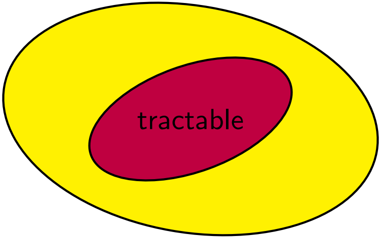
Dichotomy Theorems for Counting Graph Homomorphisms

Xi Chen
Columbia University

Complexity Dichotomy

A theorem that classifies the complexity of a collection of computational problems.

- Tractability criterion on the problem description: Problems that satisfy it are easy to solve, and are intractable otherwise.



Schaefer's Dichotomy Theorem

Theorem (Schaefer 78)

For any finite set S of Boolean relations, the decision problem **CSP**(S) *is either in P or NP-complete.*

Feder-Vardi Conjecture

For any finite set S of relations over any finite domain D, the decision problem **CSP**(S) is either in P or NP-complete.

Theorem (Bulatov 06)

A dichotomy theorem for all **CSP**(S) *of domain size* 3.

Counting Problems

- # VERTEX COVERS
- # d-COLORINGS
- # 3-SAT
- # PERFECT MATCHINGS
- ...
- # induced subgraphs with an odd number of edges

Counting Problems

Three frameworks:

1. Counting Graph Homomorphisms (this talk)
2. Counting Constraint Satisfaction Problems (tomorrow)
3. Holant Problems (talks of Jin-Yi and Heng)

Graph Homomorphisms

Given two undirected graphs G and H, a graph homomorphism from G to H is a map f from $V(G)$ to $V(H)$ such that

$$(u, v) \in E(G) \implies (f(u), f(v)) \in E(H).$$

Theorem (Lovász 67)

Two graphs H and H' are isomorphic iff for all G, the number of homomorphisms from G to H and from G to H' are the same.

Dichotomy for H-Coloring

Theorem (Hell and Nešetřil 90)

For any H, the problem of deciding if there exists a homomorphism from an input graph G to H is either in P or NP-complete.

Counting Graph Homomorphisms

EVAL(H) for a fixed graph H: Given an undirected graph G, compute the number of homomorphisms from G to H.

Theorem (Dyer and Greenhill 00)

For any H, EVAL(H) is either solvable in P-time or #P-complete.

Tractability Criterion: Solvable in P-time if each connected component of H is either an isolated vertex, a complete graph with self-loops, or a complete bipartite graph.

Examples

- #Vertex Covers:

$$V(H) = \{0,1\} \quad \text{and} \quad E(H) = \{(0,1),(1,1)\}.$$

- #d-Colorings:

$$V(H) = \{1,\ldots,d\} \quad \text{and} \quad E(H) = \{(i,j) : i \neq j\}.$$

- # induced subgraphs with an odd number of edges?

Counting Graph Homomorphisms with Weights

$\textsc{Eval}(\mathbf{A})$ for a symmetric matrix $\mathbf{A} = (A_{i,j}) \in \mathbb{C}^{m\times m}$:

- Given $G = (V, E)$ and $\xi : V \to [m]$, call

$$\mathrm{wt}_{\mathbf{A}}(\xi) = \prod_{(u,v)\in E} A_{\xi(u),\xi(v)}$$

the weight of an assignment ξ to the vertices. Compute

$$Z_{\mathbf{A}}(G) = \sum_{\xi:V\to[m]} \mathrm{wt}_{\mathbf{A}}(\xi) = \sum_{\xi:V\to[m]} \prod_{(u,v)\in E} A_{\xi(u),\xi(v)}.$$

$\textsc{Eval}(\mathbf{A}) \equiv \textsc{Eval}(H)$: $\mathbf{A}$ is the adjacency matrix of H.

Partition functions in statistical physics.

More Examples

induced subgraphs with an odd number of edges:

$$\text{EVAL}(\mathbf{A}) \quad \text{with} \quad \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Let $\xi : V \to \{1, 2\}$. Then $\text{wt}_{\mathbf{A}}(\xi) = -1$ if

subgraph induced by $\xi^{-1}(2)$ has an odd number of edges

and $\text{wt}_{\mathbf{A}}(\xi) = 1$ otherwise.

