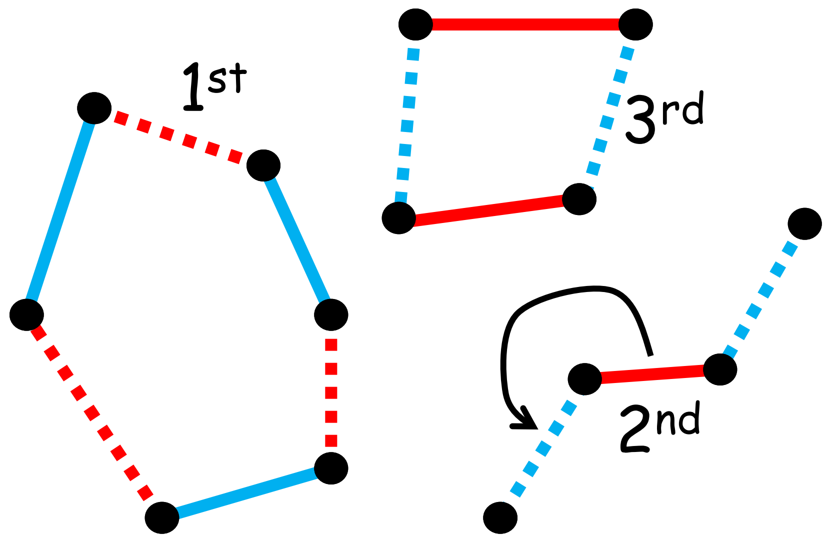

E = I ⊕ F - (M∪M')

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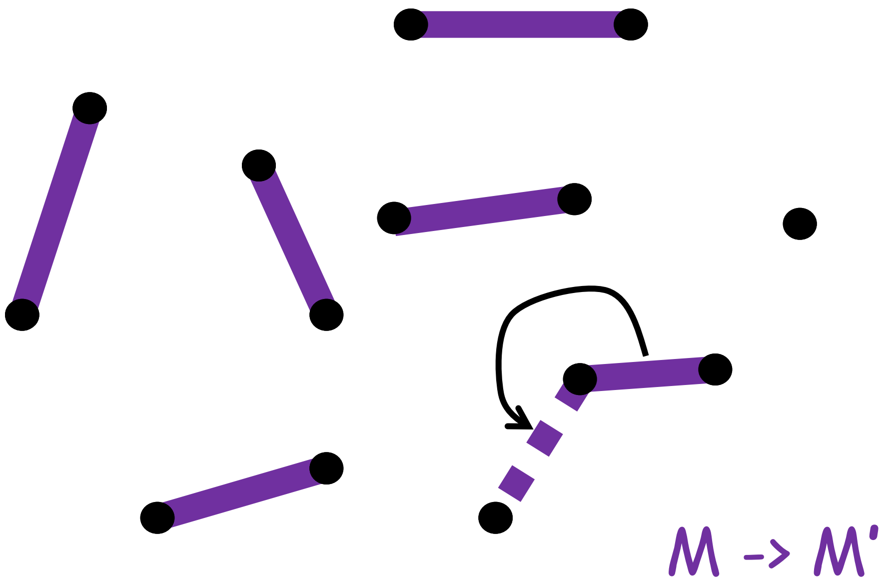
Let M->M' be a transition.
How many canonical paths go through it ? [Want $\leq |\Omega|\text{poly}(n)$]



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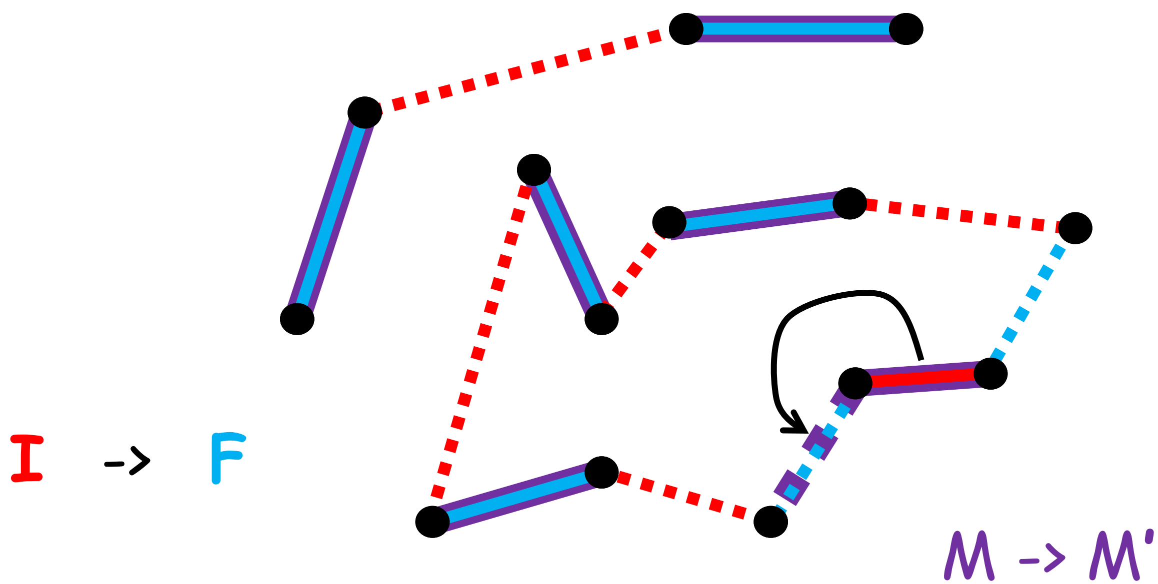

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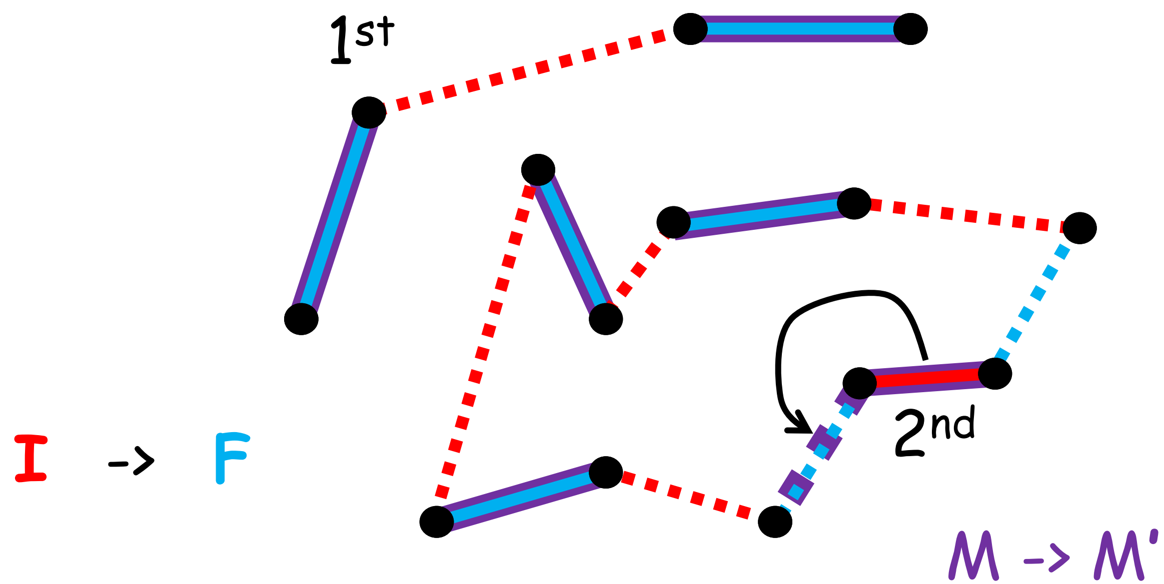

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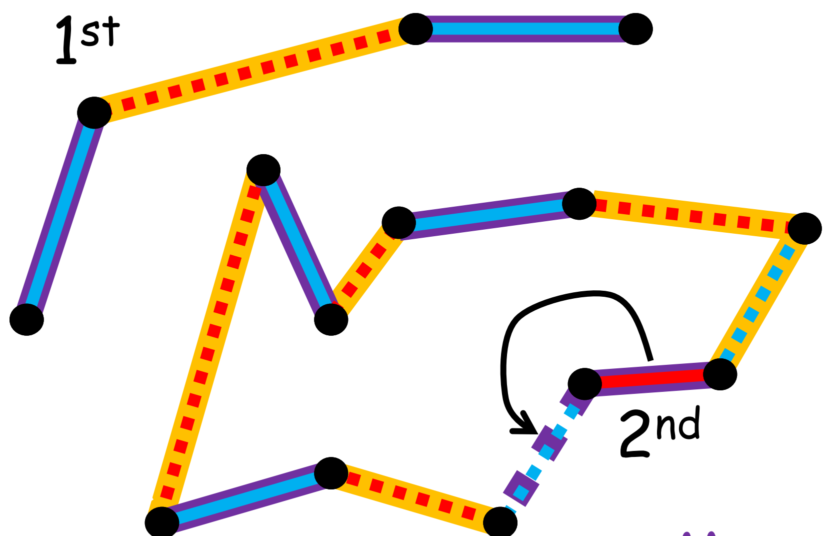

$E = I \oplus F - (M \cup M')$

Not a matching…

I -> F

M -> M'

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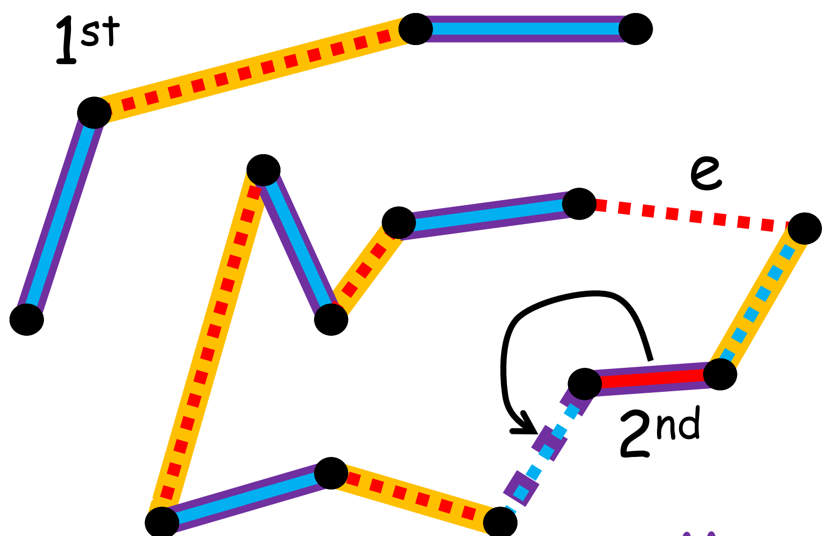

I -> F

M -> M'

$E = I \oplus F - (M \cup M')$
$- \{e\}$

Now a matching:
$E \in \Omega$

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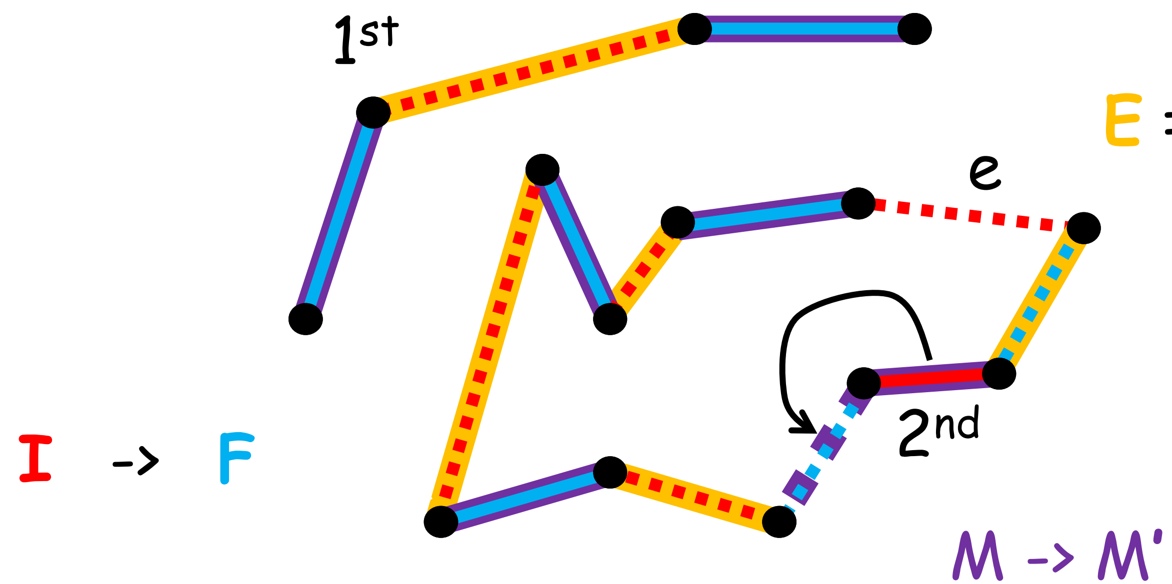

$E = I \oplus F - (M \cup M') - \{e\}$

Now a matching:
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Can "decode" E into I,F?

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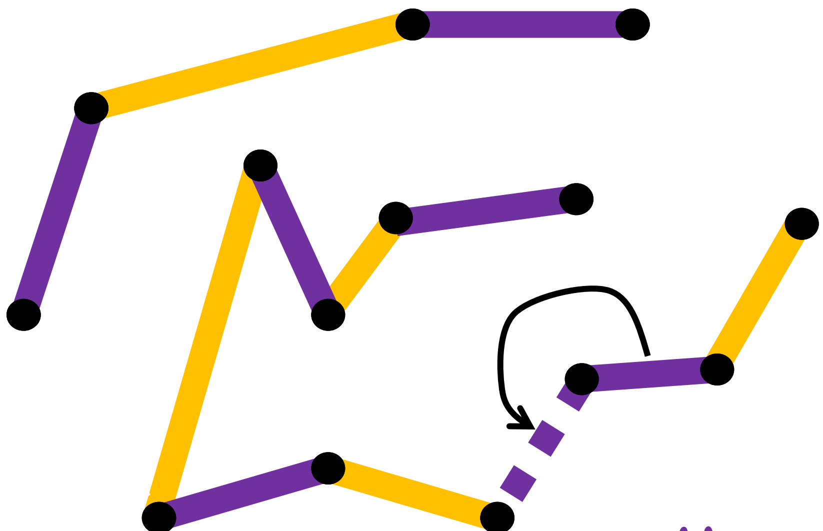

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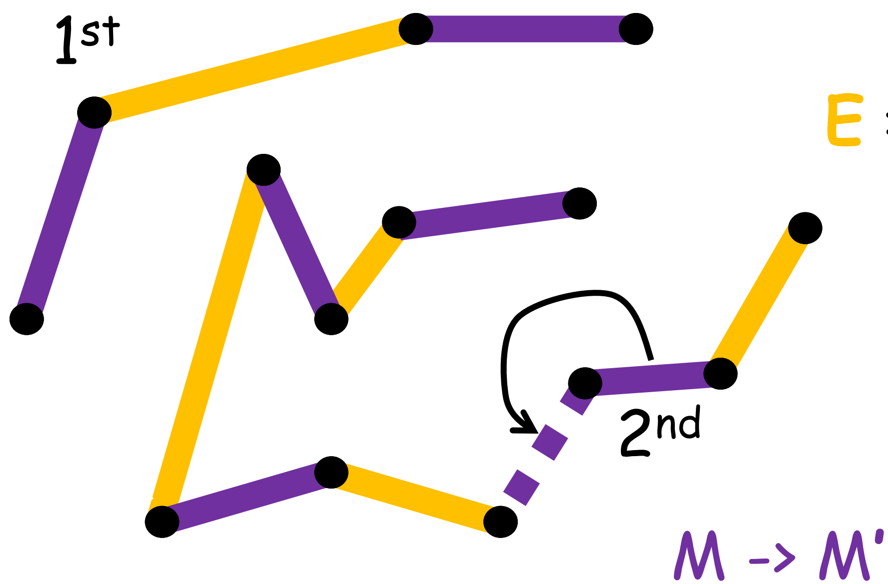

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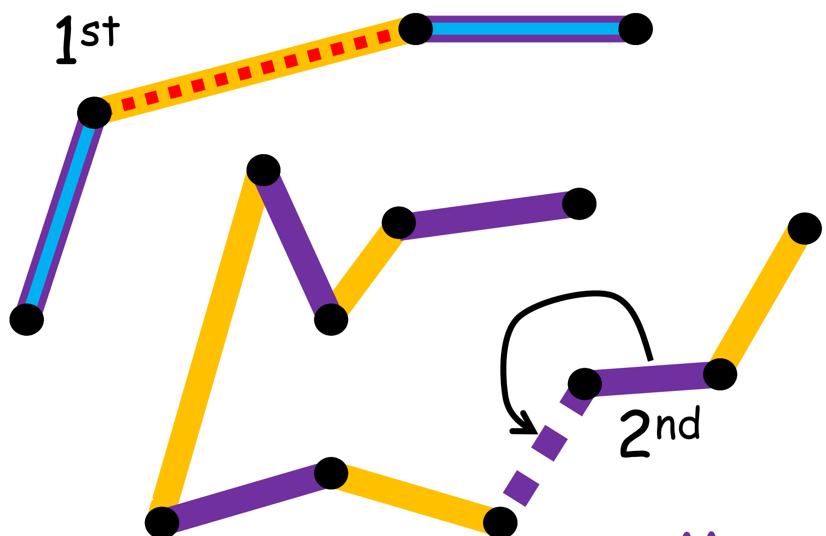

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Now a matching: $E \in \Omega$

I -> F

M -> M'

Can "decode" E into I,F?

