Nondeterministic extensions of the Strong Exponential Time Hypothesis and consequences for non-reducibility

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2015

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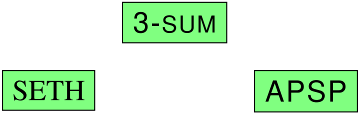

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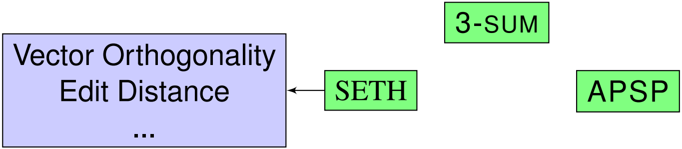

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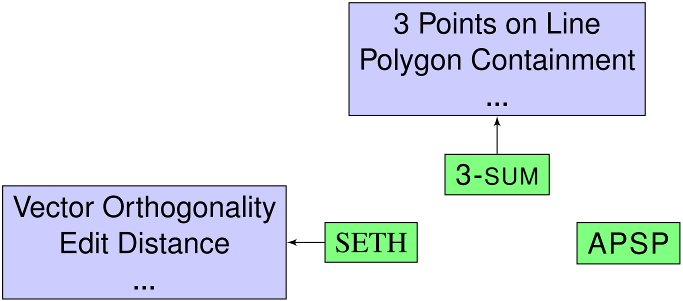

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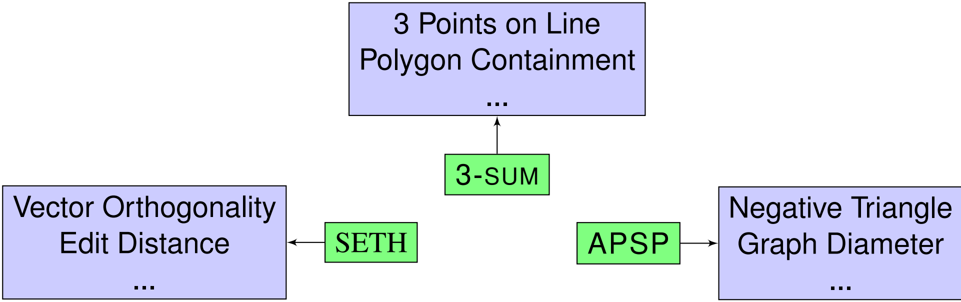

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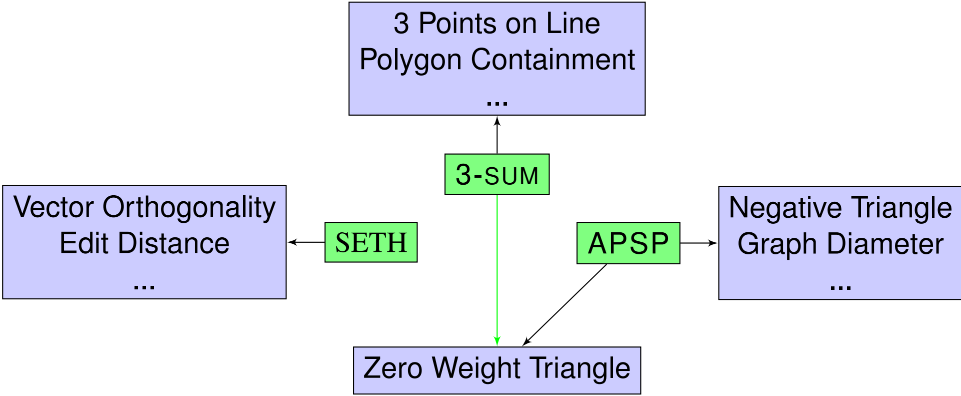

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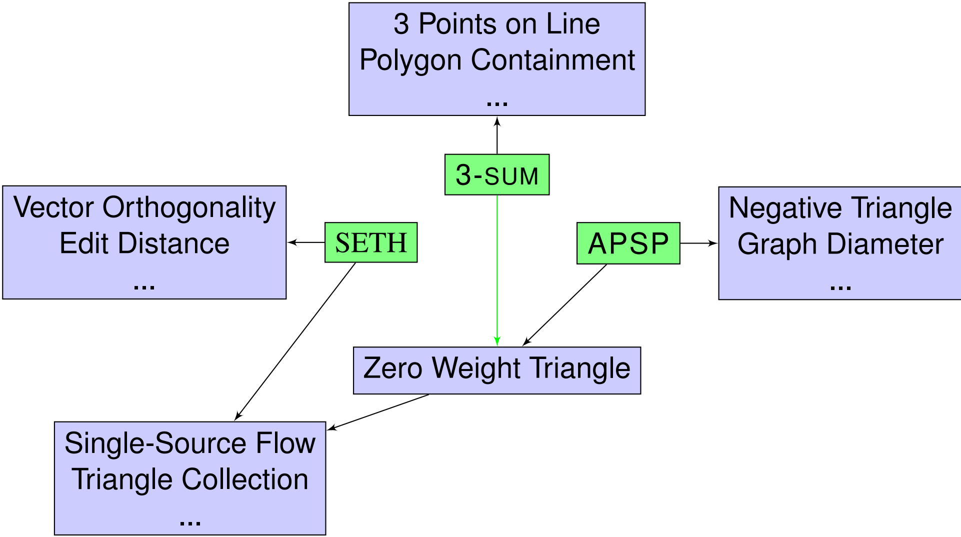