More Examples

Let $\omega = e^{2\pi\sqrt{-1}/3}$ and $\zeta = e^{2\pi\sqrt{-1}/5}$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 & \zeta^3 & \zeta^4 \\ 1 & \zeta^2 & \zeta^4 & \zeta & \zeta^3 \\ 1 & \zeta^3 & \zeta & \zeta^4 & \zeta^2 \\ 1 & \zeta^4 & \zeta^3 & \zeta^2 & \zeta^1 \end{pmatrix}$$

All these matrices are tractable!

Dichotomy for Nonnegative Matrices

Theorem (Bulatov and Grohe 05)

Given any symmetric nonnegative matrix $\mathbf{A} \in \mathbb{R}_{\mathbb{A}}^{m\times m}$, $\textsc{Eval}(\mathbf{A})$ *is either solvable in P-time or #P-hard.*

Tractability Criterion: in P-time if $\mathbf{A}$ is a block diagonal matrix and every block is either rank-1 or has the form

$$\begin{pmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{0} \end{pmatrix}, \qquad \text{where } \mathbf{B} \text{ is rank-1.}$$

Many applications. [Grohe and Thurley 11] for a new exposition.

Real or Complex Domains

Cancellations (e.g., $\{\pm 1\}$ or even roots of unity) may sometimes lead to efficient algorithms and more tractable cases (Permanent vs Determinant and Holographic algorithms [Valiant 04]).

Dichotomy Theorems Arise

Theorem (Goldberg, Grohe, Jerrum and Thurley 09)

Given any symmetric matrix $\mathbf{A} \in \mathbb{R}_{\mathbb{A}}{}^{m \times m}$, $\textsc{Eval}(\mathbf{A})$ *is either solvable in P-time or #P-hard.*

Theorem (Cai, C and Lu 11)

Given any symmetric matrix $\mathbf{A} \in \mathbb{C}_{\mathbb{A}}{}^{m \times m}$, $\textsc{Eval}(\mathbf{A})$ *is either solvable in P-time or #P-hard.*

Tractability Criterion

Roughly speaking, tractable matrices **A** correspond to rank one modifications of tensor products of Fourier matrices.

Plan of the Talk

1. Algorithms for Counting Graph Homomorphisms
2. The Group Condition
    - Graph gadget
    - Interpolation

Rank-1 Matrices

When **A** is rank-1, there exists a **b** such that $A_{i,j} = b_i \cdot b_j$.

$$
\begin{aligned}
Z_{\mathbf{A}}(G) &= \sum_{\xi : V \to [m]} \prod_{(u,v) \in E} A_{\xi(u),\xi(v)} \\
&= \sum_{x_1,\ldots,x_n \in [m]} \prod_{(u,v) \in E} b_{x_u} \cdot b_{x_v} \\
&= \sum_{x_1,\ldots,x_n \in [m]} \left(\prod_{i \in [n]} (b_{x_i})^{\deg(i)} \right) \\
&= \prod_{i \in [n]} \left(\sum_{x_i \in [m]} (b_{x_i})^{\deg(i)} \right).
\end{aligned}
$$

Similar for

$$\mathbf{A} = \begin{pmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{0} \end{pmatrix}$$

Direct Sum of Tractable Matrices

Let $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ be $m_1 \times m_1$ and $m_2 \times m_2$, and

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}^{(1)} & \\ & \mathbf{A}^{(2)} \end{pmatrix}$$

Assume WOLG that G is connected. Then

$$\begin{aligned} Z_{\mathbf{A}}(G) &= \sum_{\xi:V\to[m_1+m_2]} \prod_{(u,v)\in E} A_{\xi(u),\xi(v)} \\ &= \sum_{\xi:V\to[m_1]} \prod_{(u,v)\in E} A^{(1)}_{\xi(u),\xi(v)} + \sum_{\xi:V\to[m_2]} \prod_{(u,v)\in E} A^{(2)}_{\xi(u),\xi(v)} \end{aligned}$$

Done with all tractable cases for nonnegative $\mathbf{A}$. Hooray!

$$\text{What about } \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} ?$$

Observed in [Goldberg, Grohe, Jerrum and Thurley 09]:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \implies A_{x,y} = (-1)^{xy}$$

when the rows and columns are indexed by $x, y \in \mathbb{Z}_2$. Thus,

$$Z_{\mathbf{A}}(G) = \sum_{x_1,\ldots,x_n \in \mathbb{Z}_2} \prod_{(u,v)\in E} (-1)^{x_u x_v} = \sum_{x_1,\ldots,x_n \in \mathbb{Z}_2} (-1)^{\sum_{(u,v)\in E} x_u x_v}$$

for some quadratic polynomial in the exponent.