<u>Legend:</u>

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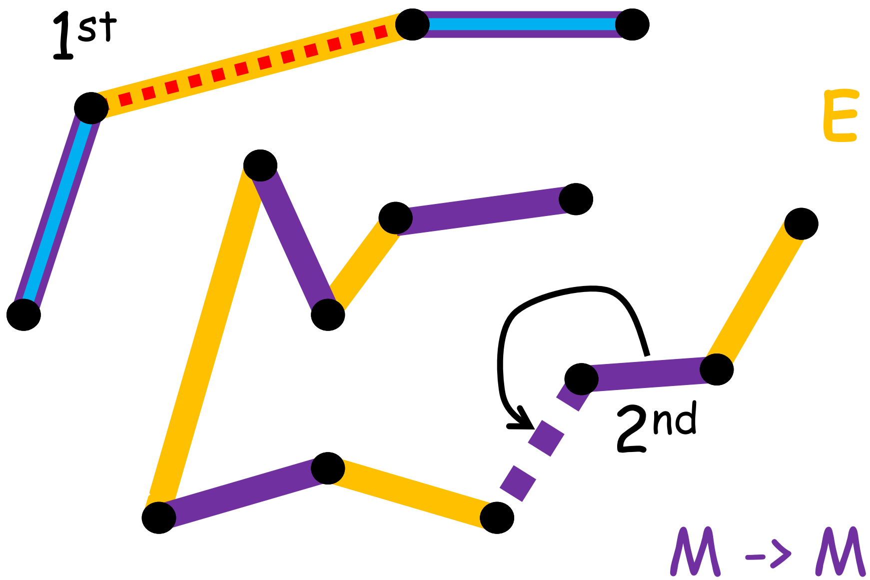

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Now a matching:
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I -> F

Can "decode" E into I,F?
- Is 2nd component a path?

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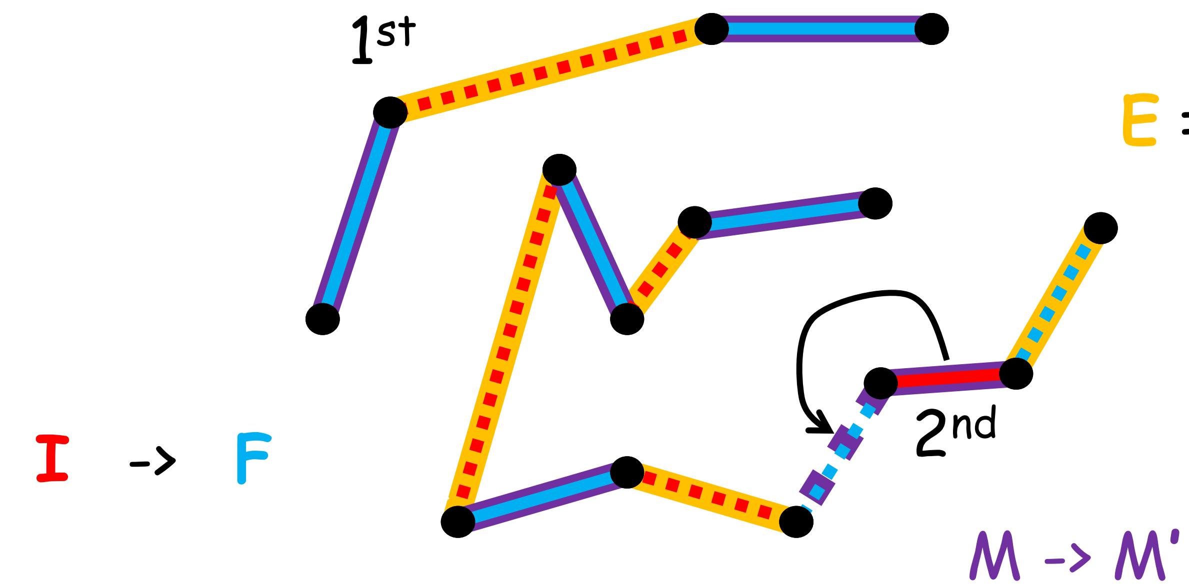

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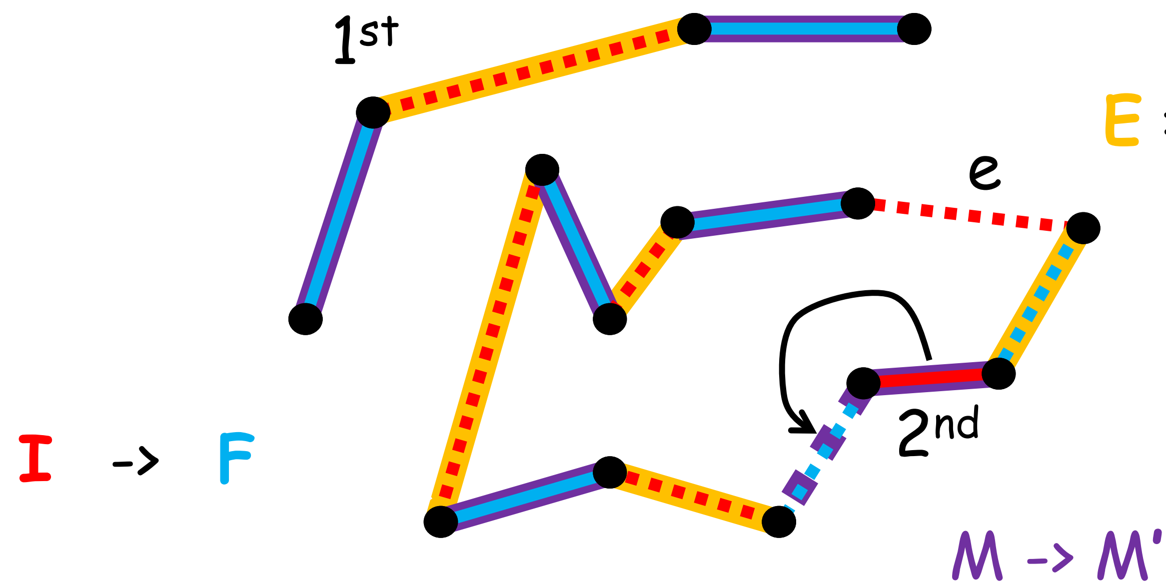

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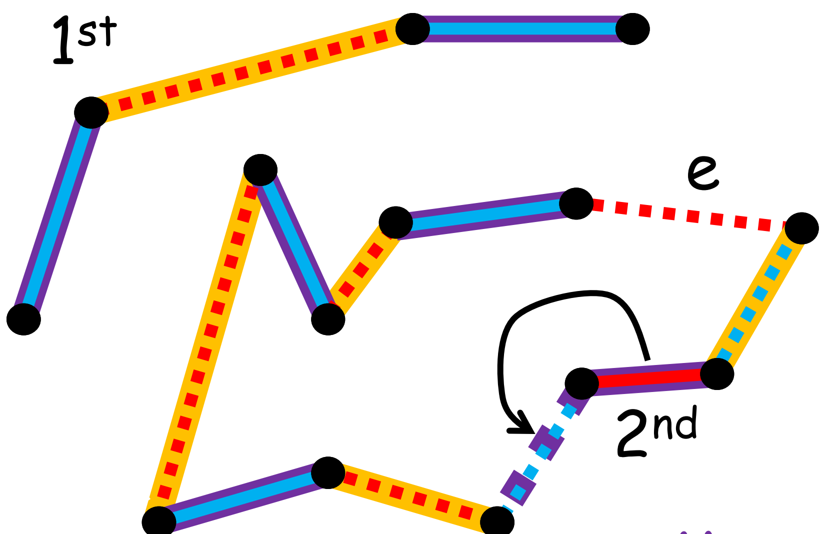

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I -> F

M -> M'

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Redefine encoding:
$E' := (E,0)$
$E' := (E,1)$
$\in \Omega \times \{0,1\}$

Bounding the Congestion: Encoding

Let M->M' be a transition.
How many canonical paths go through it ? [Want $\leq |\Omega|$poly(n)]

- For the sliding transition: $\leq 2|\Omega|$

- Need to analyze the add and remove transitions

<u>Bound on the congestion:</u>

$$\rho \leq \frac{2mn}{|\Omega|} (\text{\# can. paths through M->M'}) = \frac{2mn}{|\Omega|} 2|\Omega| = 4mn$$

<u>Mixing time:</u> $t_{mix}(\epsilon) = O(mn \log(1/(\epsilon\pi_{min})))$
$= O^*(mn^2)$ [O* - ignore polylog]

<u>FPRAS:</u> $O(T(n,m,\epsilon/(6m))\ m^2/\epsilon^2) = O^*(m^3n^2/\epsilon^2)$

A Markov Chain for Perfect Matchings

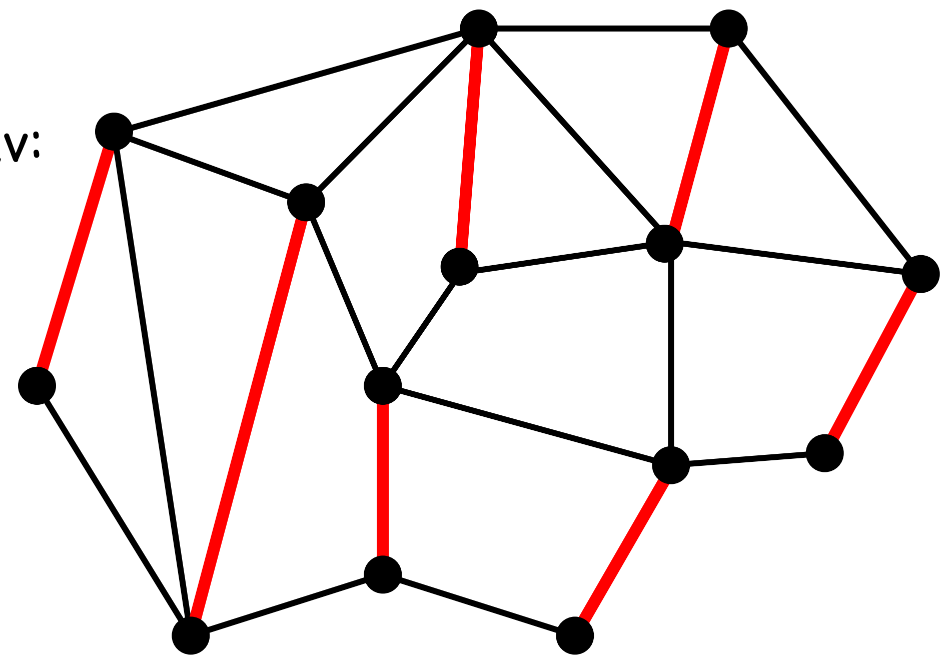
Input: a graph G

Exactly 2 vertices not matched

State space Ω: all perfect and near-perfect matchings of G

Markov chain (slide chain):
Let M be the current matching, we get the next state by choosing a random vertex w and:

- if M is perfect: remove w's edge
- if M is near-perfect with holes u,v:
  - if w=u or v, add (u,v) if can
  - else, randomly choose u or v, replace w's current edge by (u,w) or (v,w)



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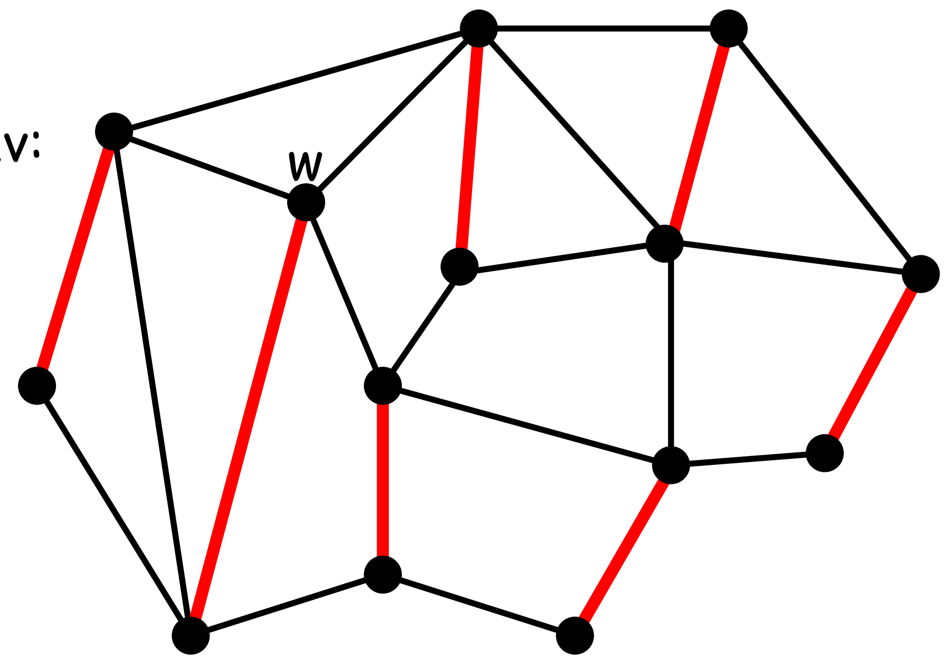

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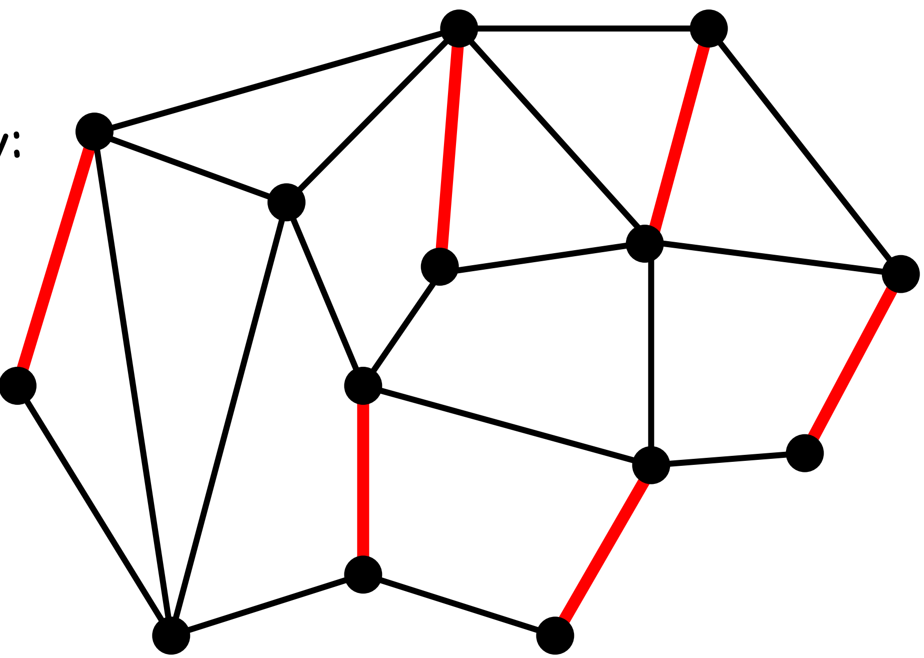

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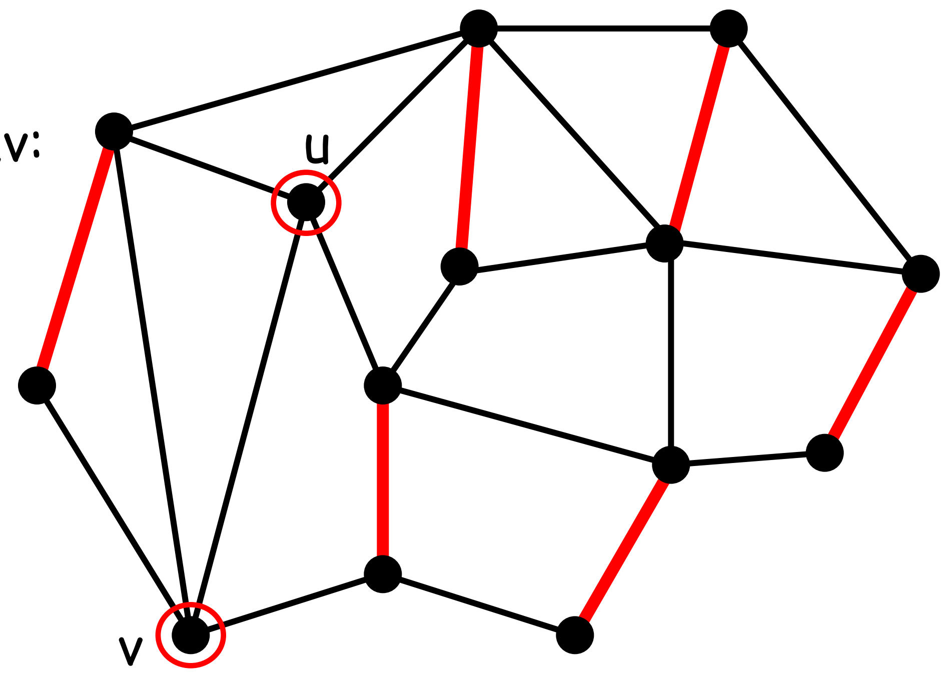

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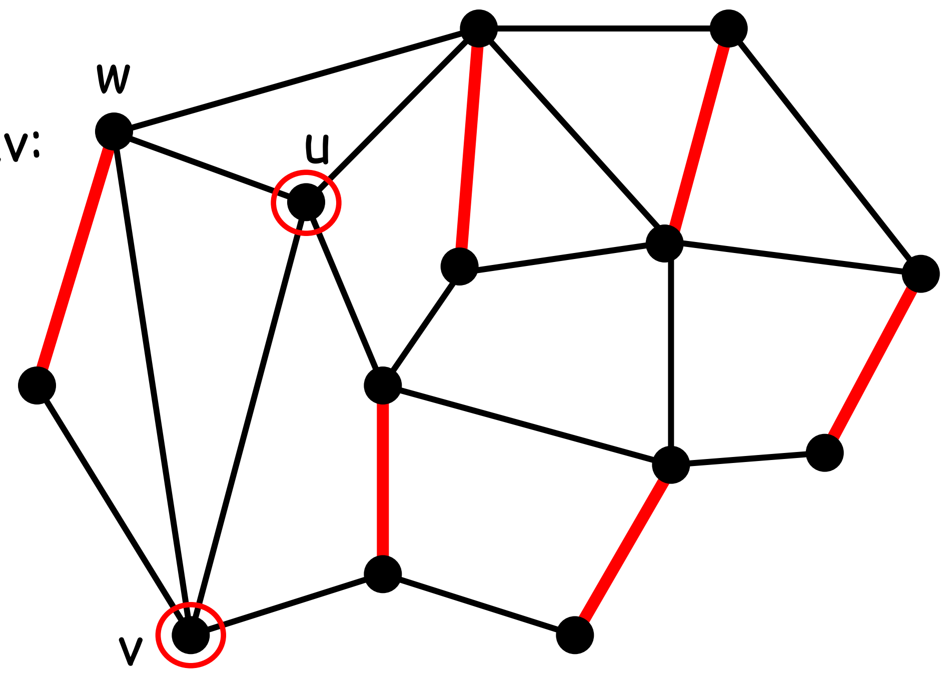