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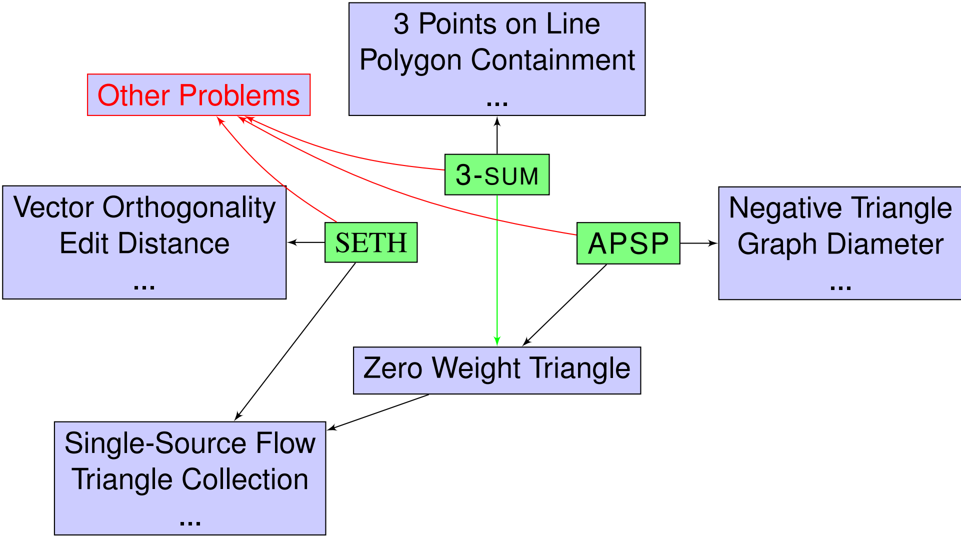

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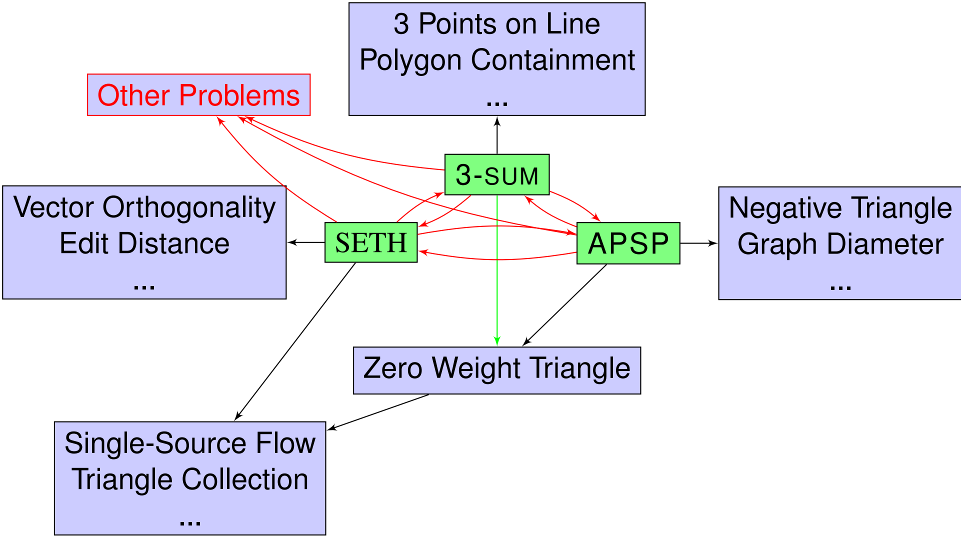