This can be computed in polynomial time!

Theorem (e.g., see [Lidl and Niederreiter 97])

Given a quadratic polynomial $f(x_1, \ldots, x_n)$ *over* $\mathbb{Z}_2$,

$$\sum_{x_1,\ldots,x_n \in \mathbb{Z}_2} (-1)^{f(x_1,\ldots,x_n)}$$

can be computed in polynomial time.

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

are in P-time: $(-1)^{x_1 y_2 + x_2 y_1}$ by indexing the rows by $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Theorem (Cai, C, Lipton and Lu 10)

Given $q \geq 1$ and a quadratic polynomial $f(x_1, \ldots, x_n)$ over $\mathbb{Z}_q$,

$$\sum_{x_1,\ldots,x_n \in \mathbb{Z}_q} \left(e^{2\pi\sqrt{-1}/q} \right)^{f(x_1,\ldots,x_n)}$$

can be computed in P-time in $\log q$ and n (without knowing the prime factorization of q).

All $m \times m$ Fourier matrices

$$A_{x,y} = e^{\frac{2\pi\sqrt{-1}}{m} \cdot xy}, \qquad \text{for } x, y \in \mathbb{Z}_m$$

such as

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 & \zeta^3 & \zeta^4 \\ 1 & \zeta^2 & \zeta^4 & \zeta & \zeta^3 \\ 1 & \zeta^3 & \zeta & \zeta^4 & \zeta^2 \\ 1 & \zeta^4 & \zeta^3 & \zeta^2 & \zeta^1 \end{pmatrix}$$

are solvable in P-time as well as their tensor products. Most of the tractable cases in real and complex graph homomorphisms.

Tensor Product of Tractable Matrices

Let $\mathbf{A} = \mathbf{A}^{(1)} \otimes \mathbf{A}^{(2)}$, where $\mathbf{A}^{(1)}$ is $m_1 \times m_1$ and $\mathbf{A}^{(2)}$ is $m_2 \times m_2$.

$$
\begin{aligned}
Z_{\mathbf{A}}(G) &= \sum_{\xi: V \to [m_1]\times[m_2]} \prod_{(u,v)\in E} A_{\xi(u),\xi(v)} \\
&= \sum_{\xi_1: V \to [m_1]} \sum_{\xi_2: V \to [m_2]} \prod_{(u,v)\in E} A^{(1)}_{\xi_1(u),\xi_1(v)} \cdot A^{(2)}_{\xi_2(u),\xi_2(v)} \\
&= \left(\sum_{\xi_1} \prod_{(u,v)} A^{(1)}_{\xi_1(u),\xi_1(v)} \right) \left(\sum_{\xi_2} \prod_{(u,v)} A^{(2)}_{\xi_2(u),\xi_2(v)} \right)
\end{aligned}
$$

Proof of the $q = 2$ Case

Theorem (e.g., see [Lidl and Niederreiter 97])

Given a quadratic polynomial $f(x_1, \ldots, x_n)$ over $\mathbb{Z}_2$,

$$\sum_{x_1,\ldots,x_n \in \mathbb{Z}_2} (-1)^{f(x_1,\ldots,x_n)}$$

can be computed in polynomial time.

Proof.

Two cases: f has an x_i^2 or every quadratic term is $x_i x_j$, $i \neq j$.

Case 1: $f = x_1 \cdot \ell(x_2, \ldots, x_n) + f'(x_2, \ldots, x_n)$. Then

$$\sum_{x_1,\ldots,x_n} (-1)^f = \sum_{x_2,\ldots,x_n} (-1)^{f'} \cdot \sum_{x_1} (-1)^{x_1 \cdot \ell}$$

Since

$$\sum_{x_1} (-1)^{x_1 \cdot \ell} = \begin{cases} 2 & \text{if } \ell = 0 \\ 0 & \text{if } \ell = 1 \end{cases}$$

It suffices to compute

$$2 \cdot \sum_{x_2,\ldots,x_n : \ell = 0} (-1)^{f'},$$

which reduces the number of variables by two.