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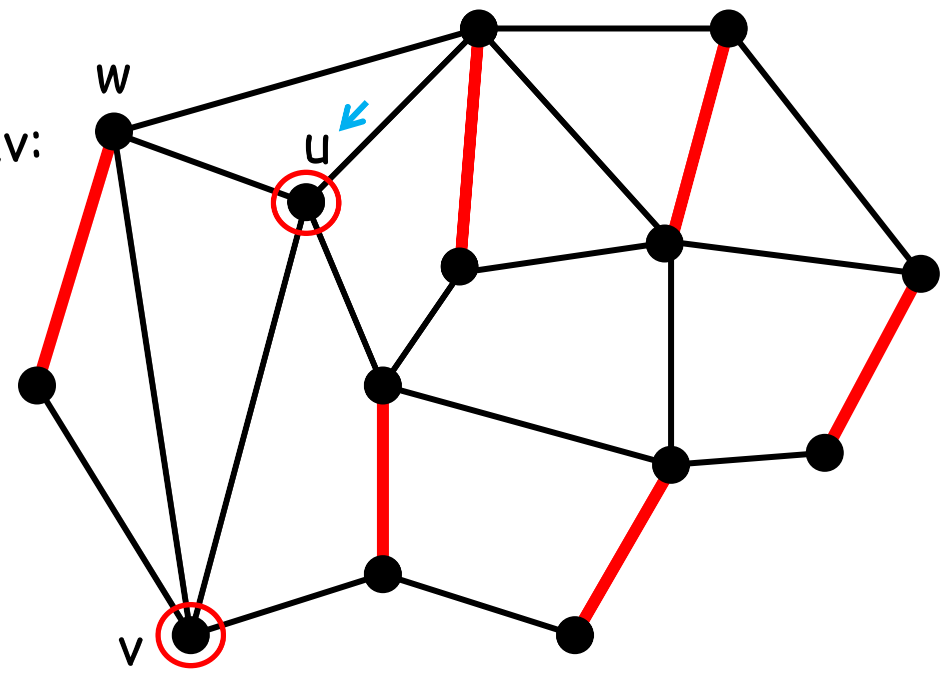

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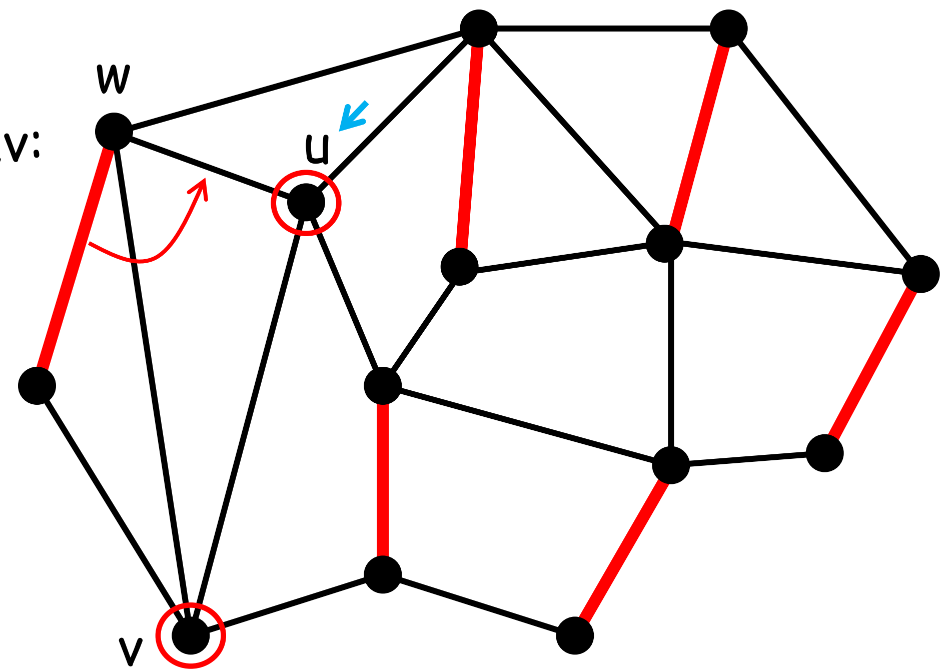

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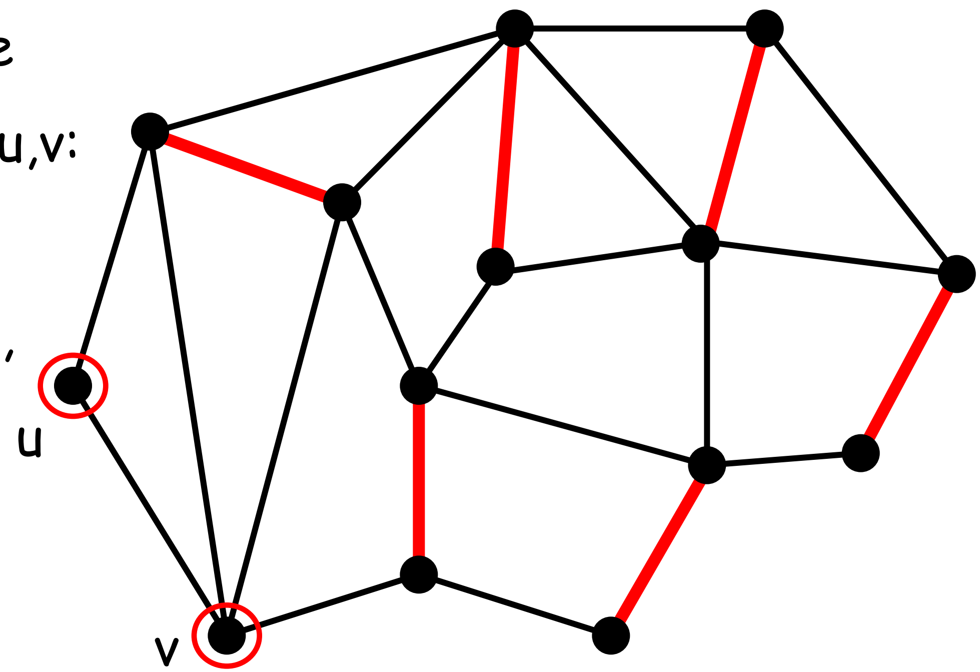
<u>Markov chain</u> (slide chain):
Let M be the current matching, we get the next state by choosing a random vertex w and:

- if M is perfect: remove w's edge
- if M is near-perfect with holes u,v:
    - if w=u or v, add (u,v) if can
    - else, randomly choose u or v, replace w's current edge by (u,w) or (v,w)



A Markov Chain for Perfect Matchings

Input: a graph G

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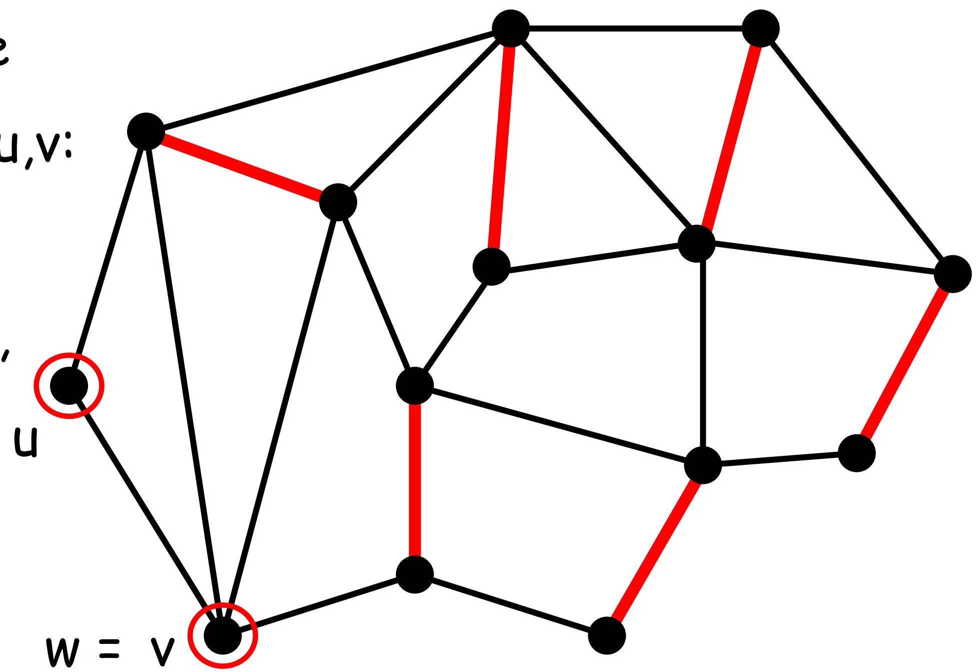

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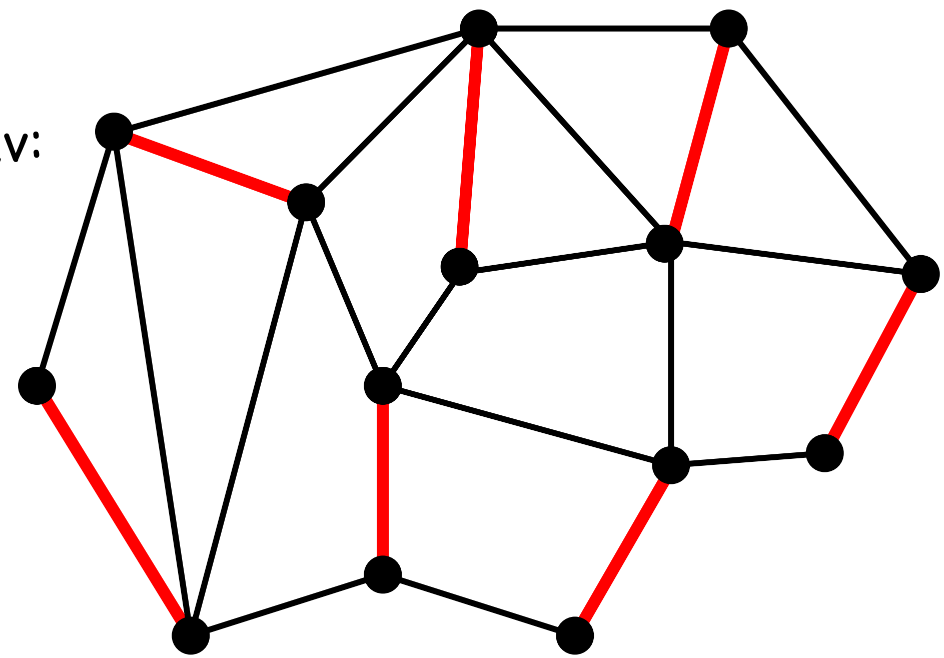
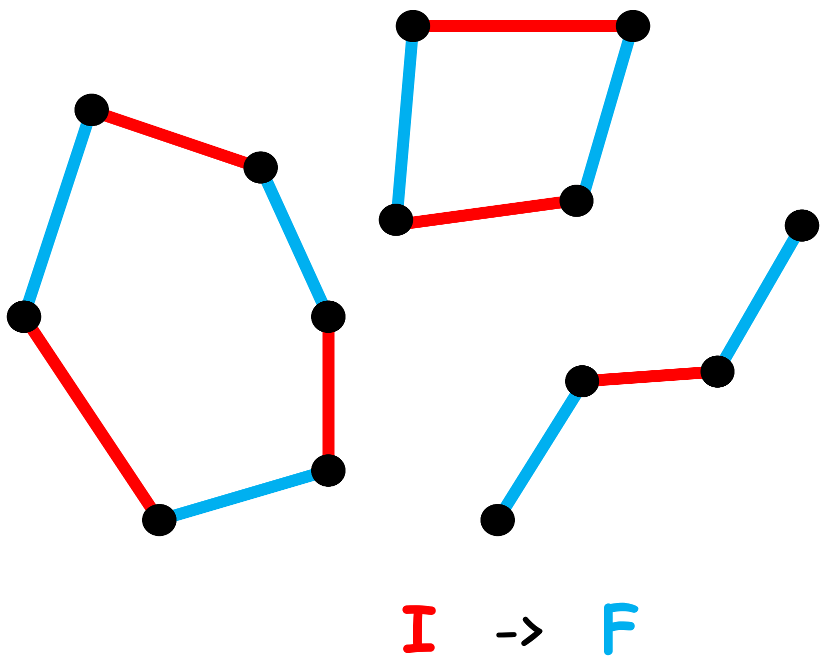


A Markov Chain for Perfect Matchings

Analysis of this MC:

- canonical paths as before for perfect to perfect
- for near to perfect:
  exactly one alternating path – process it last



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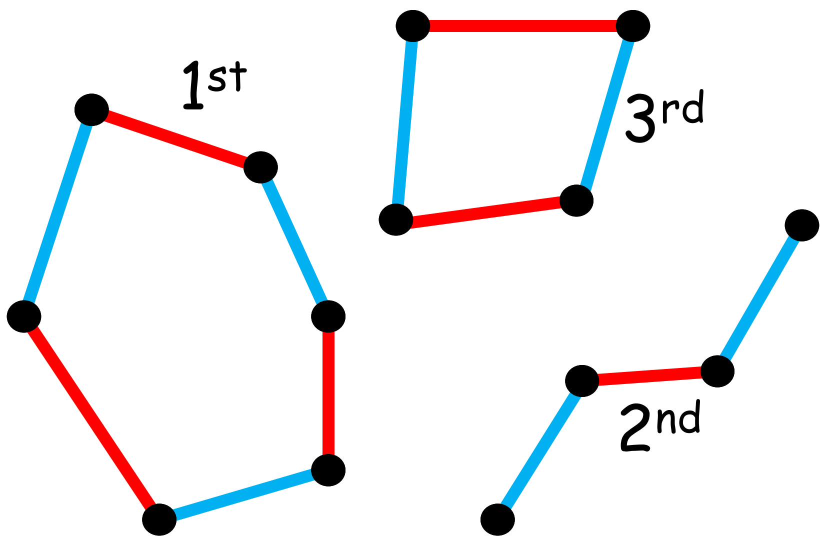

I -> F

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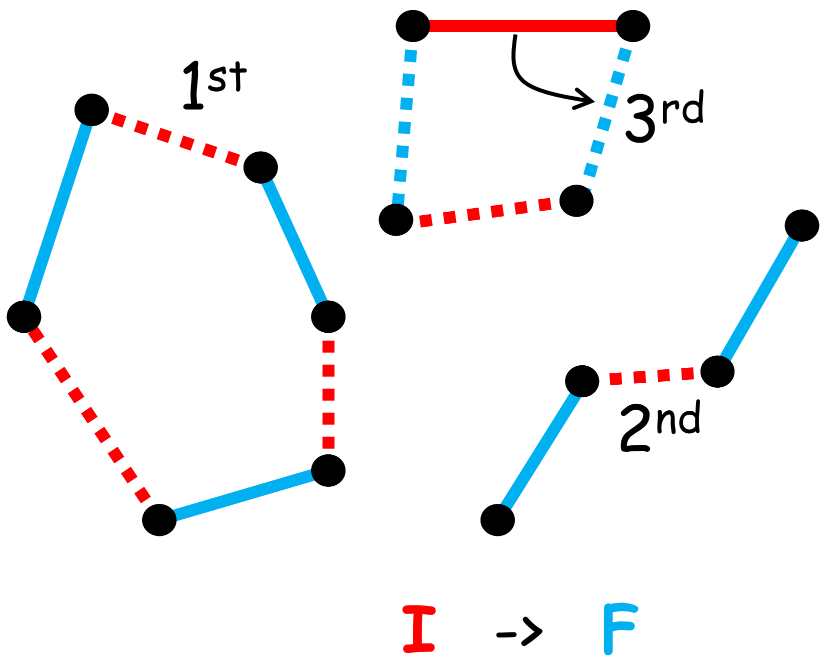

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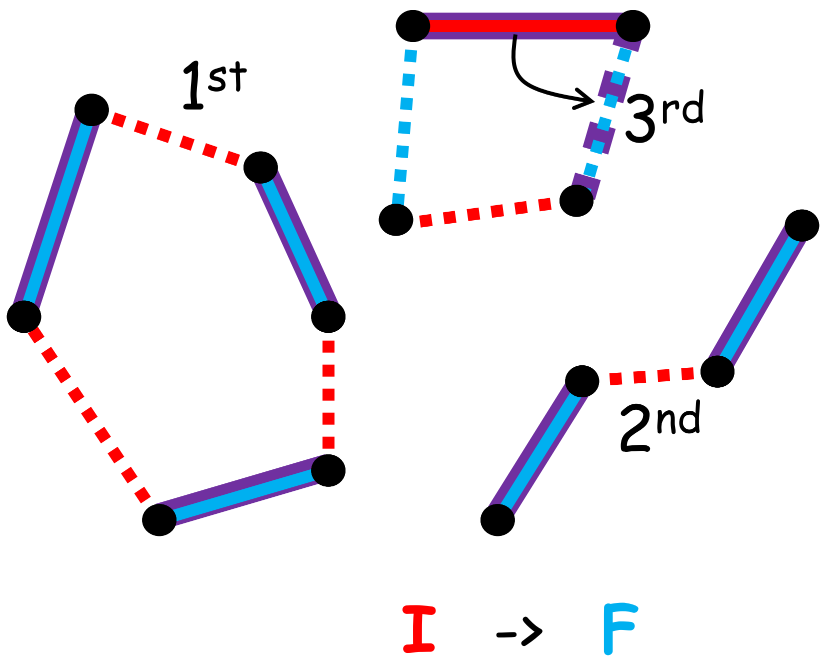

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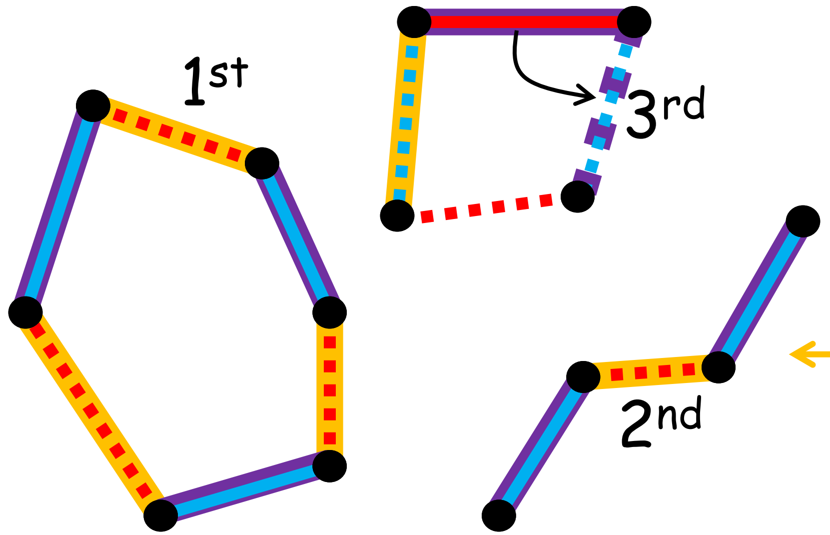

Want to process last, otherwise 4 holes -> not in Ω

I -> F

A Markov Chain for Perfect Matchings

<u>Analysis of this MC:</u>

- canonical paths as before for perfect to perfect
- for near to perfect:
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- for near to near:
  go through a random perfect matching

  (Instead of canonical paths, split into a flow.)