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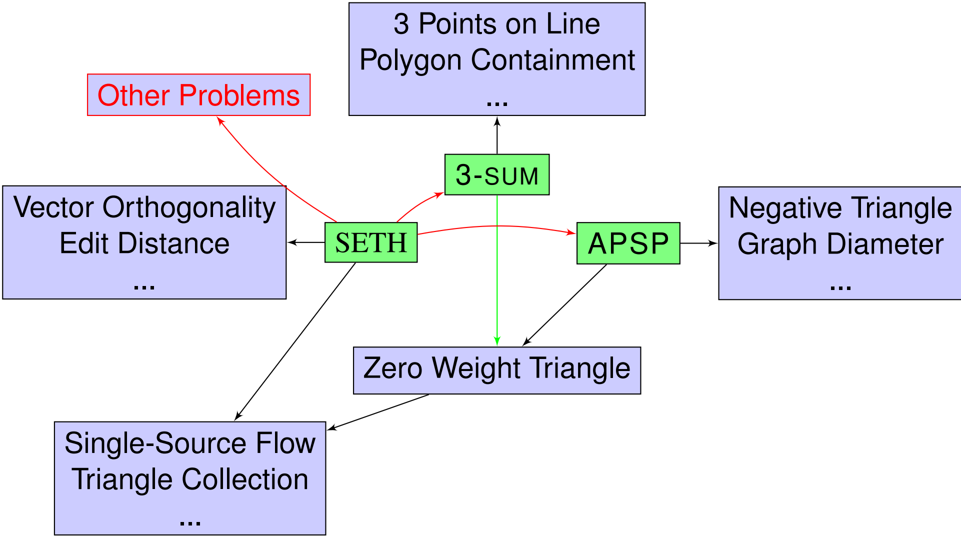

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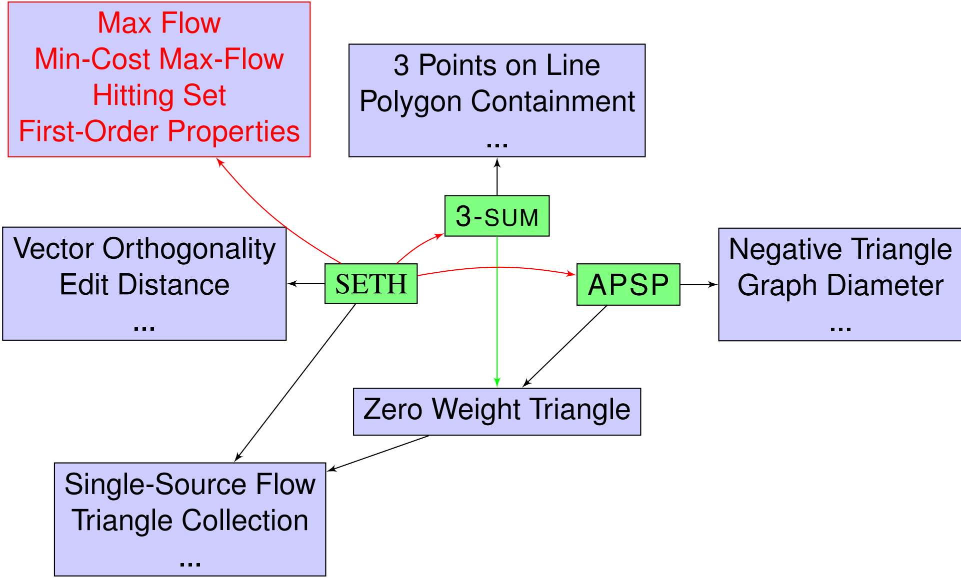

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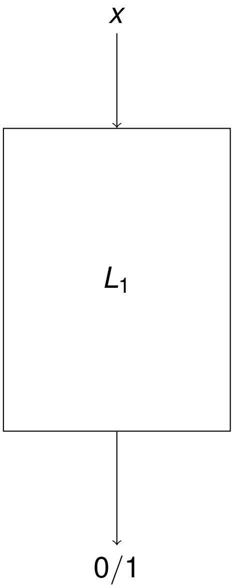

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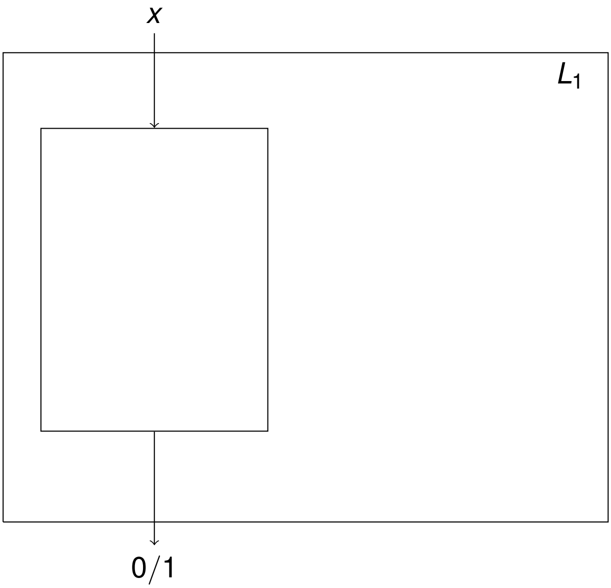