Case 2: $f = x_1^2 + x_1 \cdot \ell(x_2, \ldots, x_n) + f'(x_2, \ldots, x_n)$. Then

$$\sum_{x_1,\ldots,x_n} (-1)^f = \sum_{x_2,\ldots,x_n} (-1)^{f'} \cdot \sum_{x_1} (-1)^{x_1^2 + x_1 \cdot \ell}$$

Since

$$\sum_{x_1} (-1)^{x_1^2 + x_1 \cdot \ell} = \begin{cases} 0 & \text{if } \ell = 0 \\ 2 & \text{if } \ell = 1 \end{cases}$$

It suffices to compute

$$2 \cdot \sum_{x_2,\ldots,x_n : \ell = 1} (-1)^{f'},$$

which reduces the number of variables by two.

Theorem (Cai, C, Lipton and Lu 10)

Given $q \geq 1$ and a quadratic polynomial $f(x_1, \ldots, x_n)$ over $\mathbb{Z}_q$,

$$\sum_{x_1,\ldots,x_n \in \mathbb{Z}_q} \left(e^{2\pi\sqrt{-1}/q}\right)^{f(x_1,\ldots,x_n)}$$

can be computed in P-time in $\log q$ and n (without knowing the prime factorization of q).

1. Each round of the algorithm reduces either the number of variables by at least one, or reduce q significantly.
2. P-time even when q is given in binary, where Gauss sums form the basic building blocks of the algorithm.

Plan of the Talk

1. Algorithms for Counting Graph Homomorphisms
2. The Group Condition
   - Graph gadget
   - Interpolation

Discrete Unitary Matrices

Definition

We say $\mathbf{A} = (A_{i,j}) \in \mathbb{C}^{m\times m}$ *is a symmetric M-discrete unitary matrix, for some positive integer* M*, if*

1. *Each* $A_{i,j}$ *is a power of* $\omega_M = e^{2\pi\sqrt{-1}/M}$;
2. $A_{1,j} = 1$ *for all* $j \in [m]$;
3. *For all* $i \neq j \in [m]$, $\langle \mathbf{A}_{i,*}, \mathbf{A}_{j,*}\rangle = 0$ *where*

$$\langle \mathbf{A}_{i,*}, \mathbf{A}_{j,*}\rangle = \sum_{k=1}^{m} A_{i,k} \cdot \overline{A_{j,k}}.$$

The Goal

Lemma (The Group Condition Lemma)

Let $\mathbf{A} \in \mathbb{C}^{m \times m}$ be a symmetric M-discrete unitary matrix. Then either $\mathbf{A}$ satisfies the Group Condition or $\text{EVAL}(\mathbf{A})$ is #P-hard.

Definition (Group Condition)

For all i, j, there exists a $k \in [m]$ such that $\mathbf{A}_{k,*} = \mathbf{A}_{i,*} \circ \mathbf{A}_{j,*}$, where $\circ$ is the Hadamard product with the ℓth entry $= A_{i,\ell} \cdot A_{j,\ell}$.

Definition (Group Condition)

For all i, j, there exists a $k \in [m]$ such that $\mathbf{A}_{k,*} = \mathbf{A}_{i,*} \circ \mathbf{A}_{j,*}$, where $\circ$ is the Hadamard product with the ℓth entry $= A_{i,\ell} \cdot A_{j,\ell}$.

All $m \times m$ Fourier matrices $\mathbf{A}$, where

$$A_{x,y} = \omega^{(2\pi\sqrt{-1}/m)\cdot xy}, \quad \text{for all } x, y \in \mathbb{Z}_m$$

satisfy the Group Condition.

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$$

Lemma

If $\mathbf{A}$ *is discrete unitary and satisfies the Group Condition, then it is the tensor product of Fourier and generalized Fourier matrices.*

The Group Condition was introduced in [Goldberg, Grohe, Jerrum and Thurley 09] for $\{\pm 1\}$-matrices and generalized to complex-valued matrices in [Cai, C and Lu 11].