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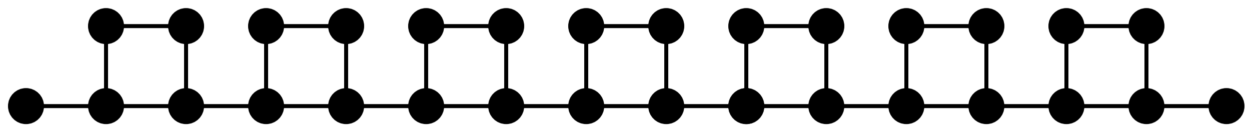
  (Instead of canonical paths, split into a flow.)

Mixing time: $t_{mix}(\epsilon) = O^*(n^3\,(\#nears/\#perfects))$

Polynomial if # near-perfect / # perfect matchings is polynomial…
E.g. for dense graphs: every vertex of degree > n/2.

A Markov Chain for Perfect Matchings

What if improve mixing time analysis?



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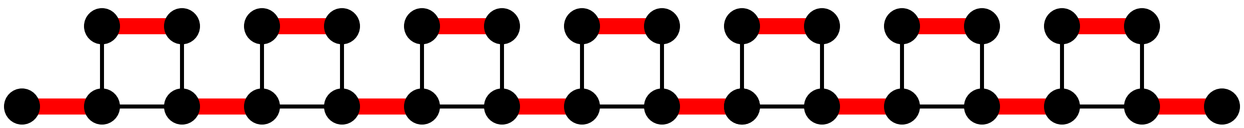

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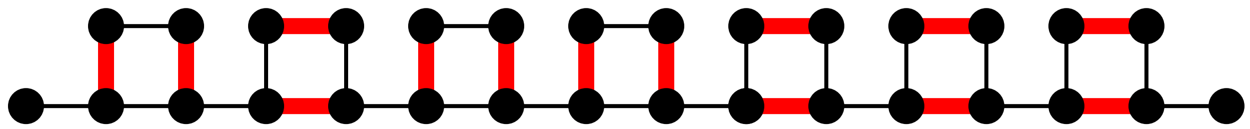

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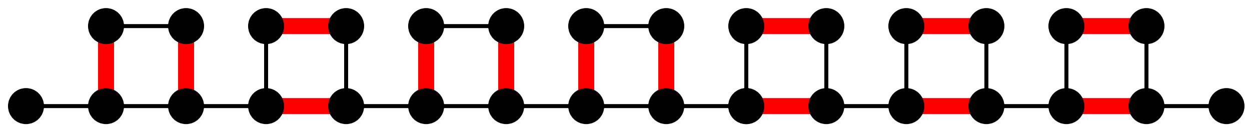

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This MC good only if #nears/#perfects polynomial...

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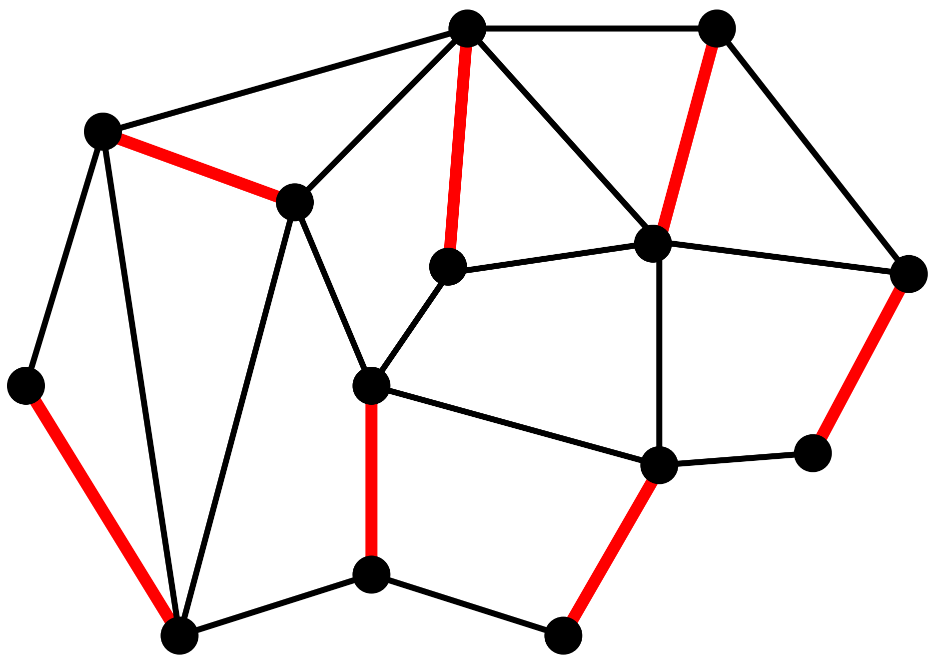
E.g. for dense graphs: every vertex of degree > n/2.

Another MC for Perfect Matchings

Input: a graph G

State space Ω: all perfect matchings of G

Markov chain (swap chain):
Let M be the current matching, choose two random edges (u,v) and (x,y) in M, replace them with (u,y) and (v,x) if can.



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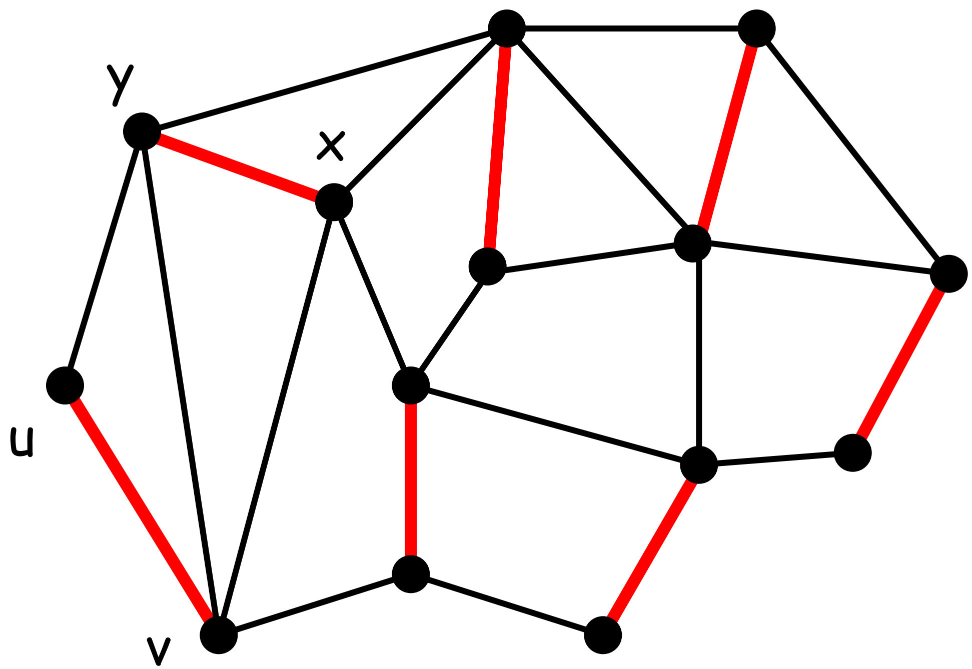

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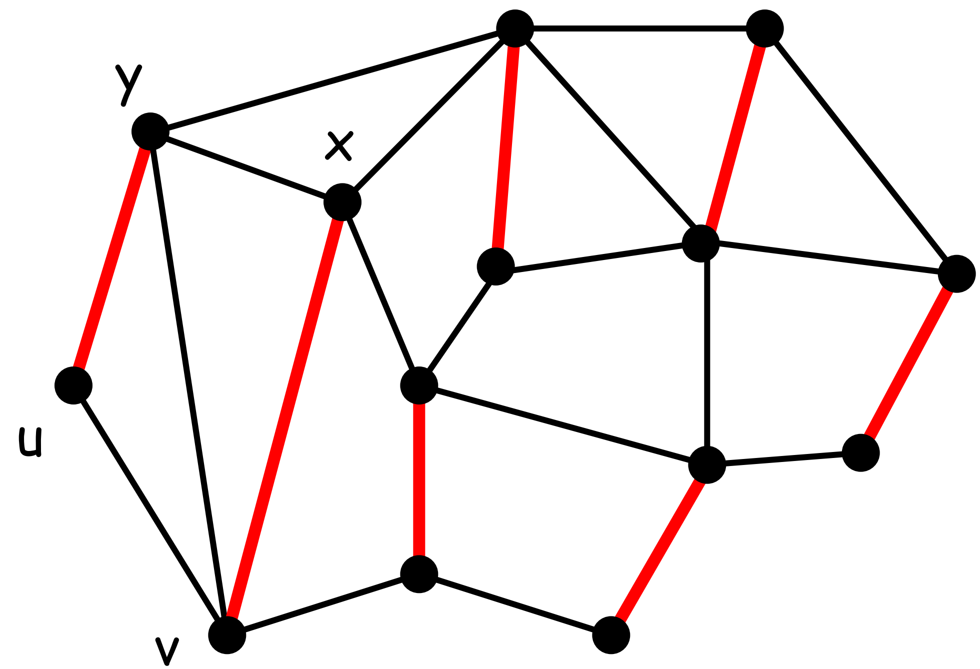

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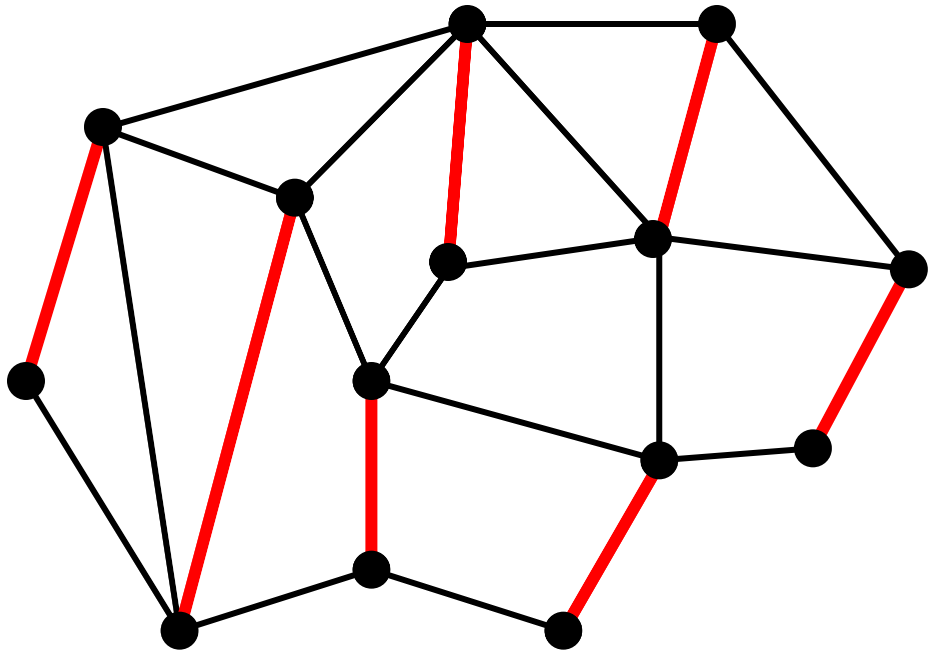

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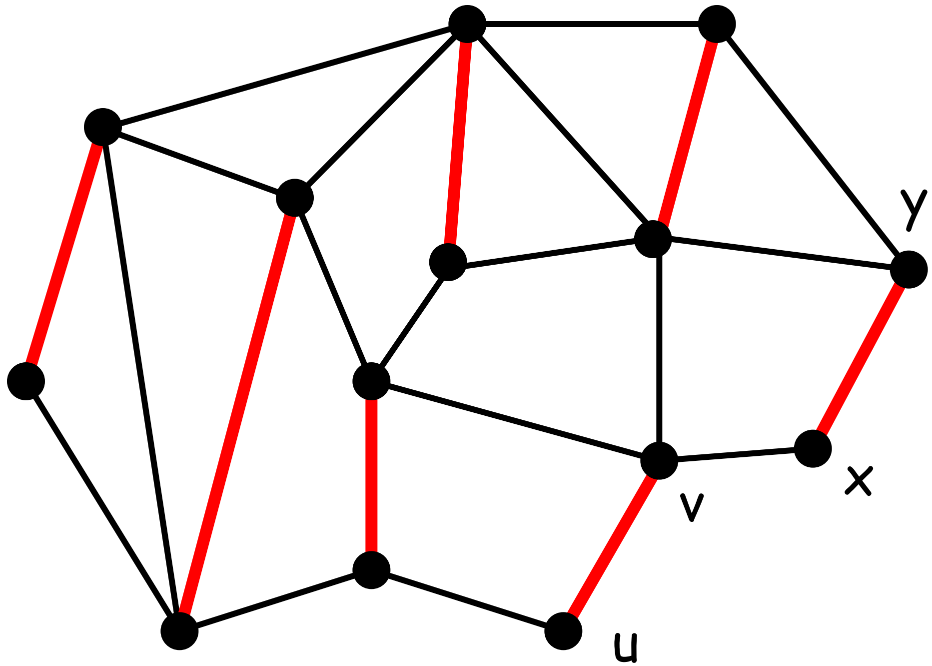

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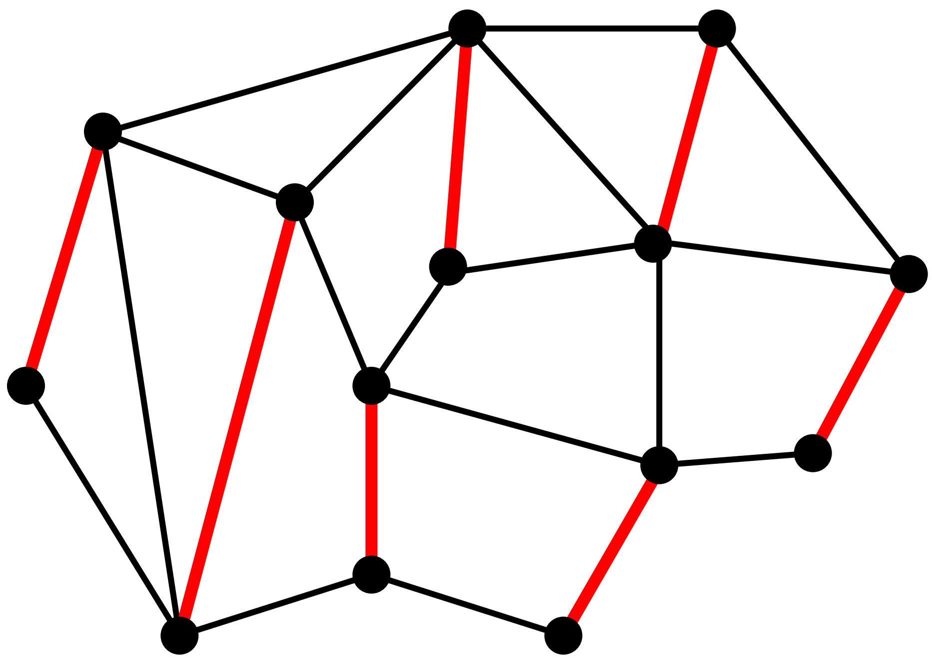

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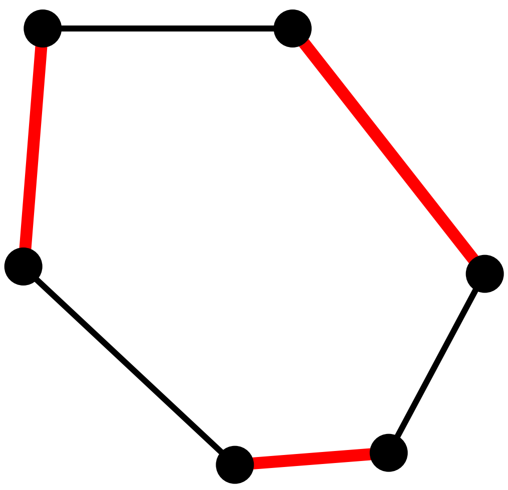

Symmetric but state space disconnected…

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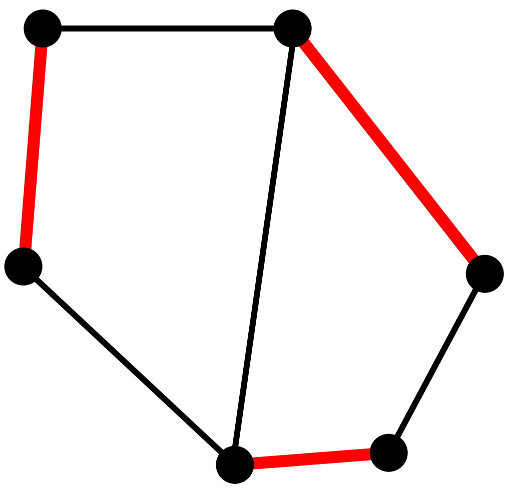
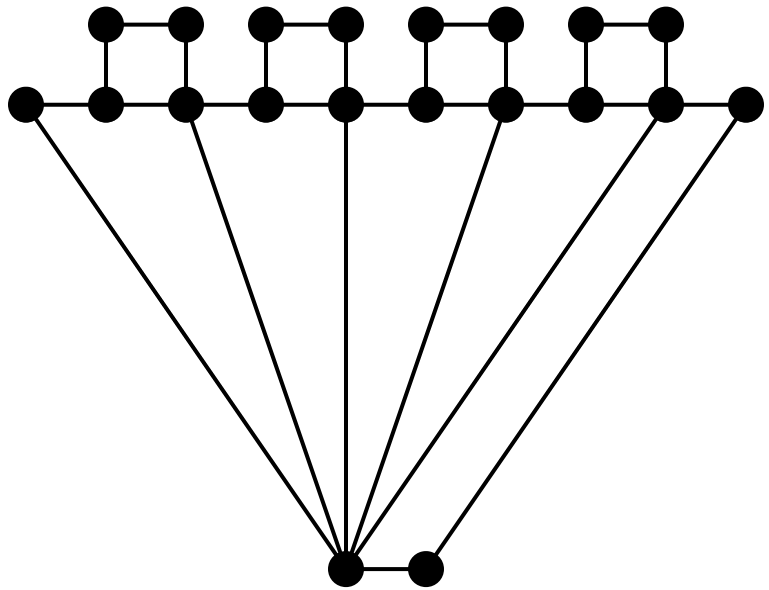


What if have an instance with connected state space:

Does it then mix rapidly?