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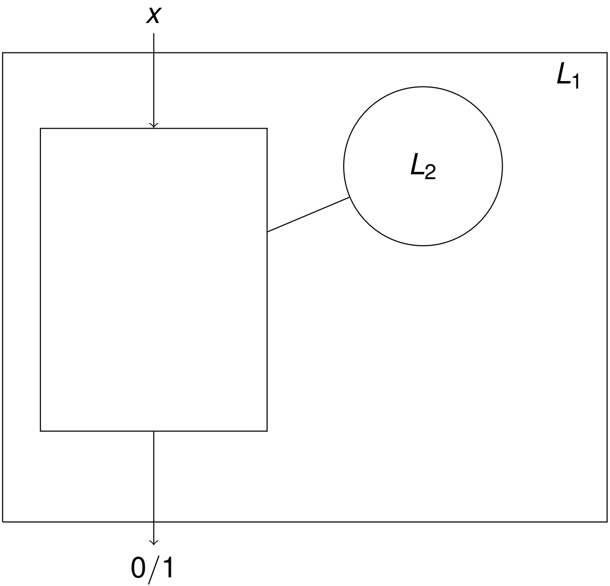

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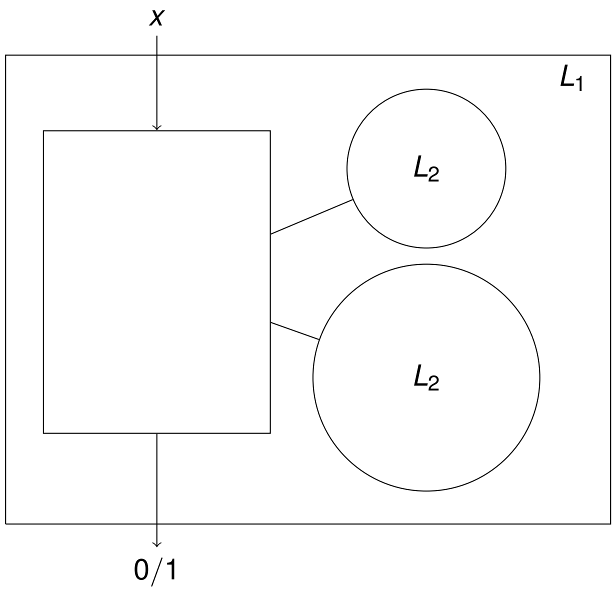

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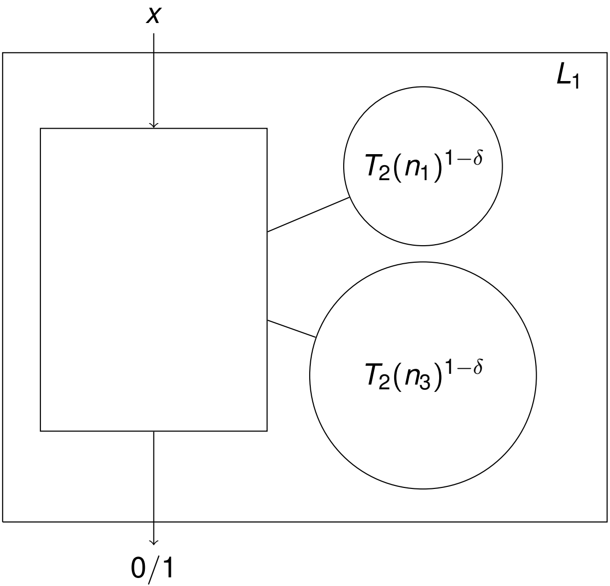

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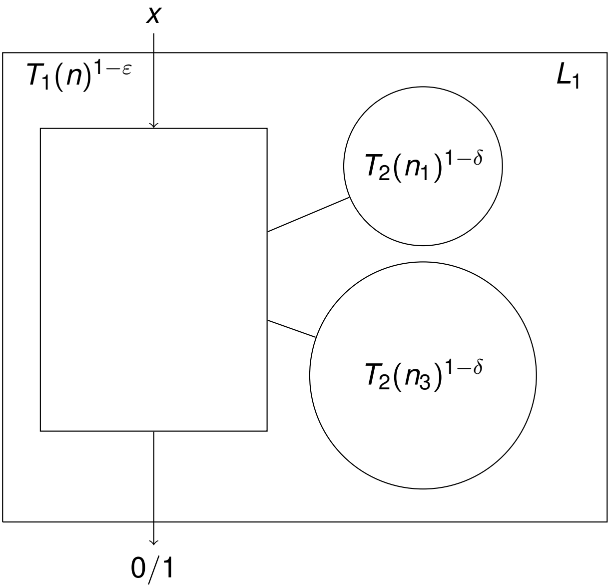