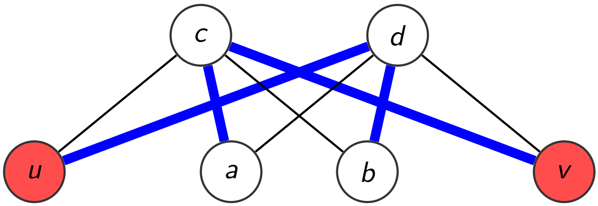
Lemma (The Group Condition Lemma)

Let $\mathbf{A} \in \mathbb{C}^{m\times m}$ *be a symmetric M-discrete unitary matrix. Then either* $\mathbf{A}$ *satisfies the Group Condition or* $\textsc{Eval}(\mathbf{A})$ *is #P-hard.*

Proof Sketch

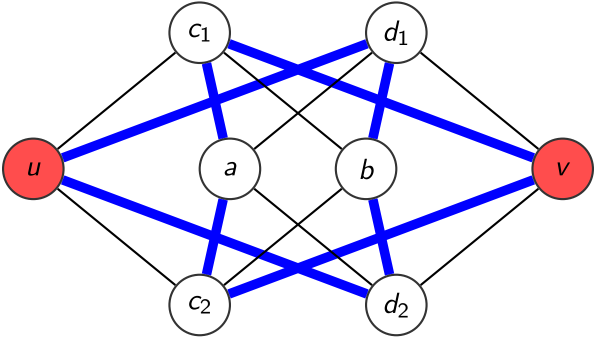
Construct a sequence of nonnegative symmetric matrices $\mathbf{B}^{[p]}$ such that each $\textsc{Eval}(\mathbf{B}^{[p]})$ is polynomial-time reducible to $\textsc{Eval}(\mathbf{A})$. Show that either 1) one of $\textsc{Eval}(\mathbf{B}^{[p]})$ is #P-hard (by [Bulatov and Grohe 05]), or 2) $\mathbf{A}$ satisfies the Group Condition.

First gadget:



Each blue thick edge: $M - 1$ parallel edges.

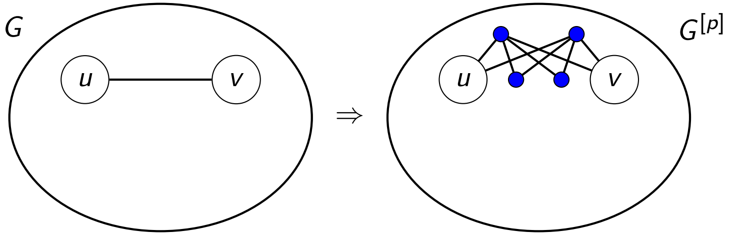
Second gadget:



In general, pth gadget for all $p \geq 1$.

Reduction Using a Gadget

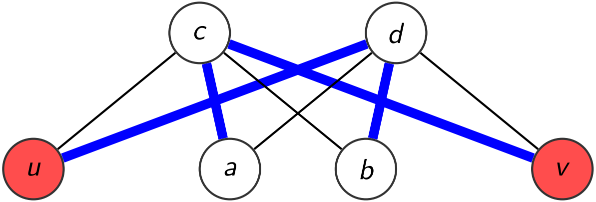
Replacing each edge e by the pth gadget: $G \Rightarrow G^{[p]}$



There is a symmetric matrix $\mathbf{B}^{[p]} \in \mathbb{C}^{m \times m}$ such that

$$Z_{\mathbf{B}^{[p]}}(G) = Z_{\mathbf{A}}(G^{[p]}).$$

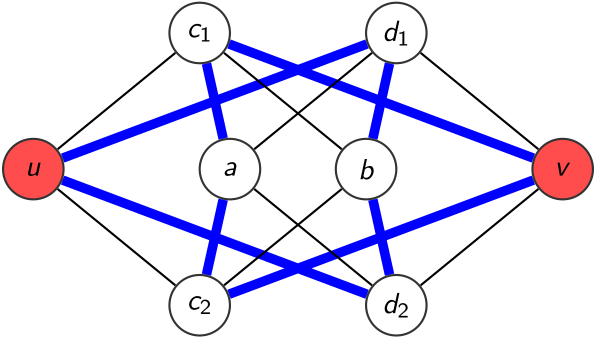
So $\textsc{Eval}(\mathbf{B}^{[p]})$ is polynomial-time reducible to $\textsc{Eval}(\mathbf{A})$.