Another MC for Perfect Matchings

Consider this instance (family of instances):



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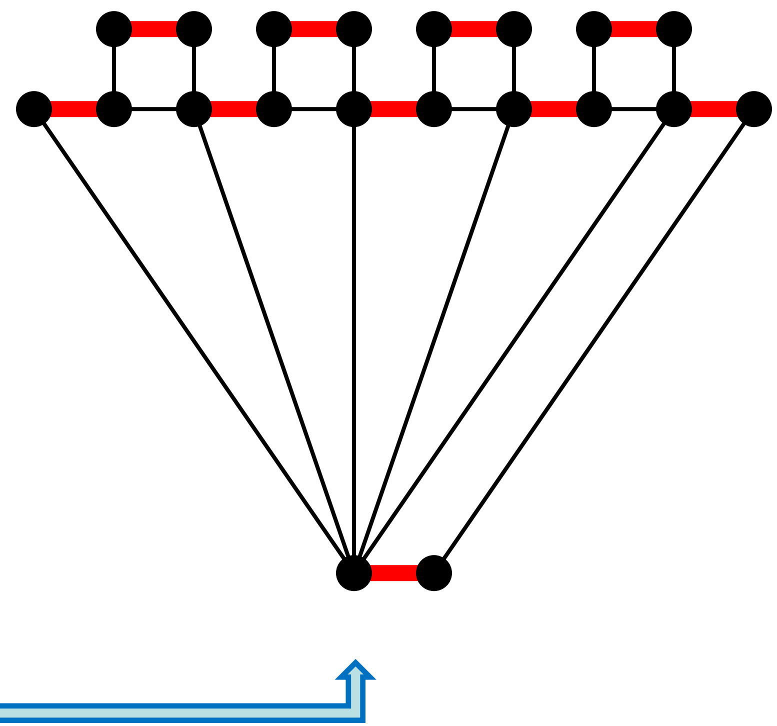

From any matching can get to

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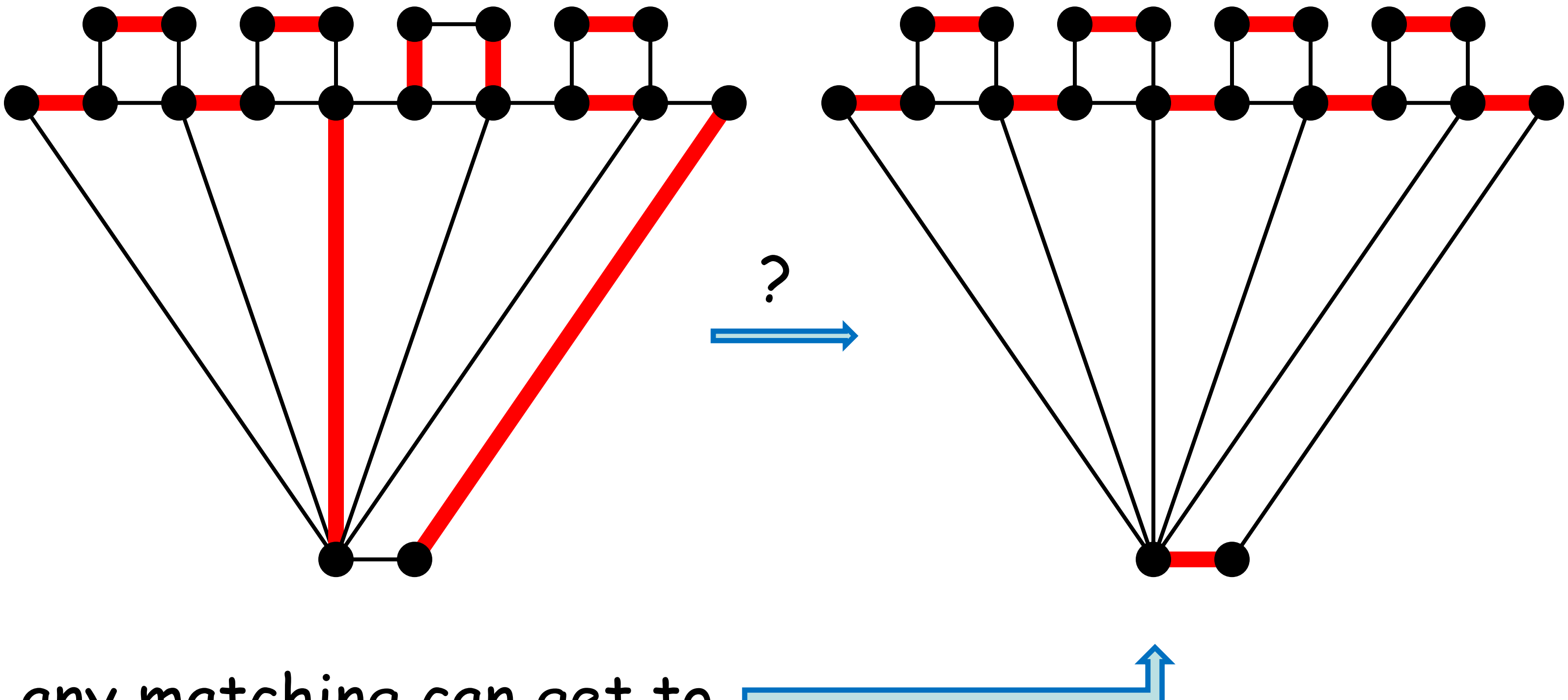

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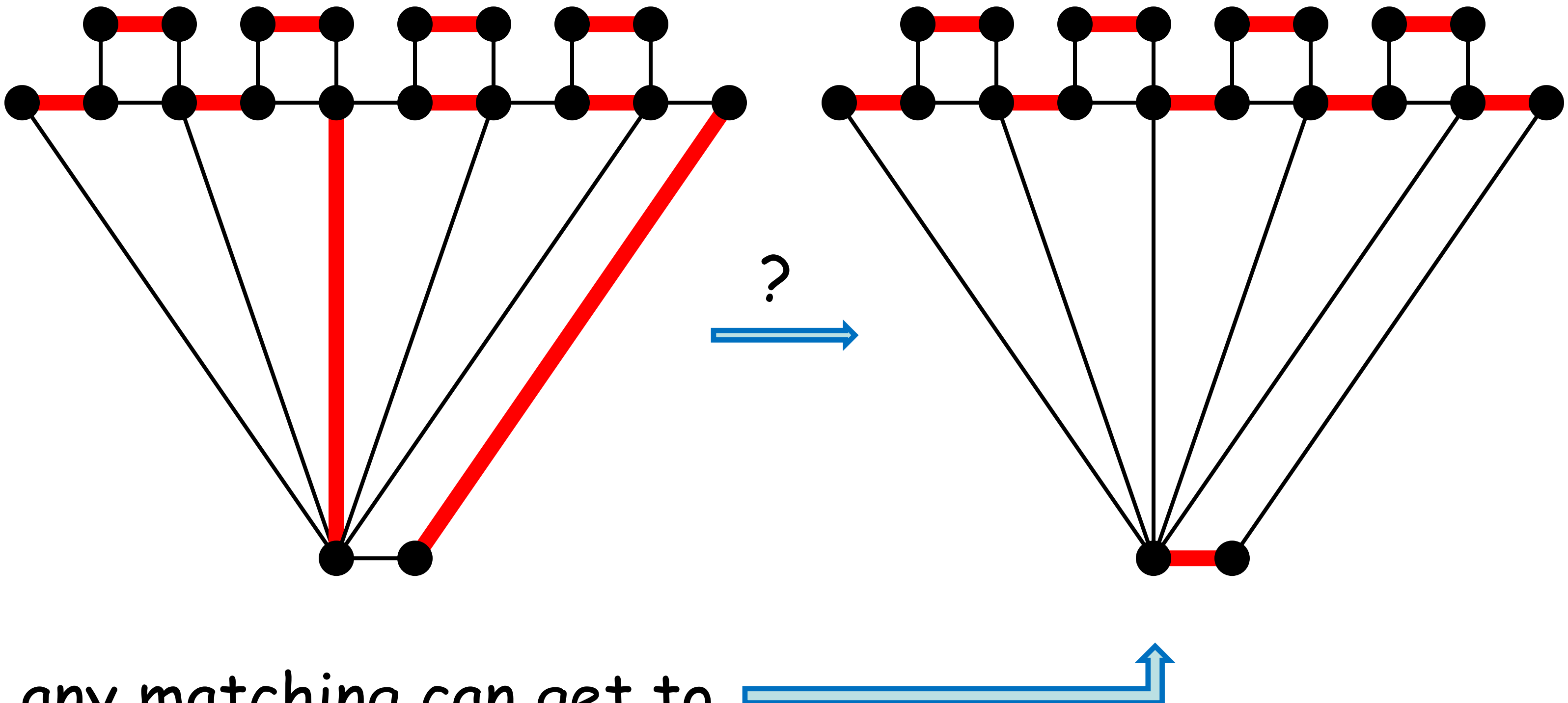

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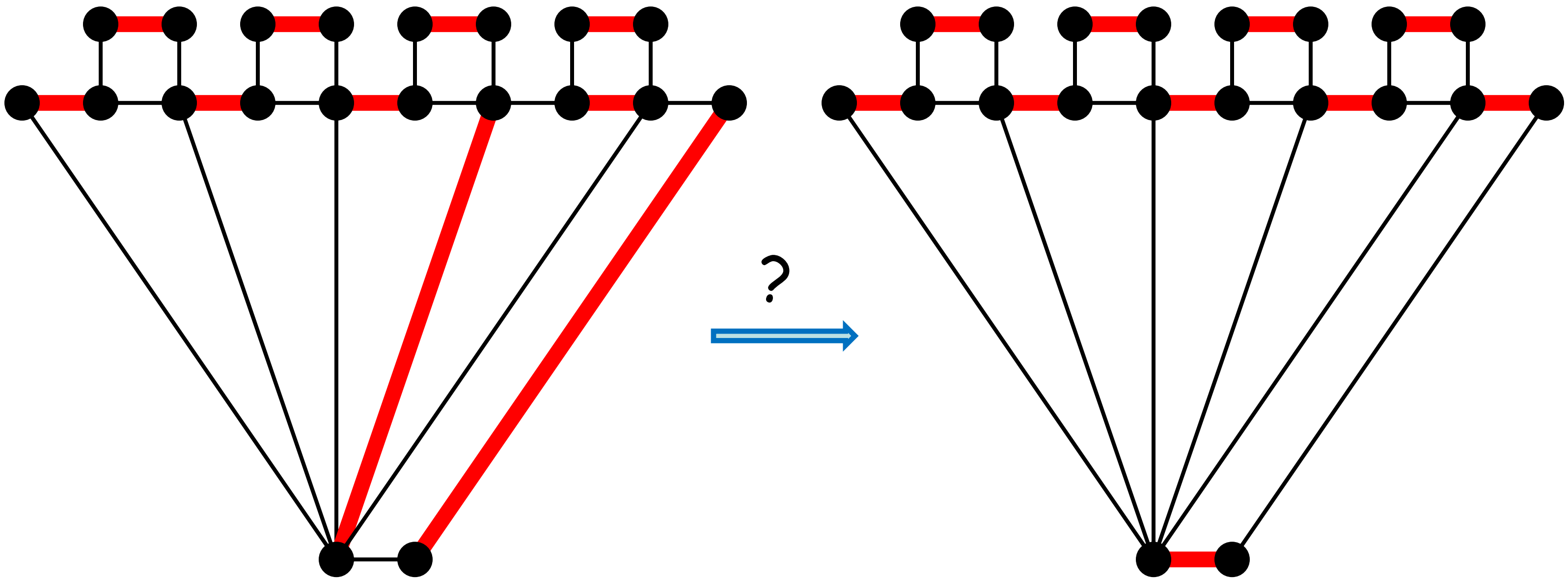

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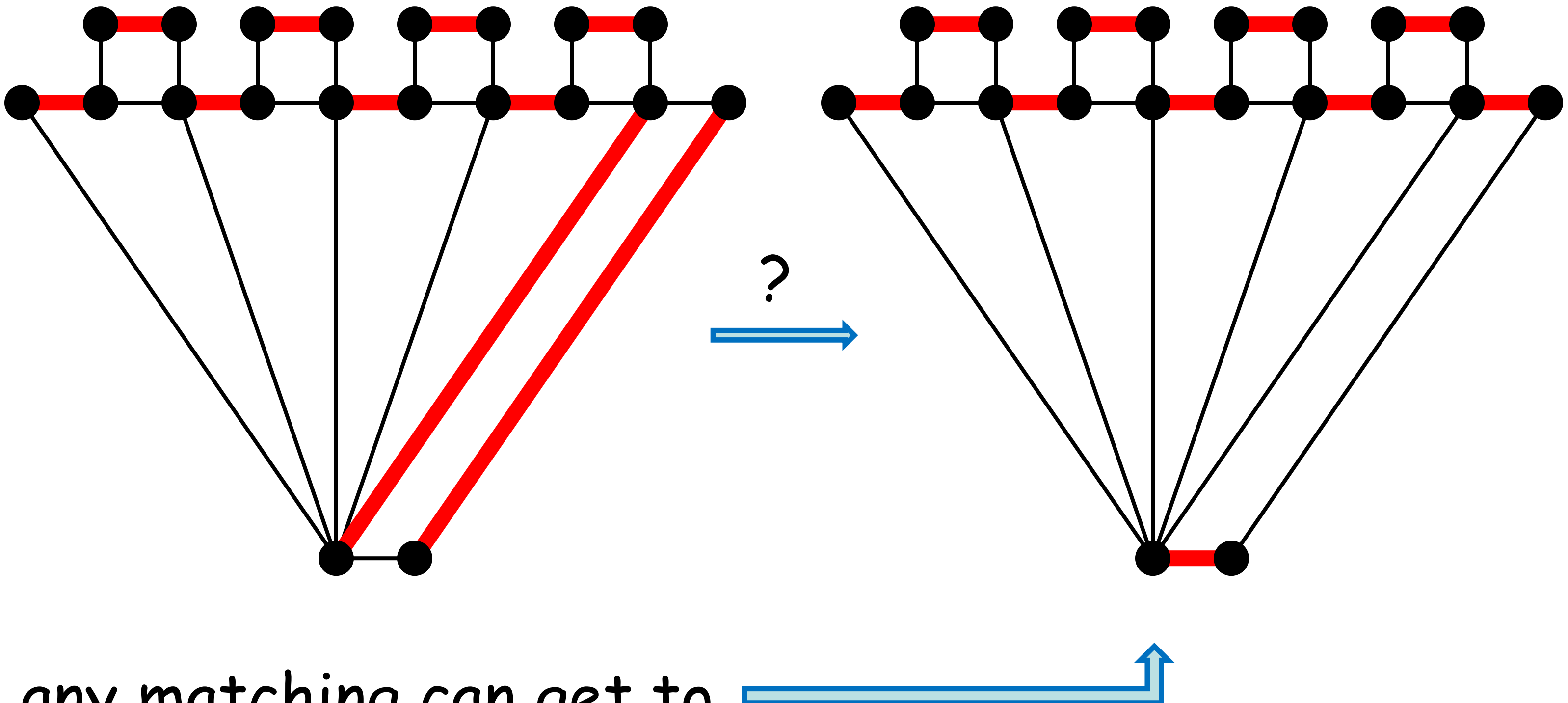