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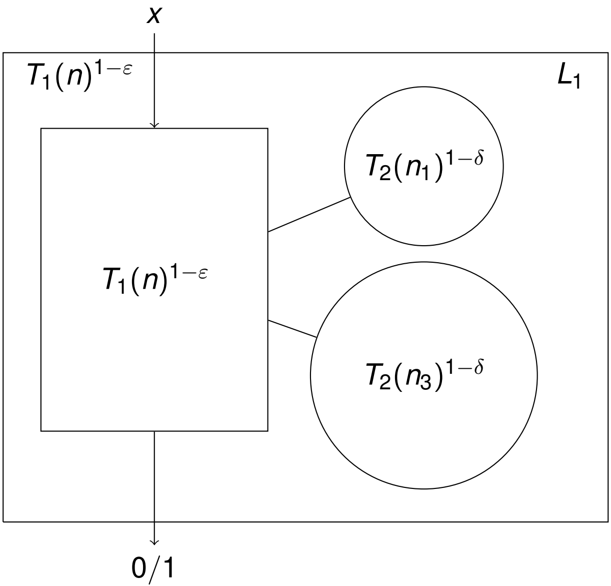

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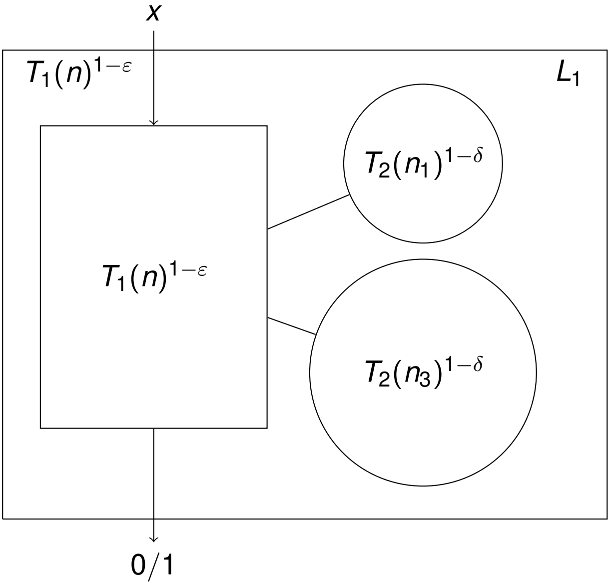

$\sum_i T_2(n_i)^{1-\delta}$
$\leq T_1(n)^{1-\varepsilon}$

Proof Idea

Lemma (Basic Idea)

Every SETH*-hard problem has property X.* 3-SUM *and* APSP *do not have property X.*

NSETH

Nondeterministic Strong Exponential Time Hypothesis

For every $s > 0$, there is a k such that k-SAT cannot be solved in *co-nondeterministic* time $2^{(1-s)n}$

Property X

Det.	Nondet.	Co-Nondet.	X	Example
T	$T^{1-\varepsilon}$	T	yes	CNF-SAT
T	T	$T^{1-\varepsilon}$	yes	DNF-TAUT
T	T	T	yes	Exact-Max-SAT
T	$T^{1-\varepsilon}$	$T^{1-\varepsilon}$	no	3-SUM

Property X

Lemma

Assuming NSETH, any problem that is SETH-hard with time T under deterministic reductions either

- *Cannot be solved in nondeterministic time $T^{(1-s)}$*
- *Cannot be solved in co-nondeterministic time $T^{(1-s)}$*

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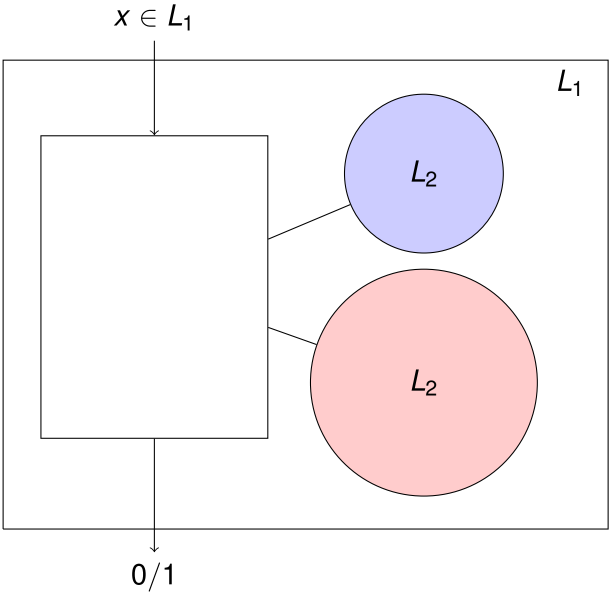