$$B^{[1]}_{i,j} = \sum_{a,b,c,d} A_{i,c} \cdot \overline{A_{a,c}} \cdot A_{b,c} \cdot \overline{A_{j,c}} \cdot \overline{A_{i,d}} \cdot A_{a,d} \cdot \overline{A_{b,d}} \cdot A_{j,d}$$

$$= \sum_{a,b} \left(\sum_{c} A_{i,c} \cdot \overline{A_{a,c}} \cdot A_{b,c} \cdot \overline{A_{j,c}} \right) \left(\sum_{d} \overline{A_{i,d}} \cdot A_{a,d} \cdot \overline{A_{b,d}} \cdot A_{j,d} \right)$$

$$= \sum_{a,b \in [m]} \left| \sum_{c \in [m]} A_{i,c} \cdot \overline{A_{a,c}} \cdot A_{b,c} \cdot \overline{A_{j,c}} \right|^2$$



$$B^{[2]}_{i,j} = \sum_{a,b} \left(\sum_{c} A_{i,c} \cdot \overline{A_{a,c}} \cdot A_{b,c} \cdot \overline{A_{j,c}} \right)^2 \left(\sum_{d} \overline{A_{i,d}} \cdot A_{a,d} \cdot \overline{A_{b,d}} \cdot A_{j,d} \right)^2$$

$$= \sum_{a,b \in [m]} \left| \sum_{c \in [m]} A_{i,c} \cdot \overline{A_{a,c}} \cdot A_{b,c} \cdot \overline{A_{j,c}} \right|^4$$

In general for $p \geq 1$:

$$B_{i,j}^{[p]} = \sum_{a,b \in [m]} \left| \sum_{c \in [m]} A_{i,c} \cdot \overline{A_{a,c}} \cdot A_{b,c} \cdot \overline{A_{j,c}} \right|^{2p}$$

$$= \sum_{a,b \in [m]} \left| \langle \mathbf{A}_{i,*} \circ \overline{\mathbf{A}_{j,*}}, \mathbf{A}_{a,*} \circ \overline{\mathbf{A}_{b,*}} \rangle \right|^{2p}.$$

So $\mathbf{B}^{[p]}$ is both symmetric and positive (setting $a = i$, $b = j$).

Diagonal Entries

Diagonal entries of $\mathbf{B}^{[p]}$:

$$B_{i,i}^{[p]} = \sum_{a,b} \left|\langle \mathbf{1}, \mathbf{A}_{a,*} \circ \mathbf{A}_{b,*} \rangle\right|^{2p} = \sum_{a,b} \left|\langle \mathbf{A}_{a,*}, \mathbf{A}_{b,*} \rangle\right|^{2p} = m^{2p+1}.$$

If $B_{i,j}^{[p]} \neq m^{2p+1}$ for some p and $i \neq j$, $\textsc{Eval}(\mathbf{B}^{[p]})$ is #P-hard by using [Bulatov and Grohe 05], and so is $\textsc{Eval}(\mathbf{A})$. Done!

Otherwise, every entry of $\mathbf{B}^{[p]}$ must equal to m^{2p+1}.
Show in this case that $\mathbf{A}$ satisfies the Group Condition.

Off-Diagonal Entries

Fix a pair $i \neq j$. Then

$$B_{i,j}^{[p]} = \sum_{a,b \in [m]} \left| \langle \mathbf{A}_{i,*} \circ \overline{\mathbf{A}_{j,*}}, \mathbf{A}_{a,*} \circ \overline{\mathbf{A}_{b,*}} \rangle \right|^{2p} = \sum_{x \in X_{i,j}} S_{i,j}^{[x]} \cdot x^{2p},$$

where

1. $X_{i,j}$ is the set of possible values of $\left| \langle \mathbf{A}_{i,*} \circ \overline{\mathbf{A}_{j,*}}, \mathbf{A}_{a,*} \circ \overline{\mathbf{A}_{b,*}} \rangle \right|$;
2. For each $x \in X_{i,j}$, $S_{i,j}^{[x]}$ is the number of (a, b) such that

$$\left| \langle \mathbf{A}_{i,*} \circ \overline{\mathbf{A}_{j,*}}, \mathbf{A}_{a,*} \circ \overline{\mathbf{A}_{b,*}} \rangle \right| = x.$$

Also $\{0, m\} \in X_{i,j}$ (setting $(a, b) = (i, j'), (i, j)$ for some $j' \neq j$).

