Theory of Data Streams

S. *Muthu* Muthukrishnan

Familiar Puzzle: Missing Number

- A shows B numbers $1, \ldots, n$ but in a permuted order and leaves out one of the numbers.
- B has to determine the missing number.
- Key: B has only $O(\log n)$ bits.

Familiar Puzzle: Missing Number

- A shows B numbers $1, \ldots, n$ but in a permuted order and leaves out one of the numbers.
- B has to determine the missing number.
- Key: B has only $O(\log n)$ bits.
- Solution: B maintains the running sum s of numbers seen. Missing number is $\frac{n(n+1)}{2} - s$.

A New Puzzle: One Word Median

- A sees items $i_1, i_2, \ldots$ arrive in a stream.
- A has to maintain the median m_j of the items $i_1, \ldots, i_j$.
- Key: A is allowed to store only one word of memory (of $\log n$ bits).

A New Puzzle: One Word Median

- A sees items $i_1, i_2, \ldots$ arrive in a stream.
- A has to maintain the median m_j of the items $i_1, \ldots, i_j$.
- Key: A is allowed to store only one word of memory (of $\log n$ bits).
- Each i_j generated independently and randomly from some unknown distribution $\mathcal{D}$ over integers $[1, n]$.

A New Puzzle: One Word Median

- A sees items $i_1, i_2, \ldots$ arrive in a stream.
- A has to maintain the median m_j of the items $i_1, \ldots, i_j$.
- Key: A is allowed to store only one word of memory (of $\log n$ bits).
- Each i_j generated independently and randomly from some unknown distribution $\mathcal{D}$ over integers $[1, n]$.
- Solution. Maintain μ_j.
  If $i_{j+1} > \mu_j$, $\mu_{j+1} \leftarrow \mu_j + 1$.
  If $i_{j+1} < \mu_j$, $\mu_{j+1} \leftarrow \mu_j - 1$.

This Talk

- Two basic primitives and applications with data stream algorithms.

This Talk

- Two basic primitives and applications with data stream algorithms.
  - Count-Min sketch, applications to compressed sensing
  - L_0 sampling, applications to graph problems
- Some topics:
  - L_2 sketches and applications
  - Nonstreaming applications of streaming results
  - Distributed streaming
  - Pan-privacy
  - Cryptography

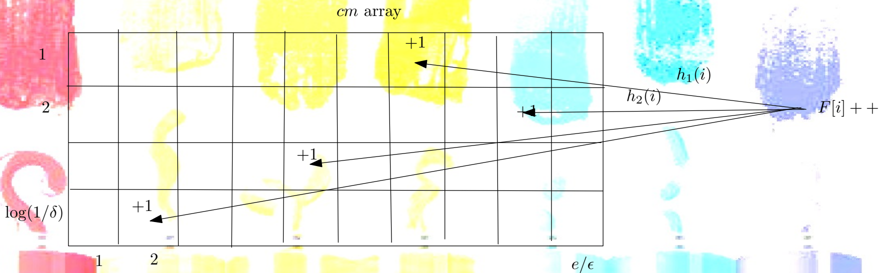
A Basic Problem: Indexing

- Imagine a virtual array $F[1 \cdots n]$
- Updates: $F[i]++$, $F[i]--$
- Assume $F[i] \geq 0$ at all times
- Query: $F[i] = ?$

- Key: Use $o(n)$ space, may be $O(\log n)$ space

Count-Min Sketch

- For each update $F[i]++$,
  - for each $j = 1, \ldots, \log(1/\delta)$, update $cm[h_j(i)]++$.
- Estimate $\tilde{F}(i) = \min_{j=1,\ldots,\log(1/\delta)} cm[h_j(i)]$.