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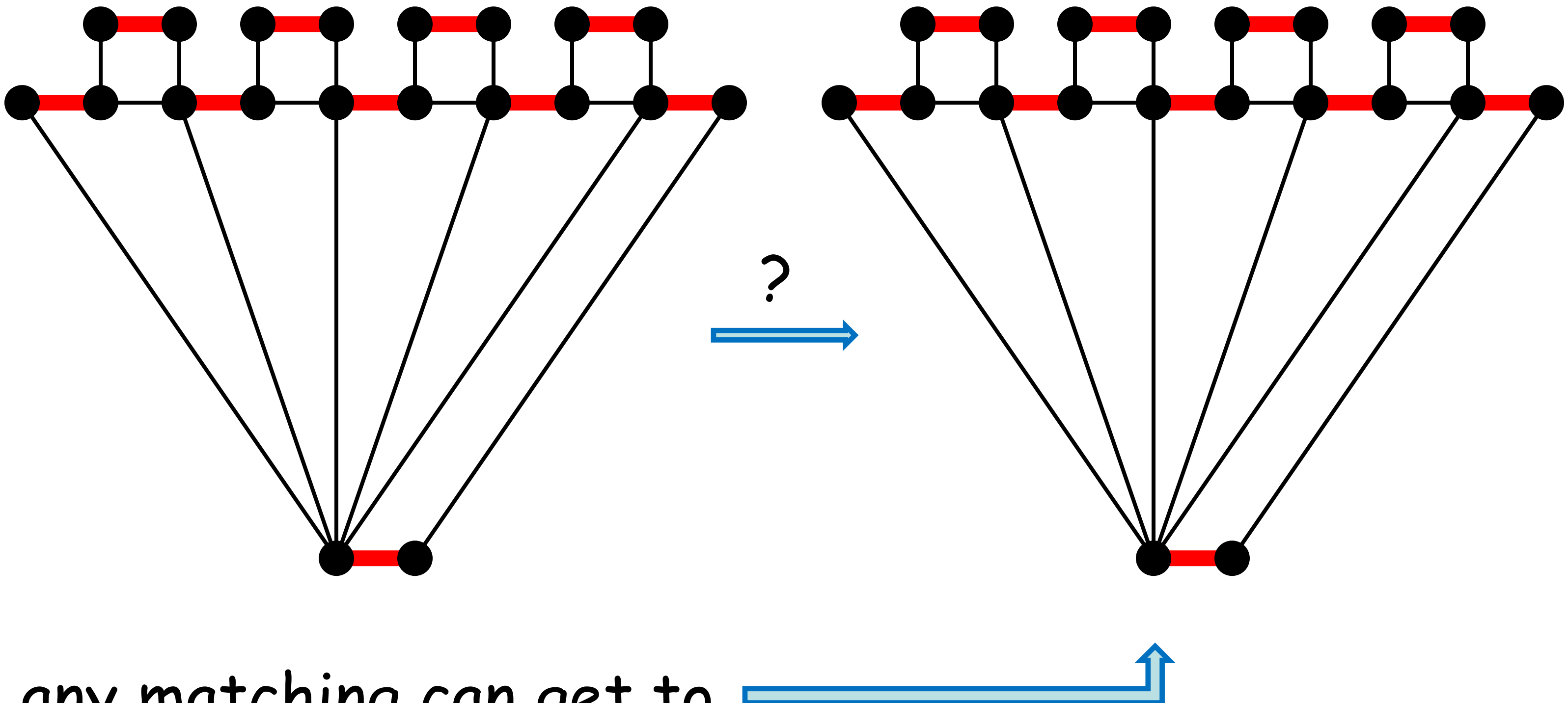

From any matching can get to

-> State space is connected

Another MC for Perfect Matchings

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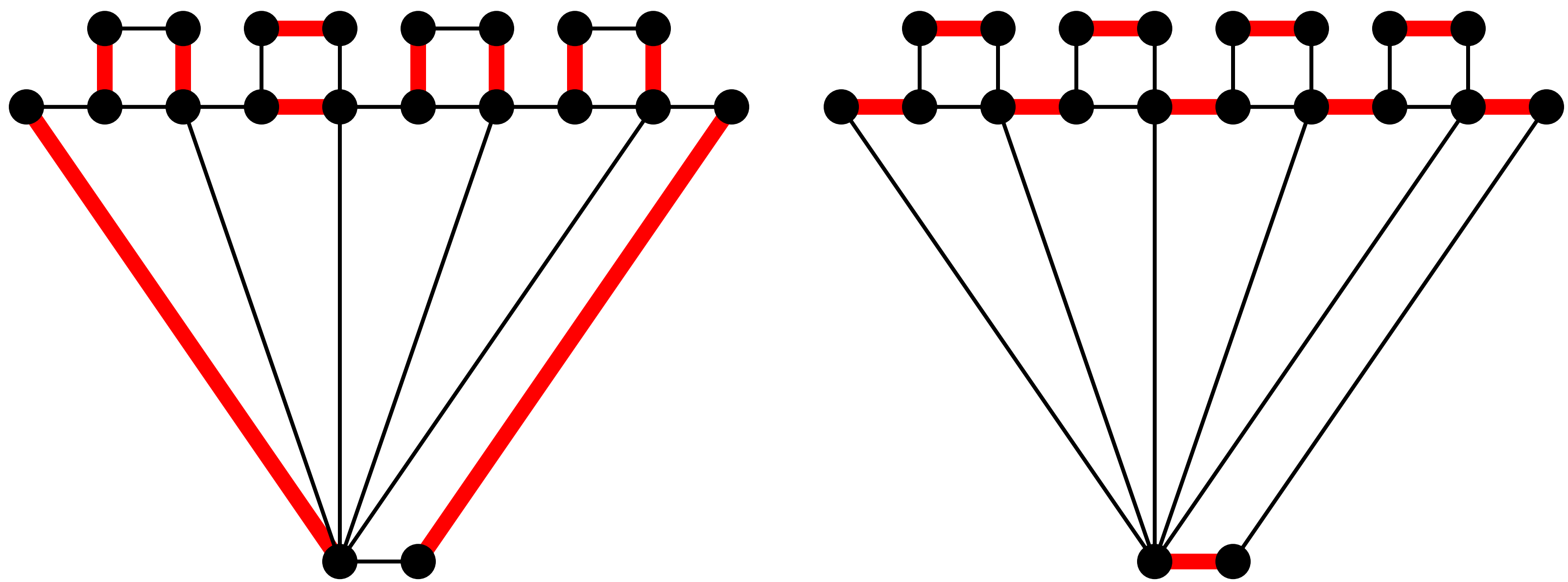

matchings that do not use the bottom edge: $\geq 2^{n/4-1}$

matchings that use the bottom edge: 1

Another MC for Perfect Matchings

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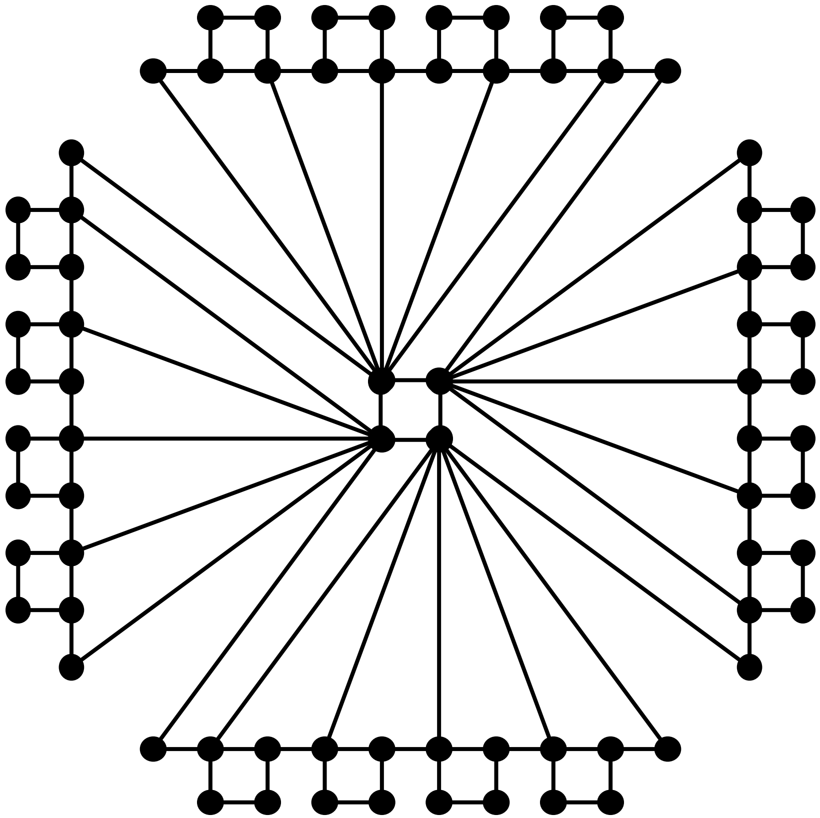

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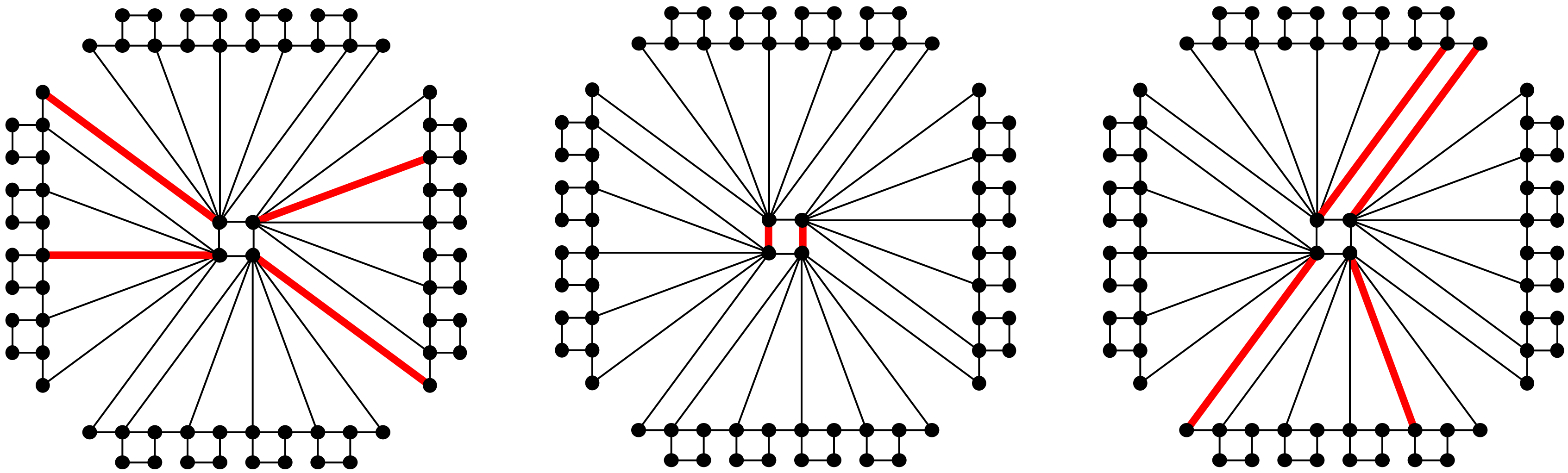

Conductance:

$$\Phi := \min_{S:S\subseteq\Omega,\pi(S)\le 1/2} \frac{\sum_{x\in S, y\notin S} \pi(x)P(x,y)}{\pi(S)} \le \frac{\frac{1}{|\Omega|}\frac{1}{2n(n-1)}}{1/2} \le \frac{1}{2^{n/2-1}n(n-1)}$$

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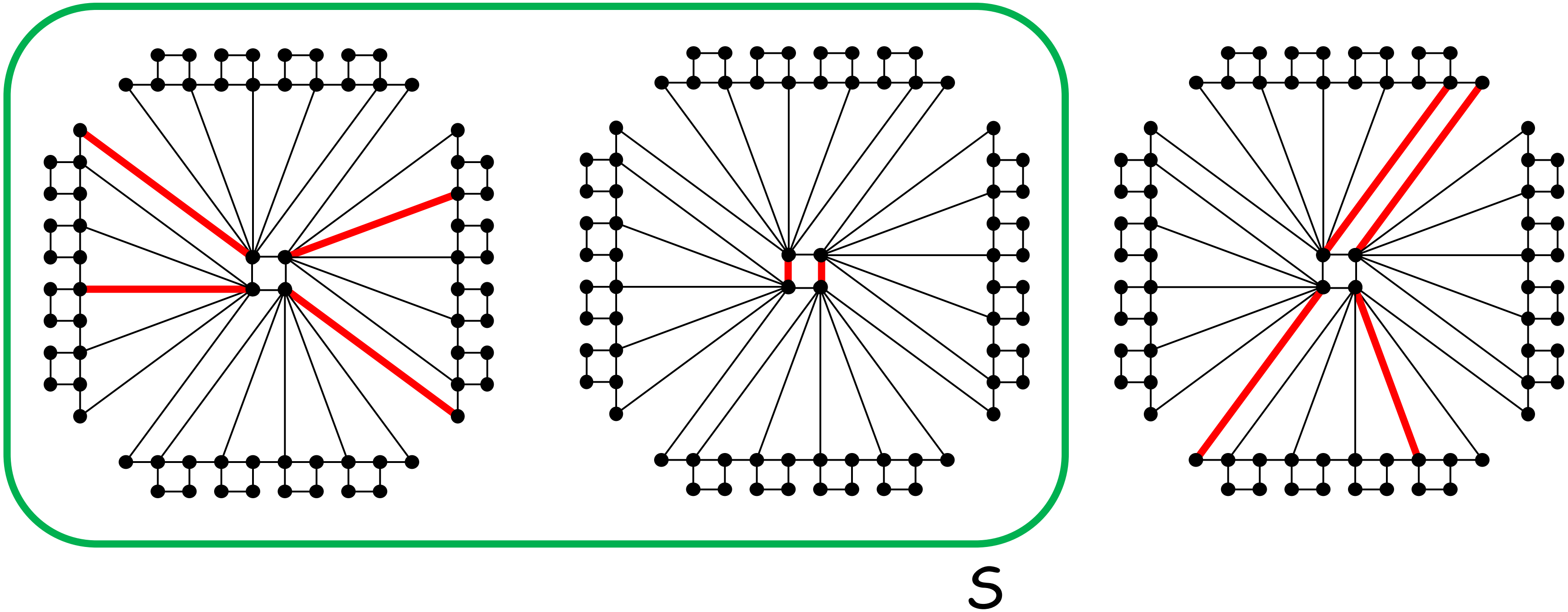

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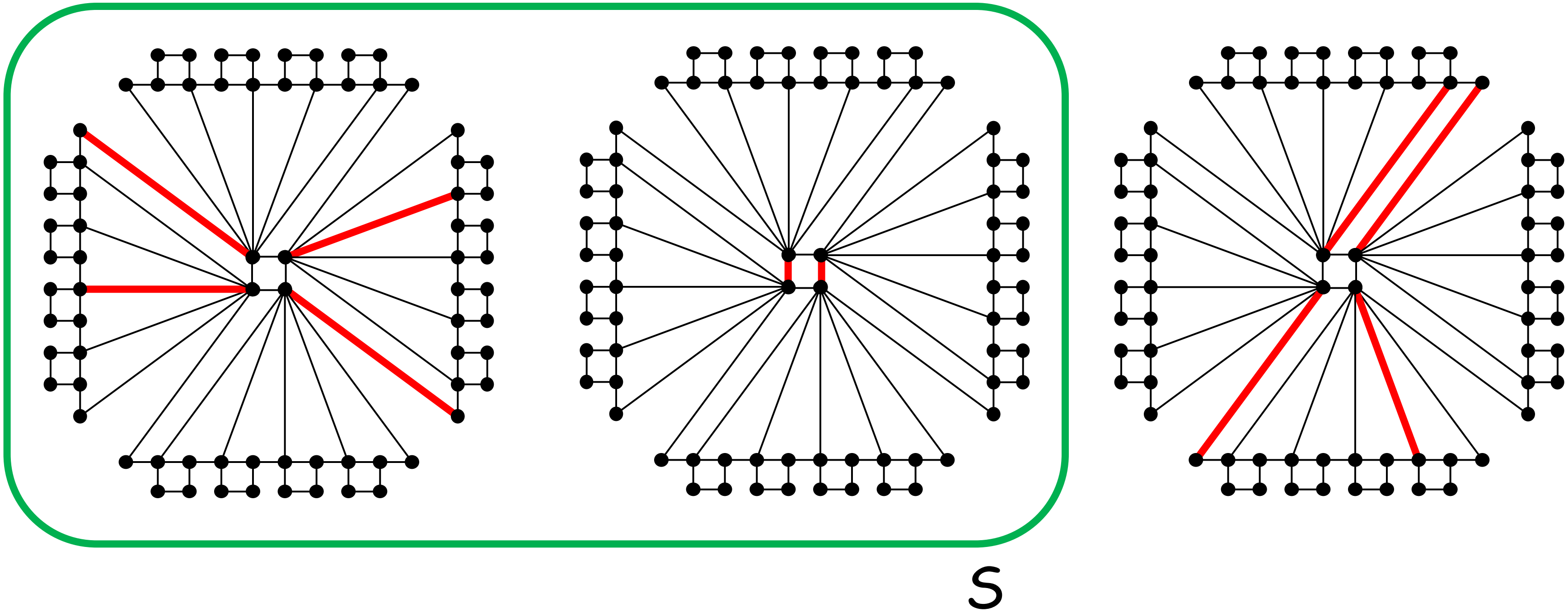

Conductance:

$$t_{mix}(\varepsilon) \geq \frac{1}{2}\left(\frac{1}{2\Phi}-1\right)\log\left(\frac{1}{2\varepsilon}\right) \geq \left(2^{n/2-3}n(n-1)-\frac{1}{2}\right)\log\left(\frac{1}{2\varepsilon}\right)$$