A Vandermonde System

Since $B_{i,j}^{[p]} = m^{2p+1}$, we have

$$\sum_{x \in X_{i,j}} S_{i,j}^{[x]} \cdot x^{2p} = m^{2p+1}, \quad \text{for } p = 1, \ldots, |X_{i,j}| - 1.$$

In addition, there are m^2 many pairs (a, b) so

$$\sum_{x \in X_{i,j}} S_{i,j}^{[x]} = m^2.$$

A Vandermonde system, with a unique solution:

$$X_{i,j} = \{0, m\}, \quad S_{i,j}^{[m]} = m \quad \text{and} \quad S_{i,j}^{[0]} = m^2 - m.$$

The Consequence

For all $i, j, a, b \in [m]$, we have

$$\left|\langle \mathbf{A}_{i,*} \circ \overline{\mathbf{A}_{j,*}}, \mathbf{A}_{a,*} \circ \overline{\mathbf{A}_{b,*}} \rangle\right| \in \{0, m\}.$$

Use this to establish the Group Condition.

Almost There . . .

Fix $i, b \in [m]$. Set $j = 1$. As $\mathbf{A}_{1,*} = \mathbf{1}$,

$$\left|\langle \mathbf{A}_{i,*} \circ \mathbf{1}, \mathbf{A}_{a,*} \circ \overline{\mathbf{A}_{b,*}} \rangle\right| = \left|\langle \mathbf{A}_{i,*} \circ \mathbf{A}_{b,*}, \mathbf{A}_{a,*} \rangle\right| \in \{0, m\}.$$

Since $\left\{ A_{a,*} : a \in [m] \right\}$ is an orthogonal basis, by Parseval:

$$\sum_a \left|\langle \mathbf{A}_{i,*} \circ \mathbf{A}_{b,*}, \mathbf{A}_{a,*} \rangle\right|^2 = m \cdot \left\|\mathbf{A}_{i,*} \circ \mathbf{A}_{b,*}\right\|^2 = m^2.$$

The Group Condition

As a result, for all $i, b \in [m]$, there exists an $a \in [m]$ such that

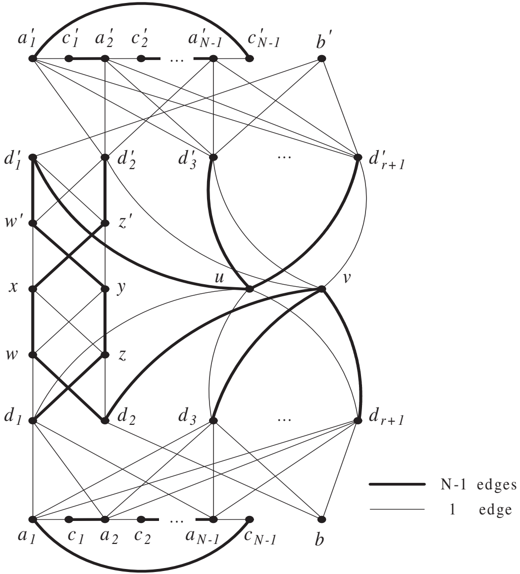
$$\left|\langle \mathbf{A}_{i,*} \circ \mathbf{A}_{b,*}, \mathbf{A}_{a,*} \rangle\right| = m.$$

The first entries of $\mathbf{A}_{i,*} \circ \mathbf{A}_{b,*}$ and $\mathbf{A}_{a,*}$ are 1:

$$\mathbf{A}_{a,*} = \mathbf{A}_{i,*} \circ \mathbf{A}_{b,*}.$$

Conclusion

1. Algorithms for Counting Graph Homomorphisms
2. The Group Condition
   - Graph gadget
   - Interpolation


a'1
c'1
a'2
c'2
...
a'N-1
c'N-1
b'
d'1
d'2
d'3
...
d'r+1
w'
z'
x
y
u
v
w
z
d1
d2
d3
...
dr+1
a1
c1
a2
c2
...
aN-1
cN-1
b
N-1 edges
1 edge

Plan for Tomorrow

1. Dichotomy for Unweighted #CSP:
    - Tractability criterion: Strong balance
    - Mal'tsev polymorphisms and Witness functions
    - The main counting algorithm
2. Dichotomy for Nonnegative and Complex #CSP

Thanks!