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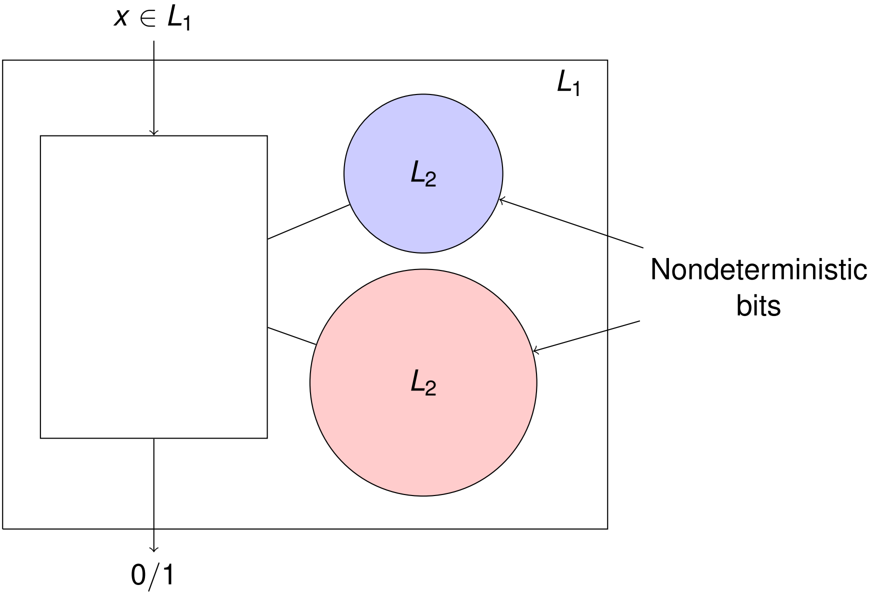

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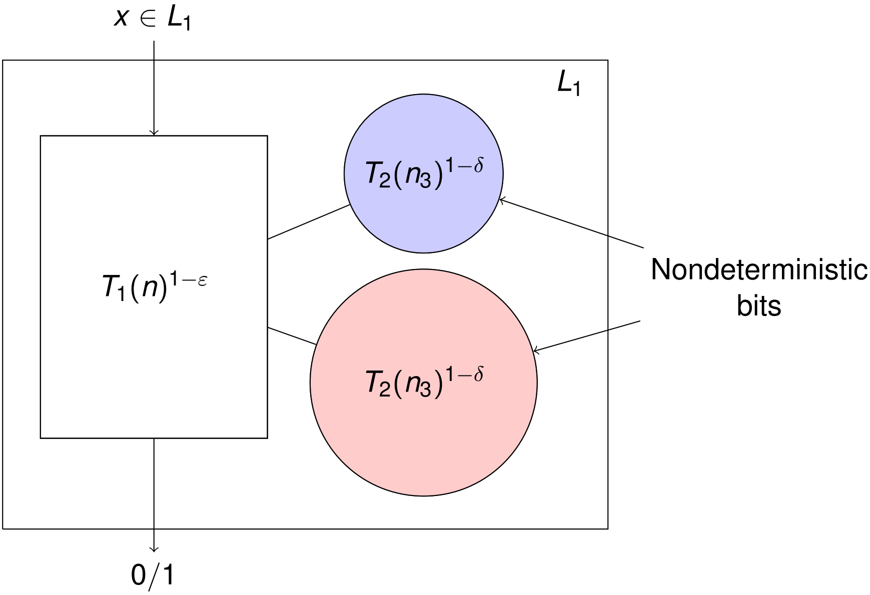
$x \in L_1$
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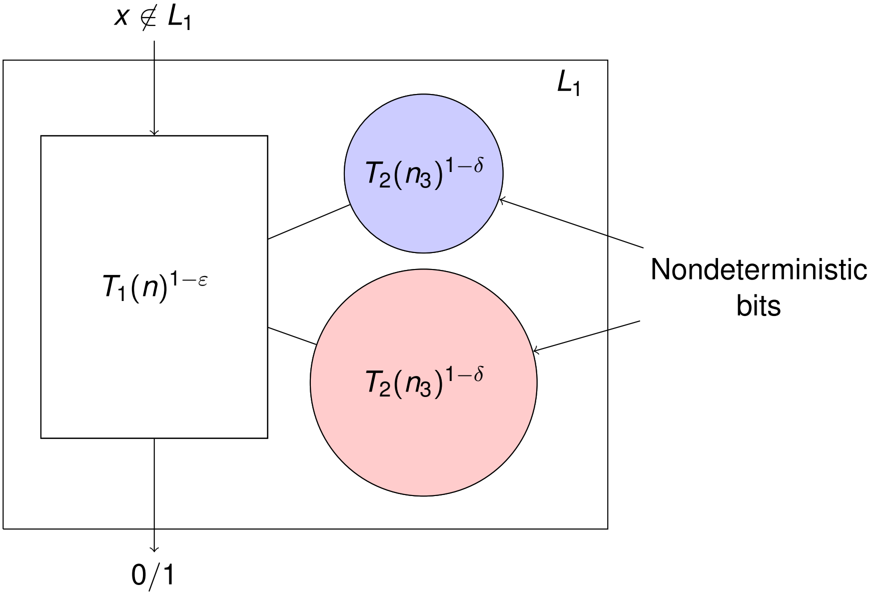
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- Problems with fast nondeterministic and co-nondeterministic algorithms ($\neg X$)
- First-order graph properties