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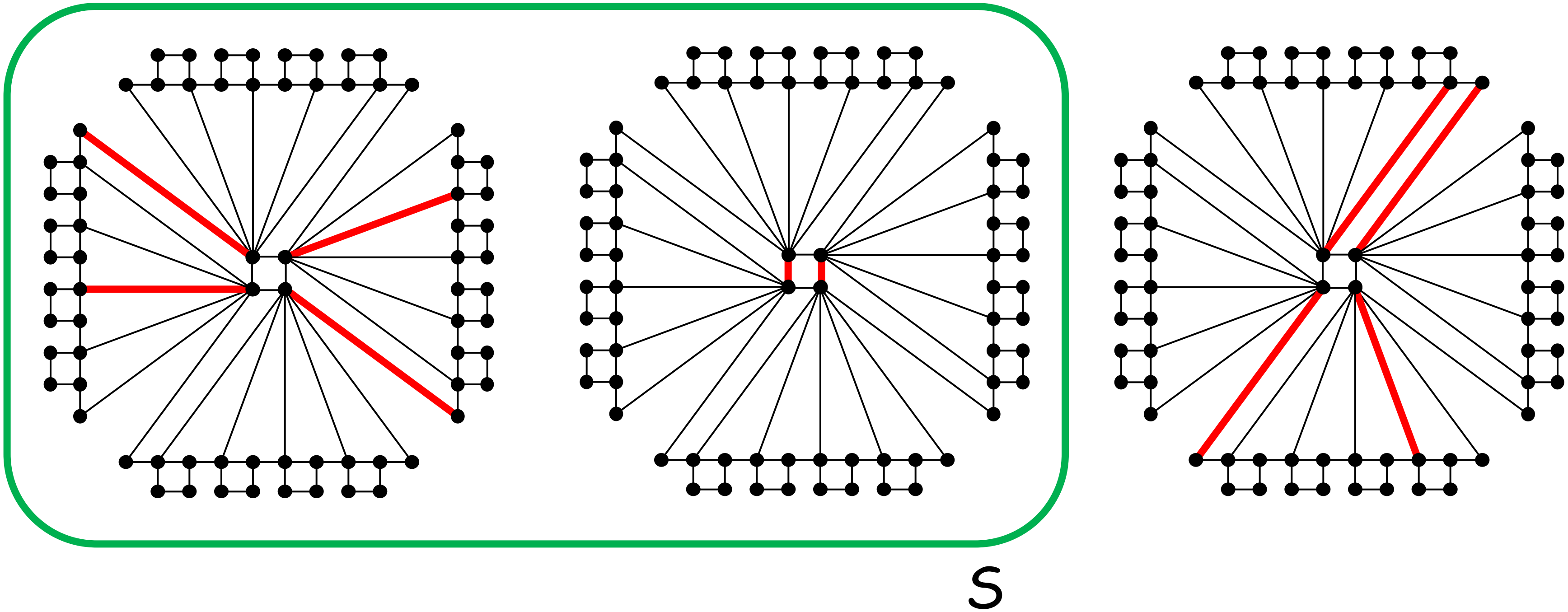

Conductance:

More on this chain:
Dyer-Jerrum-Müller

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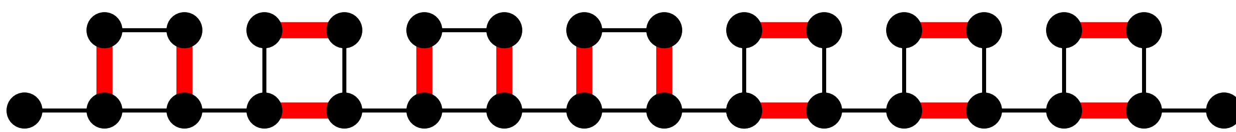
Back to the Sliding Chain: Permanent

<u>What if improve mixing time analysis?</u>

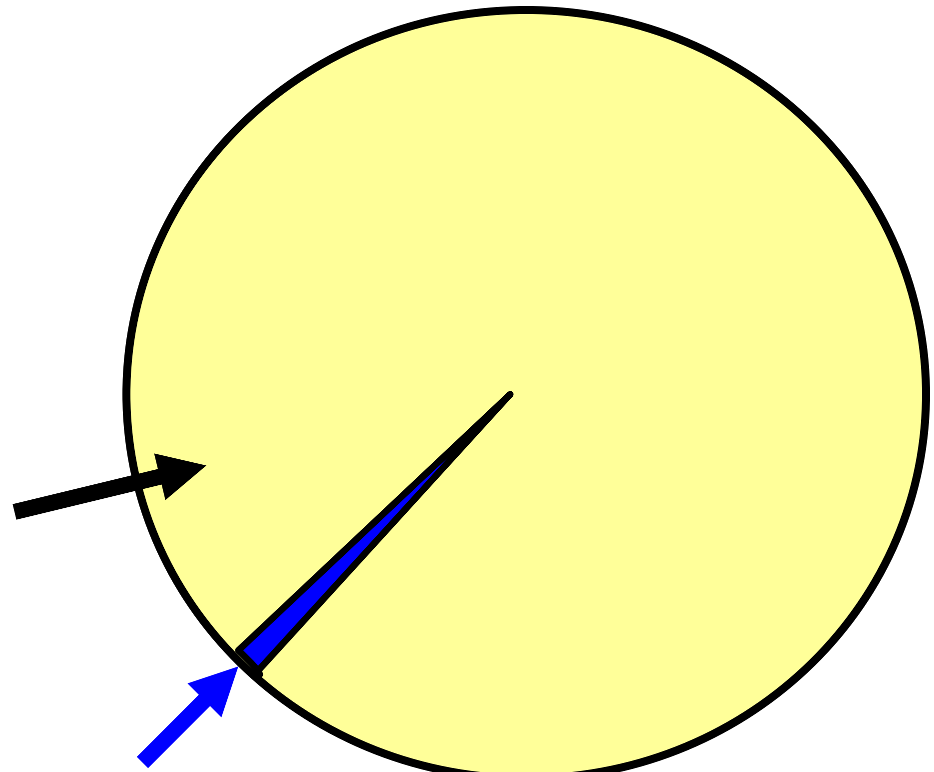
$$\boxed{\mathrm{Per}(A) = \sum_{\pi \in S_n} \prod_{i=1}^{n} a_{i,\pi(i)}}$$

Counts perfect matchings in <u>bipartite</u> graphs



Just one perfect matching…

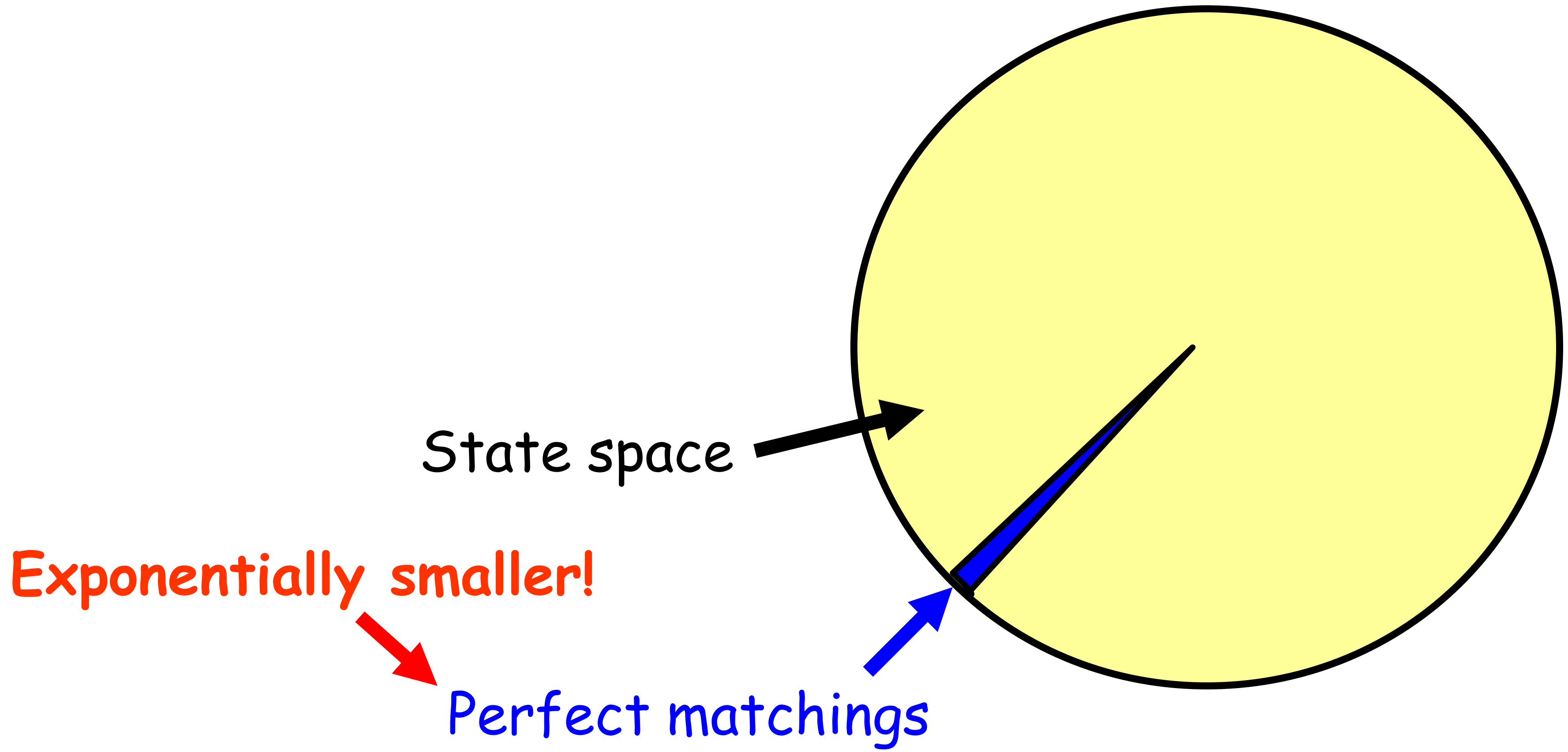
But exponentially many nears!



Back to the Sliding Chain: Permanent

Idea [Jerrum-Sinclair-Vigoda]:

Change the weights of the states
(change stationary distribution).



Back to the Sliding Chain: Permanent

Idea [Jerrum-Sinclair-Vigoda]:

Change the weights of the states
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n^2+1 regions,
very different
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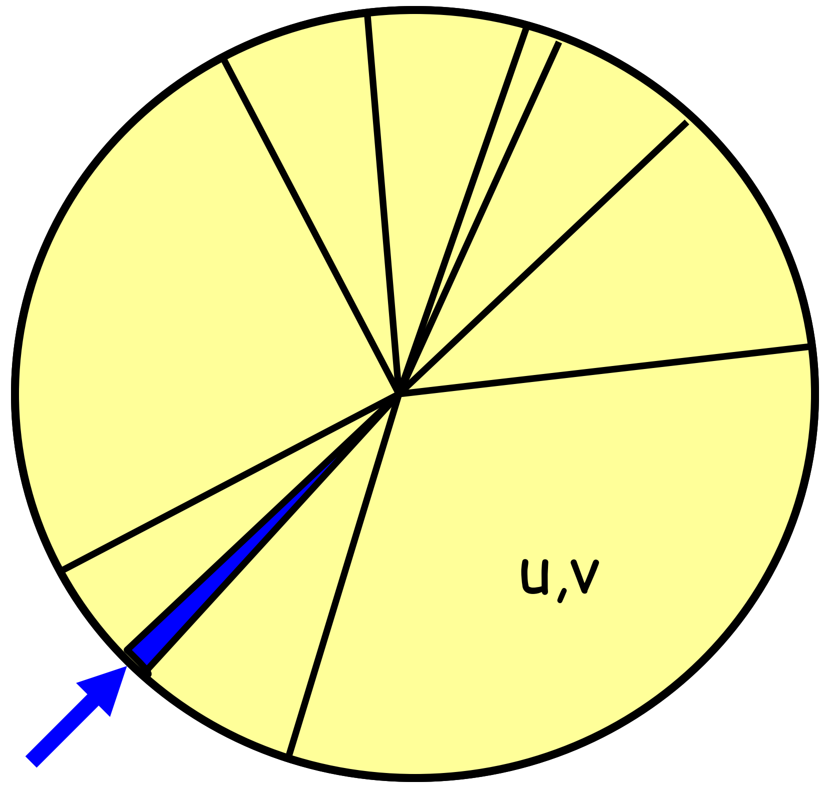

Exponentially smaller!

Perfect matchings

Back to the Sliding Chain: Permanent

<u>Idea</u> [Jerrum-Sinclair-Vigoda]:

Change the weights of the states
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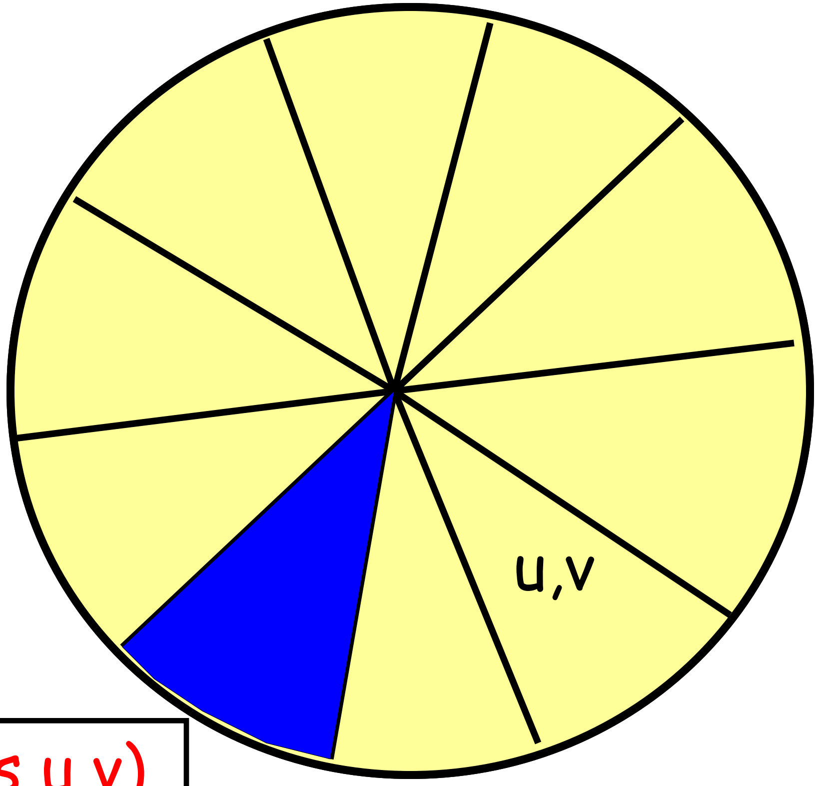
n^2+1 regions,
each about the
same weight



Ideal weights
(for a matching with holes u,v):

(# perfects) / (# nears with holes u,v)

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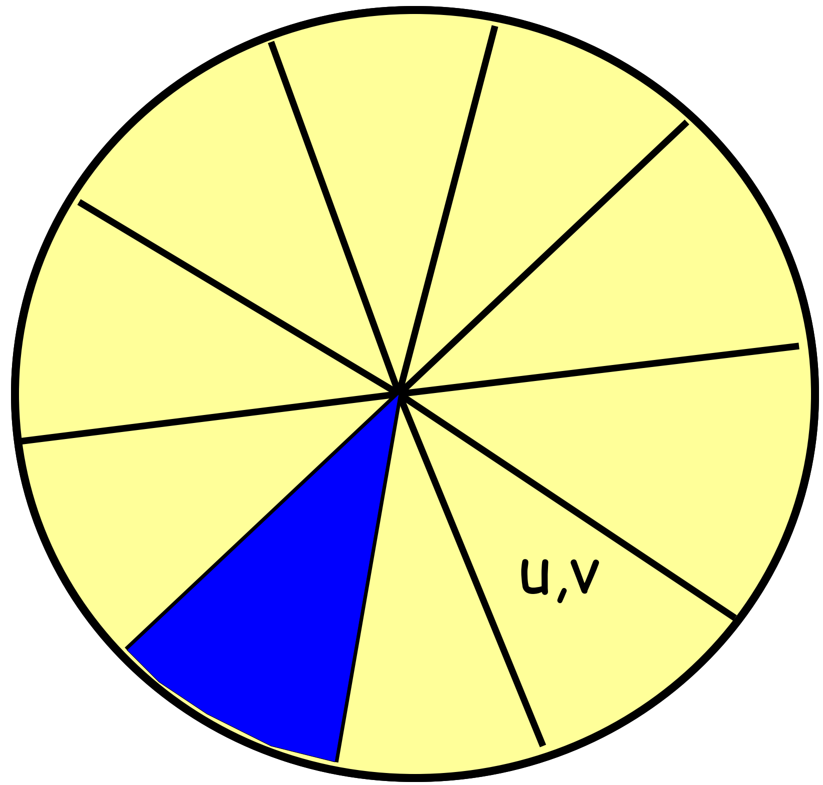
How to compute ???