Count-Min Sketch Analysis

- $F[i] \le \tilde{F}[i]$. With probability at least $1 - \delta$,

$$\tilde{F}[i] \le F[i] + \varepsilon \sum_{j \neq i} F[j].$$

Count-Min Sketch Analysis

- $F[i] \leq \tilde{F}[i]$. With probability at least $1 - \delta$,

$$\tilde{F}[i] \leq F[i] + \varepsilon \sum_{j \neq i} F[j].$$

- $X_{i,j}$ is the expected contribution of $F[j]$ to the bucket containing i, for any h.

$$E(X_{i,j}) = \frac{\varepsilon}{e} \sum_{j \neq i} F[j].$$

Count-Min Sketch Analysis

- $F[i] \leq \tilde{F}[i]$. With probability at least $1-\delta$,
$$\tilde{F}[i] \leq F[i] + \varepsilon \sum_{j \neq i} F[j].$$
- $X_{i,j}$ is the expected contribution of $F[j]$ to the bucket containing i, for any h.
$$E(X_{i,j}) = \frac{\varepsilon}{e} \sum_{j \neq i} F[j].$$
- Consider $\Pr(\tilde{F}[i] > F[i] + \varepsilon \sum_{j \neq i} F[j])$:
$$\begin{aligned} \Pr() &= \Pr(\forall j, F[i] + X_{i,j} > F[i] + \varepsilon \sum_{j \neq i} F[j]) \\ &= \Pr(\forall j, X_{i,j} \geq eE(X_{i,j})) < e^{-\log(1/\delta)} = \delta \end{aligned}$$

Count-Min Sketch

- Claim: $F[i] \leq \tilde{F}[i]$. With probability at least $1 - \delta$,

$$\tilde{F}[i] \leq F[i] + \varepsilon \sum_{j \neq i} F[j]$$

- Space used is $O(\frac{1}{\varepsilon} \log \frac{1}{\delta})$.
- Time per update is $O(\log \frac{1}{\delta})$. Indep of n.

G. Cormode and S. Muthukrishnan: An improved data stream summary: count-min sketch and its applications. *Journal of Algorithms, 55(1): 58-75 (2005).*

Improve Count-Min Sketch?

- Index Problem:
  - ALICE has n long bitstring and sends messages to BOB who wishes to compute the ith bit.
  - Needs $\Omega(n)$ bits of communication.
- Reduction of estimating $F[i]$ in data stream model.
  - $I[1 \cdots 1/(2\varepsilon)]$ such that
    - $I[i] = 1 \rightarrow F[i] = 2$
    - $I[i] = 0 \rightarrow F[i] = 0; F[0] \leftarrow F[0] + 2$
  - Observe that $||F|| = \sum_i F[i] = 1/\varepsilon$

Improve Count-Min Sketch?

- Index Problem:
  - ALICE has n long bitstring and sends messages to BOB who wishes to compute the ith bit.
  - Needs $\Omega(n)$ bits of communication.
- Reduction of estimating $F[i]$ in data stream model.
  - $I[1 \cdots 1/(2\varepsilon)]$ such that
    - $I[i] = 1 \to F[i] = 2$
    - $I[i] = 0 \to F[i] = 0; F[0] \leftarrow F[0] + 2$
  - Observe that $||F|| = \sum_i F[i] = 1/\varepsilon$
  - Estimating $F[i] \leq \tilde{F}[i] \leq F[i] + \varepsilon||F||$ implies,
    - $I[i] = 0 \to F[i] = 0 \to 0 \leq \tilde{F}[i] \leq 1$
    - $I[i] = 1 \to F[i] = 2 \to 2 \leq \tilde{F}[i] \leq 3$

    and reveals $I[i]$.
  - Therefore, $\Omega(1/\varepsilon)$ space lower bound for index problem.

Count-Min Sketch, The Challenges

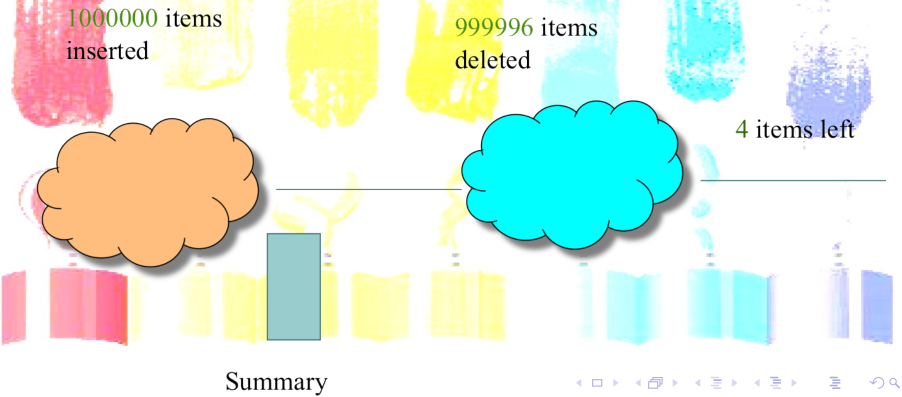
- Not all projections, dimensionality reduction are the same:
  - All prior work $\Omega(1/\varepsilon^2)$ space, via Johnson-Lindenstrauss