$\neg$SETH and $\neg$NSETH Imply Circuit Lower Bounds

The work of [JVM13] uses a two-part strategy to show $\neg\text{SETH} \implies$ Circuit Lower Bounds:

1. A tight implication from $\mathcal{C}$-CKT-SAT algorithms to $\mathcal{C}$ lower bounds
2. Decomposition of $\mathcal{C}$-circuits into $\bigvee$ *CNF* form

"$\neg\text{NSETH} \implies$ Circuit Lower Bounds" is implicit in [JVM13], following from their technical contributions and the proofs of [Williams 2013].

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Reducing $\mathcal{C}$-CKT-SAT to k-SAT?

For $\mathcal{C}$:

- Linear-size circuits
- Linear-size series-parallel circuits

There are decompositions: $\forall C \in \mathcal{C}$, we have $C = \bigvee CNF_k$.
Execute the faster k-SAT algorithm on each "leaf" of the decomposition.

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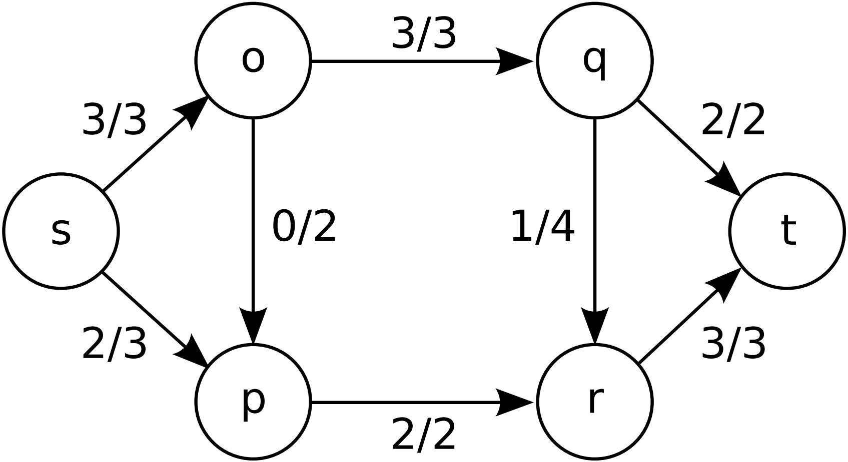
  - All-Pairs Shortest Path ($\tilde{O}(n^{2.69})$)

Maximum Flow Problem



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[1] Licensed under CC BY-SA 3.0 via Commons - https://commons.wikimedia.org/wiki/File:Max_flow.svg#/media/File:Max_flow.svg

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- Special cases: Max-Flow, Min-Cost Perfect Bipartite Matching

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- Count number of solutions and check that given list is complete

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- Nice coincidence: decision tree complexity of 3-SUM is also $\tilde{O}(n^{3/2})$.