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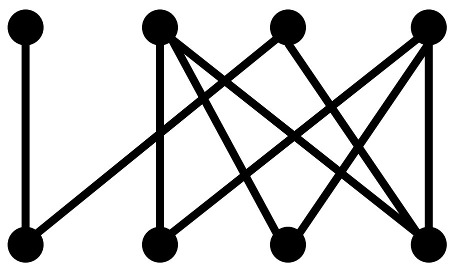
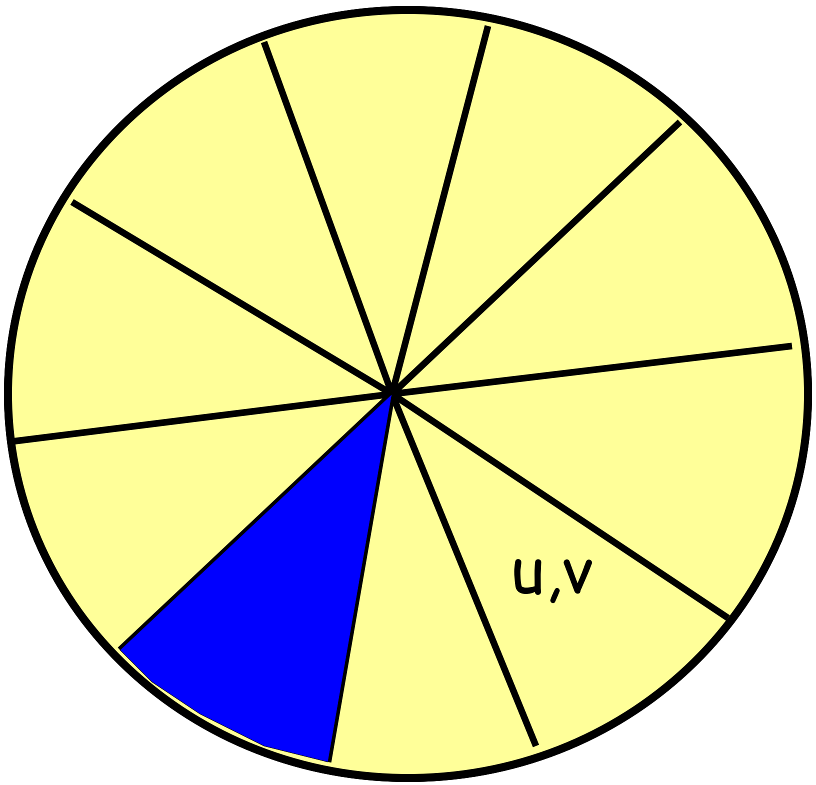
Approximate:
start with an easy graph,
gradually get to the target graph



target



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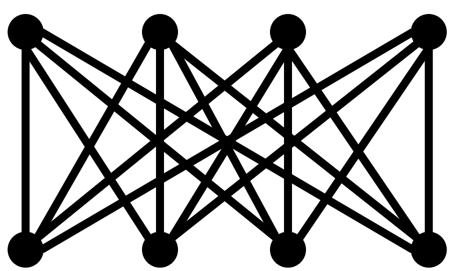

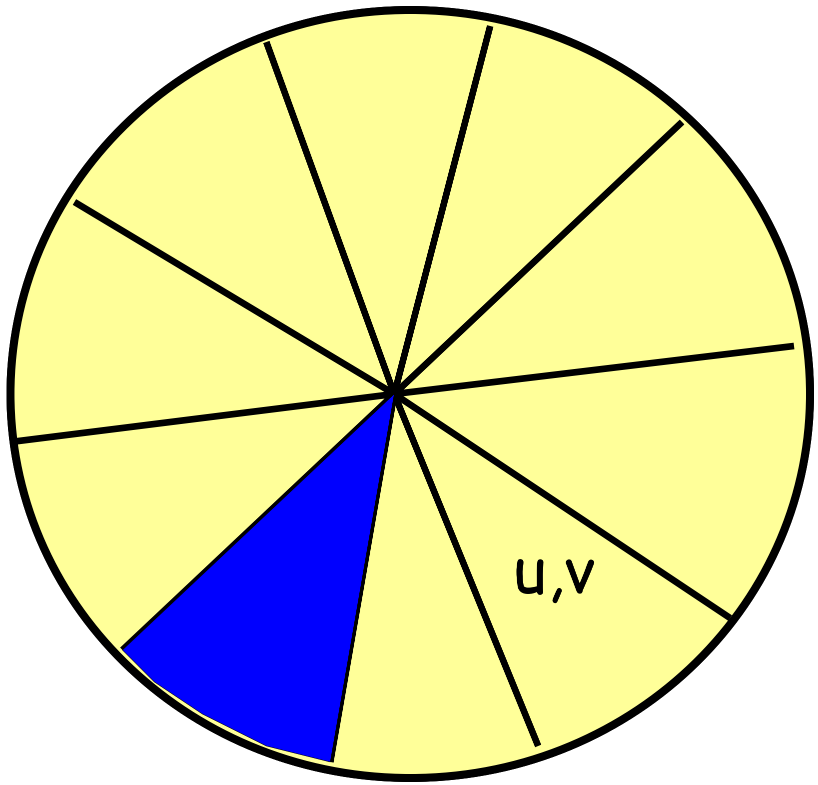

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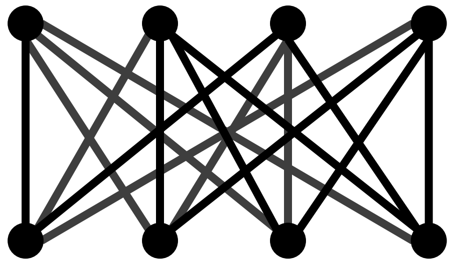

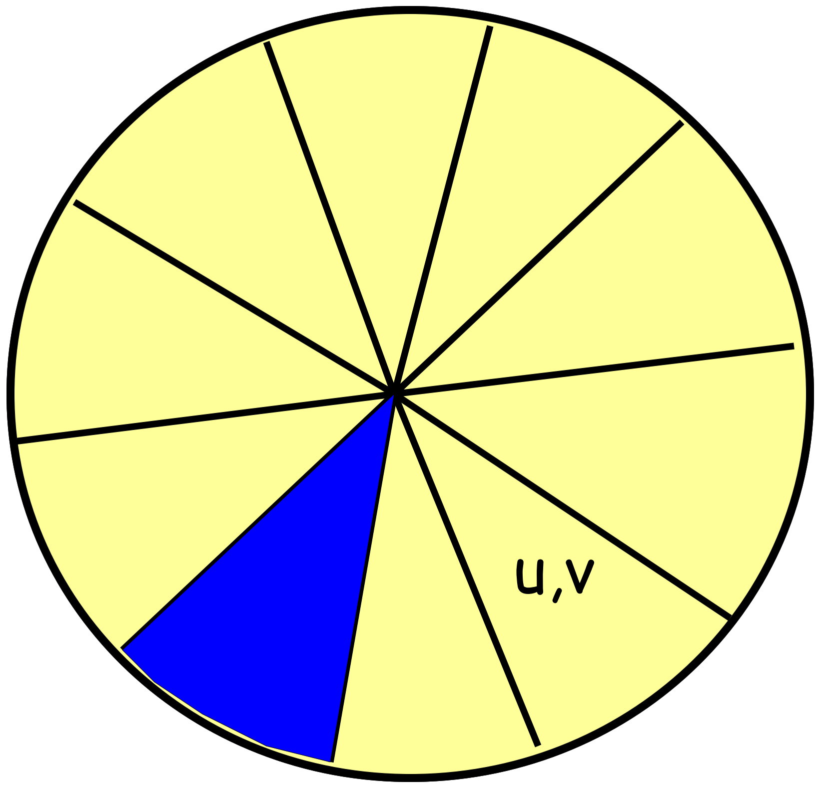

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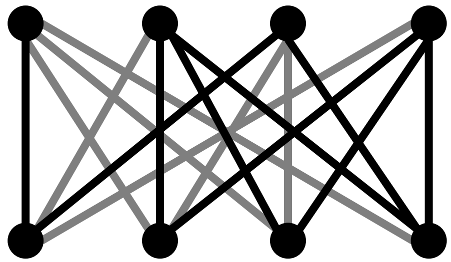
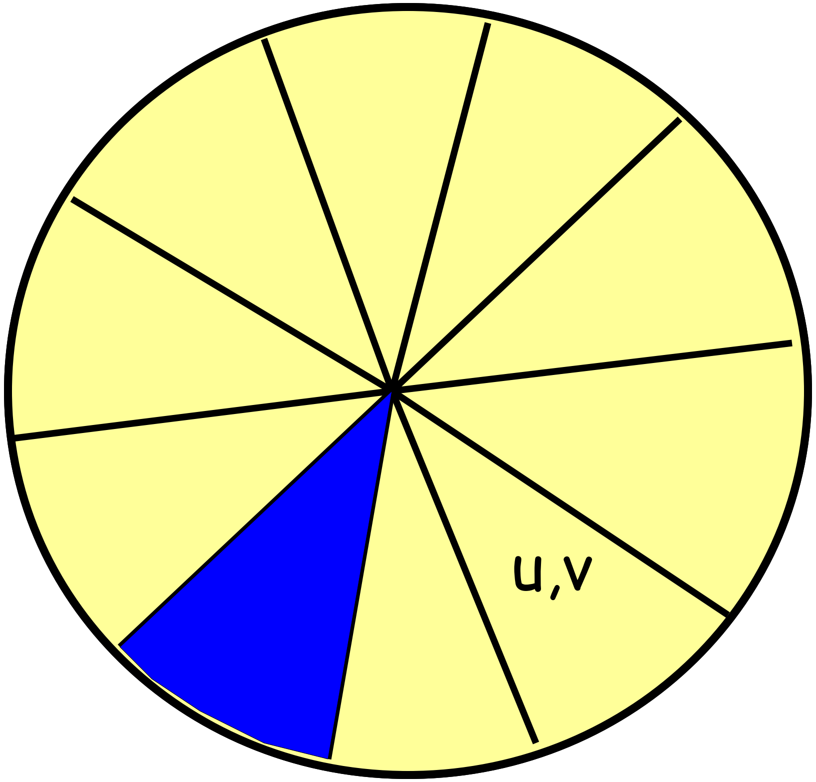




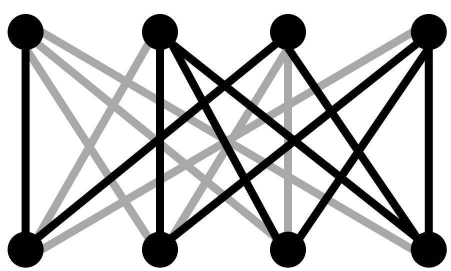
Back to the Sliding Chain: Permanent

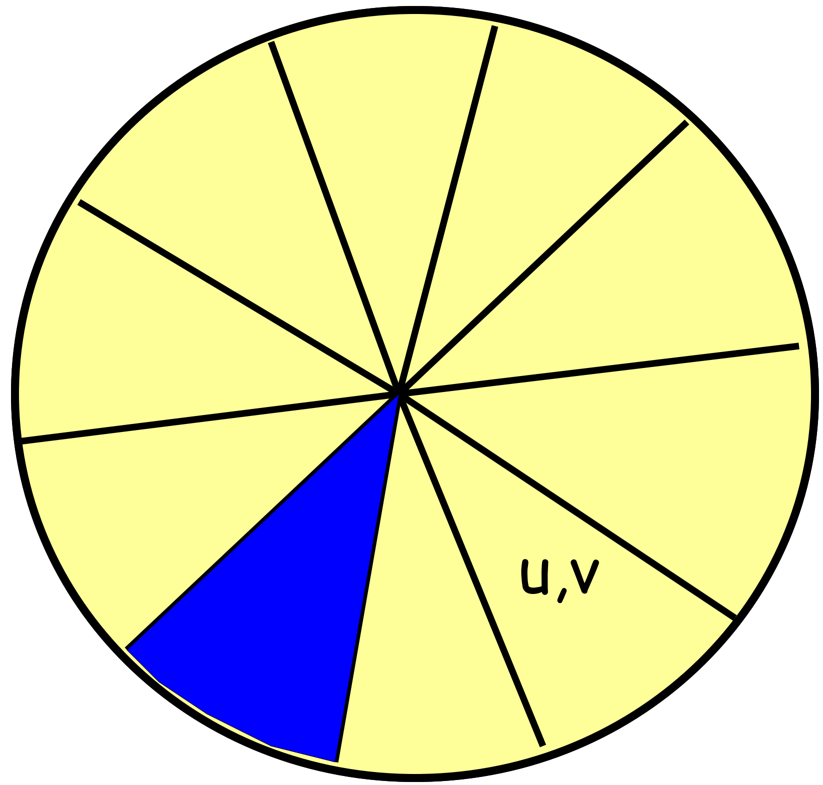
Ideal weights
(for a matching with holes u,v):

(# perfects) / (# nears with holes u,v)

How to compute ???

<u>Approximate:</u>
start with an easy graph,
gradually get to the target graph





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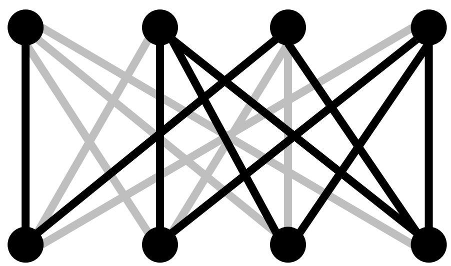

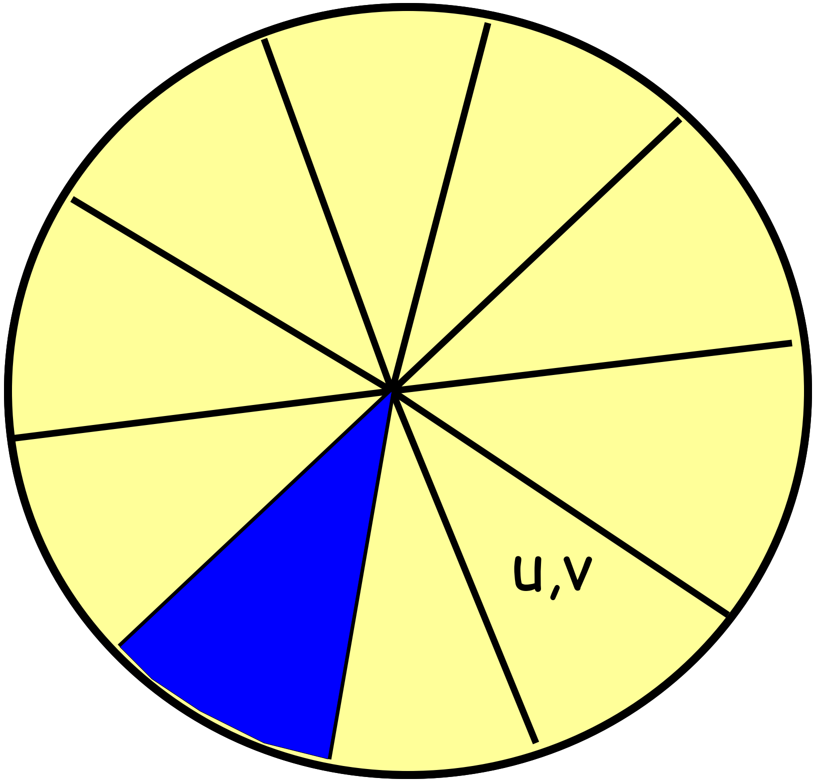

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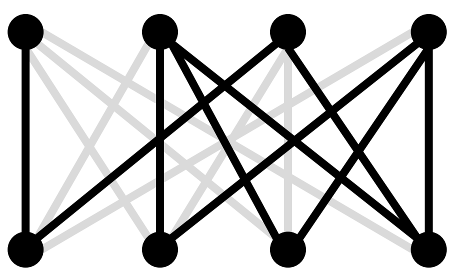
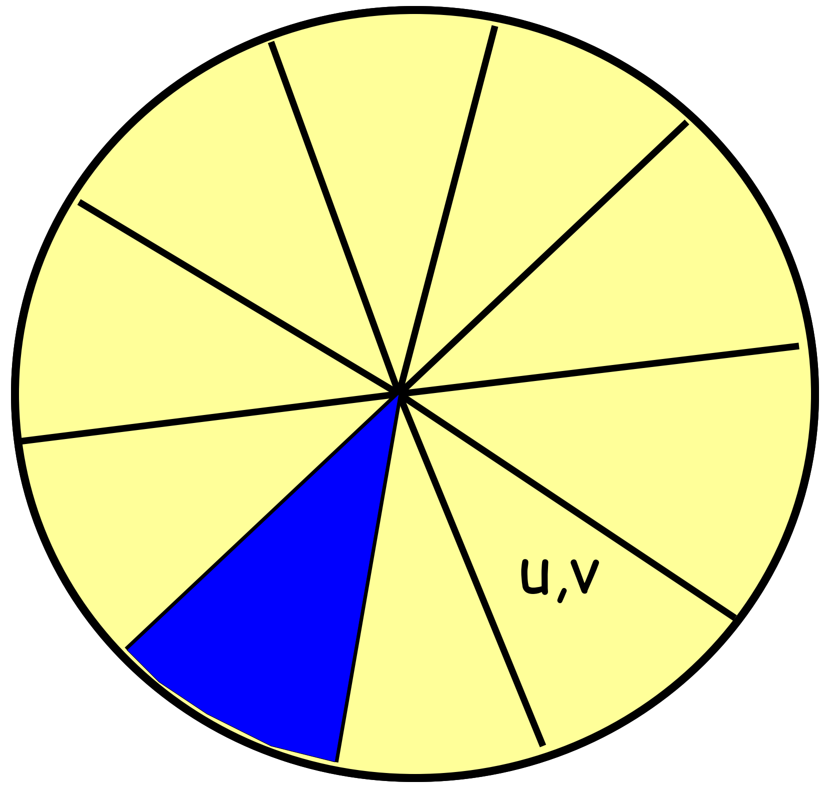




Back to the Sliding Chain: Permanent

Ideal weights

(for a matching with holes u,v):

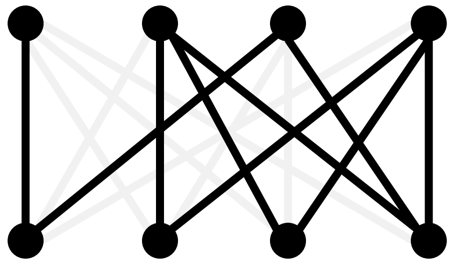
(# perfects) / (# nears with holes u,v)

Edge weights:

- 1 for edge
- λ for non-edge

- Start with λ=1:

#perfect/#nears = n!/(n-1)!



Back to the Sliding Chain: Permanent

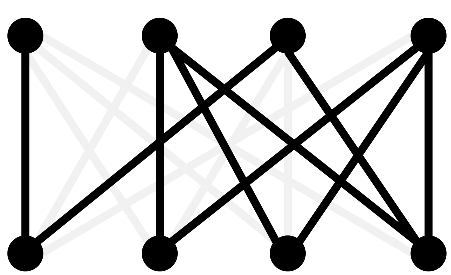
Ideal weights

(for a matching with holes u,v):

λ(perfects) / λ(nears with holes u,v)

Edge weights:

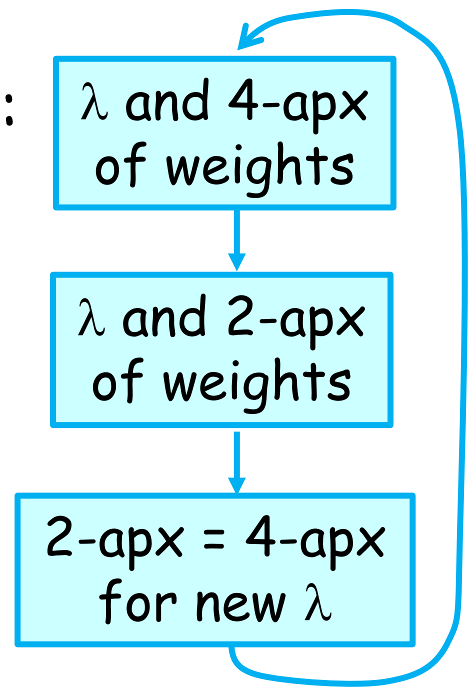
- 1 for edge
- λ for non-edge



- Start with λ=1:

  #perfect/#nears = n!/(n-1)!

- Repeat until λ < 1/n!:



Back to the Sliding Chain: Permanent

Thm [Jerrum-Sinclair-Vigoda]:

FPRAS for the permanent.

OPEN PROBLEM:

counting perfect matchings in non-bipartite graphs

$\boxed{\lambda(\text{perfects}) \,/\, \lambda(\text{nears with holes u,v})}$