Count-Min Sketch, The Challenges

- Not all projections, dimensionality reduction are the same:
  - All prior work $\Omega(1/\varepsilon^2)$ space, via Johnson-Lindenstrauss
- Not all hashing algorithms are the same:
  - Pairwise independence

Count-Min Sketch, The Challenges

- Not all projections, dimensionality reduction are the same:
  - All prior work $\Omega(1/\varepsilon^2)$ space, via Johnson-Lindenstrauss
- Not all hashing algorithms are the same:
  - Pairwise independence
- Not all approximations are sampling.
  - Recovering $F[i]$ to $\pm 0.1|F|$ accuracy will retrieve each item precisely.



Using Count-Min Sketch

- For each i, determine $\tilde{F}[i]$
- Keep the set S of heavy hitters ($\tilde{F}[i] \geq 2\varepsilon||F||$).
  - Guaranteed that S contains i such that $F[i] \geq 2\varepsilon||F||$ and no $F[i] \leq \varepsilon||F||$
  - Extra $\log n$ factor space for n queries

Problem is of database interest.

Using Count-Min Sketch

- For each i, determine $\tilde{F}[i]$
- Keep the set S of heavy hitters ($\tilde{F}[i] \geq 2\varepsilon||F||$).
  - Guaranteed that S contains i such that $F[i] \geq 2\varepsilon||F||$ and no $F[i] \leq \varepsilon||F||$
  - Extra $\log n$ factor space for n queries

  Problem is of database interest.
- Faster recovery: In each bucket, recover majority i ($F[i] > \sum_{j \text{ same bucket as } i} F[j]/2$)

Using Count-Min Sketch

- For each i, determine $\tilde{F}[i]$
- Keep the set S of heavy hitters ($\tilde{F}[i] \geq 2\varepsilon||F||$).
  - Guaranteed that S contains i such that $F[i] \geq 2\varepsilon||F||$ and no $F[i] \leq \varepsilon||F||$
  - Extra $\log n$ factor space for n queries

  Problem is of database interest.
- Faster recovery: In each bucket, recover majority i ($F[i] > \sum_{j \text{ same bucket as } i} F[j]/2$)
  - Takes $O(\log n)$ extra time, space
  - Gives compressed sensing in L_1:

$$||F - \tilde{F}_k||_1 \leq ||F - F_k^*||_1 + \varepsilon||F||_1$$

Sparse recovery experiments: http://groups.csail.mit.edu/toc/sparse/wiki/index.php?title=Sparse_Recovery_Experiments

Count-Min Sketch: Summary

- Solves many problems:
  - Heavy hitters, compressed sensing, inner products, ...
- Applications to other CS/EE areas:
  - NLP, ML, Password checking.
- Systems, code, hardware.
  - Gigascope, CMON, Sawzall, MillWheel, ...

Wiki: http://sites.google.com/site/countminsketch/

L_0 Sampling

- Imagine a virtual array $F[1 \cdots n]$
- Updates: $F[i]++$, $F[i]--$
- Assume $F[i] \geq 0$ at all times

- Query: inverse sample?
  Return i, $F[i] \neq 0$ with prob $\frac{1}{|\{i | F[i] \geq 0\}|}$

- Key: Use $o(n)$ space, may be $O(\log n)$ space

- Solutions use $O(1/\varepsilon^2)$ space.

Application of L_0 Sampling

- Graph Sketch: For node i, let a_i be vector indexed by node pairs. $a_i[i,j] = 1$ if $j > i$ and $a_i[i,j] = -1$ if $j < i$.
- For any subset $S \subset V$, $\text{support}(\sum_{i \in S} a_i) = E(S, V - S)$