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- We give a co-nondeterministic algorithm for Zero-Weight Triangle

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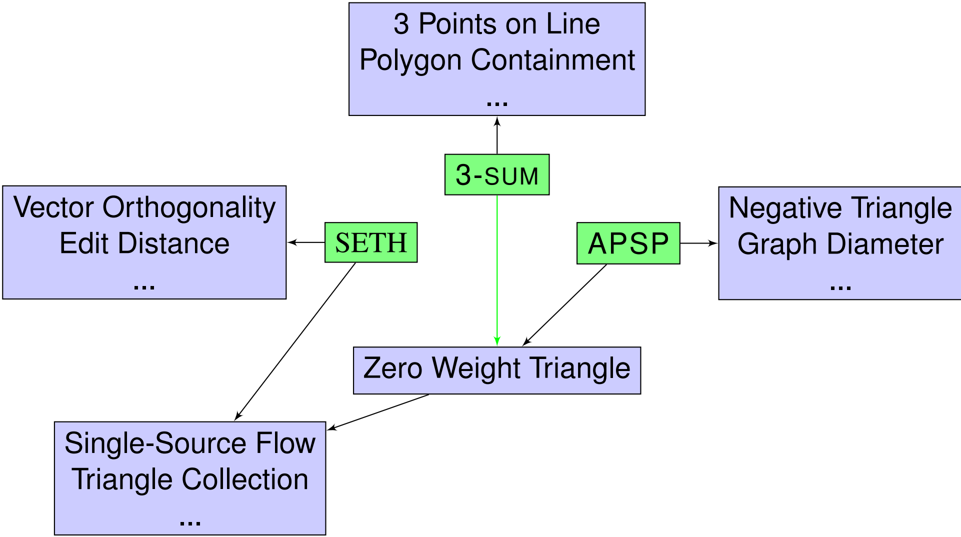

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  - Hitting Set (not known to be SETH-hard)
    $(\exists H)(\forall S)(\exists x)\,[(u \in H) \wedge (u \in S)]$

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  - If NSETH holds, there is no reduction from this quantifier structure to other quantifier structures.

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Quantifier structure	**Result**	**Hardness**
$\exists \ldots \exists\forall$	If solvable in $O(m^{k-1-\epsilon})$ co-nondeterministic time, then NSETH is false.	SETH-hard
All k quantifiers are $\exists$'s	solvable in $O(m^{k-1.5})$ time	Easy
More than one $\forall$'s	faster co-nondeterministic algorithms	Not SETH-hard under NSETH
Exactly one $\forall$, but not at Q_k	faster co-nondeterministic algorithms	Not SETH-hard under NSETH

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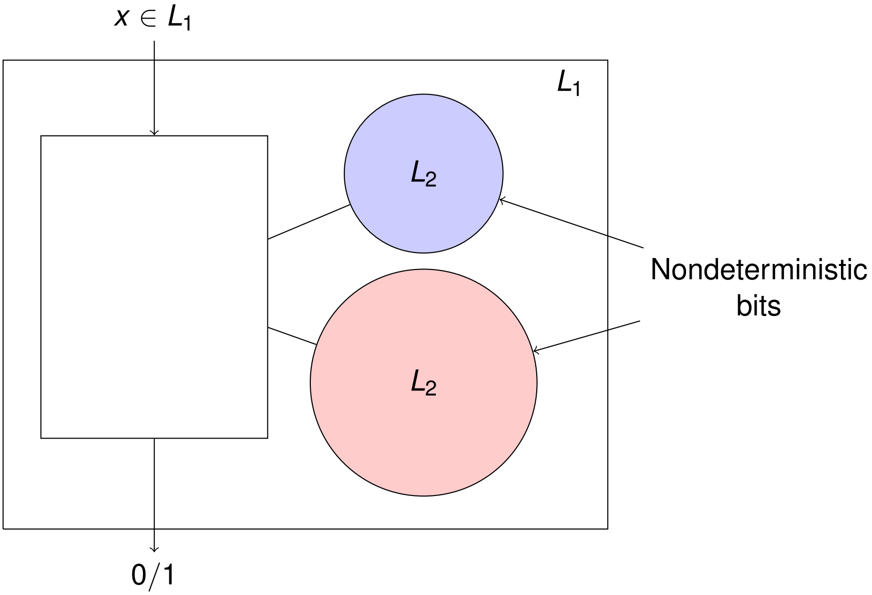
- Orthogonal Vectors $\not\leq_{FGR}$ Hitting Set, under NSETH.

Randomized Reductions



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Non-Uniform Nondeterministic Strong Exponential Time Hypothesis

For every $s > 0$, there is a k such that k-SAT does not have $2^{(1-s)n}$ size nondeterministic circuits

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- If NSETH is false, then new circuit lower bounds follow
- If NSETH is true, then non-reducibility results follow
- If there is a deterministic fine-grained reduction from vector orthogonality to hitting set, then we have new lower bounds for linear size circuits

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Thank You!