Application of L_0 Sampling

- Graph Sketch: For node i, let a_i be vector indexed by node pairs. $a_i[i,j] = 1$ if $j > i$ and $a_i[i,j] = -1$ if $j < i$.
- For any subset $S \subset V$, support$(\sum_{i \in S} a_i) = E(S, V - S)$
- Prob: Is G connected?
  - Algorithm (Spanning Forest):
    - For each node, select an incident edge
    - Contract selected edges. Repeat until no edges
  - Data structure: L_0 sketch C for each a_j.
  - Use Ca_j to get incident edge. Then, run algorithm above. Observe:

$$\sum_{j \in S} Ca_j = C(\sum_{j \in S} a_j) \to e \in \text{support}(\sum_{j \in S} E(S, V - S))$$

Ahn, Guha, McGregor: Analyzing graph structure via linear measurements. SODA12.
Dynamic graph connectivity in polylogarithmic worst case time. B. Kapron, V. King, B. Mountjoy. SODA13

Topics: L_2 approximation

- L_2 estimate: $||F||_2 = \sum_i F[i]^2$
  - $Y_j = \sum_{i=1,\ldots,w/2}(cm_F[h_j(2i-1)] - cm_F[h_j(2i)])^2$
  - Gives $\widetilde{||F||_2} \leq (1+\varepsilon)||F||_2$
  - Space $O(1/\varepsilon^2)$, update time is $O(\log 1/\delta)$

Topics: L_2 approximation

- L_2 estimate: $||F||_2 = \sum_i F[i]^2$
  - $Y_j = \sum_{i=1,\ldots,w/2}(cm_F[h_j(2i-1)] - cm_F[h_j(2i)])^2$
  - Gives $\widetilde{||F||_2} \le (1+\varepsilon)||F||_2$
  - Space $O(1/\varepsilon^2)$, update time is $O(\log 1/\delta)$
- Ex: Least squares approximation
  - Problem: Given matrix $A \in R^{n \times d}$ and a vector $b \in R^n$, find $x \in R^d$ such that $z = ||Ax - b||_2$ is minimized.
  - Result: Can find y such that $||Ay - b||_2 \le (1+\epsilon)z$ in $O(nd \log d)$ randomized time.
  - Solution: Consider TA and Tb, where $T \in R^{O(d/\epsilon \times d)}$
  - Extends to low rank matrix approx, classification,...

Low Rank Approximation and Regression in Input Sparsity Time. K. Clarkson, D. Woodruff, STOC 13

Topics: Nonstreaming Problems

- Compute the Discrete Fourier Transform of signal of size n
- Classical: $O(n \log n)$ time.
- Recent result: There exists a randomized $O(k \log n)$ time algorithm for k-sparse case.

> Nearly Optimal Sparse Fourier Transform H. Hassanieh, P. Indyk, D. Katabi, E. Price. STOC 12

Topics: Distributed Learning

- Alice has data D_A and Bob has D_B, and learn linear classifier. Minimize communication. h^* is optimal.
- $E_D(h)$ is the number of points misclassified by h on D.
- g has ε- error if $E_D(g) - E_D(h^*) \leq \varepsilon |D|$.
- There is a $O(\log 1/\varepsilon)$ round two way communication protocol with $O(1)$ bits per round and ε-error.

Efficient Protocols for Distributed Classification and Optimization. H. Daume, J. Phillips, A. Saha, S. Venkatasubramanian. ALT12
Distributed Learning, Communication Complexity and Privacy. N. Balcan, A. Blum, S. Fine, Y. Mansour. ICML12

Topics: Pan Privacy

- Well known notion of differential privacy (DP). What if the internal state is breached?
- Pan-Privacy. For every two neighboring streams, at any time, internal state and final output should be DP.
- Use count-min and L_0 sketches to get approximate pan-private algorithms.

Pan-private algorithms via statistics on sketches. D. Mir, S. Muthukrishnan, A. Nikolov and R. Wright, PODS11.

Topics: Streaming Cryptography

- Cryptography against polynomial time adversaries using a streaming algorithm?
- Recent result: Streaming algorithms for one-way functions and pseudorandom generators with $O(1)$ passes over two read-write tapes, under suitable assumptions.

Cryptography with streaming algorithms P. Papakonstantinou, G. Yang. Manuscript, 13.

Conclusions

- Two basic primitives and applications with data stream algorithms.
  - Count-Min sketch, applications to compressed sensing
  - L_0 sampling, applications to graph problems
- Some topics:
  - L_2 sketches and applications
  - Nonstreaming applications of streaming results
  - Distributed streaming
  - Pan-privacy
  - Cryptography

Area continues to grow tentacles