Parameterized Inapproximability of Max k-Subset Intersection under ETH

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One message

ETH can be used to refute the existence of exponential time approximation algorithms.

Max-k-Subset-Intersection	
Input:	A collection $\mathcal{F} = \{S_1, S_2, \cdots, S_n\}$ of subsets over $[n]$.
Solution:	k distinct subsets $S_{j_1}, S_{j_2}, \cdots, S_{j_k}$ from $\mathcal{F}$.
Cost:	$\lvert S_{j_1} \cap \cdots \cap S_{j_k} \rvert$.
Goal:	max.

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Remark

1. MAX-k-SUBSET-INTERSECTION *is* **NP**-*hard*
2. MAX-k-SUBSET-INTERSECTION *can be solved in time* $n^{O(k)}$.

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Cost:	$\lvert S_{j_1} \cap \cdots \cap S_{j_k} \rvert$.
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Let $\mathrm{OPT}_{kmsi}(\mathcal{F})$ be the maximum k-subset intersection size of $\mathcal{F}$.

Question

Is there an $f(k) \cdot n^{O(1)}$-time algorithm that, given $\mathcal{F}$, finds k distinct subsets from $\mathcal{F}$ with intersection size at least $\frac{1}{r} \cdot \mathrm{OPT}_{kmsi}$?

Previous work

Results of **Polynomial-time** inapproximability:

Problem	Ratio	Assumptions	Ref
Max-Biclique	$2^{(\log n)^\delta}$	3SAT $\notin$ DTIME$(2^{n^{3/4+\varepsilon}})$	Feige and Kogan 04
Max-Biclique	$n^{\varepsilon'}$	SAT has no randomized 2^{n^ε} algorithm	Khot 05
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It does not rule out approximate algorithms in $f(k) \cdot n^{O(1)}$-time.

Difficulity of showing parameterized inapproximability

1. Most proofs of the classical inapproximability rely on the PCP theorem.
2. Reductions based on the PCP theorem produce instances with optimal solutions of relatively large size, e.g. $k = n^{\Theta(1)}$.
3. In parameterized complexity, we assume the value of k is small, hence k should not depend on n.

A gap-producing reduction

Theorem (main)

We can construct a bipartite graph $H = (A \dot\cup B, E)$ in polynomial time on input an n-vertex graph G and $k \in \mathbb{N}$ with $(k+1)! < n^{\Theta(1/k)}$ s.t.:

1. *if $K_k \subseteq G$, then there are s vertices in A with at least $n^{\Theta(1/k)}$ common neighbors in B;*
2. *if $K_k \not\subseteq G$, every s vertices in A have at most $(k+1)!$ common neighbors in B,*

where $s = \binom{k}{2}$.

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- *This reduction does not use the PCP theorem. It is based on some extremal combinatorics construction.*

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- *It applies in case wtih small value of k.*

Consequence under **ETH**

Theorem (Chen et. al 04)

Assuming **ETH**, *k-Clique cannot be solved in $f(k) \cdot n^{o(k)}$-time for any computable function f.*

Corollary

Assuming **ETH**, Max-k-Subset-Intersection *does not admit $f(k) \cdot n^{o(\sqrt{k})}$-time approximation algorithm with ratio $n^{1/\sqrt{k}}$.*

A variant of main theorem

Fix $\Delta \in \mathbb{N}^+$.

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where $s = \binom{k}{2}$.

Trade-off between running-time and approximation ratio

Theorem

*Assuming **ETH**, MAX-k-SUBSET-INTERSECTION does not admit $f(k) \cdot n^{o(\sqrt{k/\Delta})}$-time approximation algorithm with ratio $n^{\sqrt{\Delta}/\sqrt{k}}$.*

A variant of main theorem

Let $\Delta = 2^k/s$.

Theorem

We can construct a bipartite graph $H = (A \dot\cup B, E)$ in fpt time on input an n-vertex graph G and $k \in \mathbb{N}$ with $(k+1)! < n^{\Theta(1/k)}$ s.t.:

1. *if $K_k \subseteq G$, then there are $2^k = s \cdot \Delta$ vertices in A with at least $n^{\Theta(1/k)}$ common neighbors in B;*
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Corollary

Max-k-Subset-Intersection *does not admit $f(k) \cdot n^{o(\log k)}$-time approximation algorithm to ratio $n^{1/\log k}$ under* **ETH**.

What can we do with this gap?

Inapproximability of other natural parameterized problem

Question

Find gap-preserving fpt-*reduction from* MAX-k-SUBSET-INTERSECTION *to*

- k-CLIQUE
- k-DOMINATING-SET

From $\textsc{Max-}k\textsc{-Subset-Intersection}$ to k-$\textsc{Clique}$?

Question

Is there any fpt-*algorithm* $\mathbb{A}$ *such that on input a bipartite graph* $H = (A \dot\cup B, E)$, *it construct a graph* G *satisfying:*

- *(1) if there exists* $V \in \binom{A}{s}$ *with* $n^{\Theta(1/k)}$ *common neighbors, then* G *contains a* $g(k)$ *clique;*
- *(2) if every* $V \in \binom{A}{s}$ *has at most* $(k+1)!$ *common neighbors, then* G *contains no* $\frac{g(k)}{2}$ *clique.*

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A naive idea: color A (resp. B) with s (resp. $2(k+1)!$) colors, add edges between vertices in A (resp. B) with different colors.

- in case (1), H has a $(s + 2(k+1)!)$-clique;
- in case (2), H has no clique with $> (s + (k+1)!)$ vertices.

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- in case (2), H has no clique with $> (s + (k+1)!)$ vertices.
  Wrong: there might exist $s-1$ **vertices in** A **with** $n^{\Theta(1/k)}$ **common neighbors, leading to a** $(s-1+2(k+1)!)$**-clique**.

From MAX-k-SUBSET-INTERSECTION to k-DOMINATING-SET?

Let $\gamma(G)$ be the size of its minimum dominating set.

Question

Is there any fpt-*algorithm* $\mathbb{A}$ *such that on input a bipartite graph* $H = (A \dot\cup B, E)$*, it construct a graph* G *satisfying:*

- *(i) if there exists* $V \in \binom{A}{s}$ *with* $n^{\Theta(1/k)}$ *common neighbors, then* $\gamma(G) < g(k)$*;*
- *(ii) if every* $V \in \binom{A}{s}$ *has at most* $(k+1)!$ *common neighbors, then* $\gamma(G) > 2g(k)$*.*

where $s = \binom{k}{2}$*.*

Constant inapproximability of dominating set

Theorem (Chen and Lin 15)

There is an algorithm $\mathbb{A}$ such that on input a bipartite graph $H = (A \dot\cup B, E)$, it construct a graph G in $f(k,d) \cdot |H|^{O(c)}$-time satisfying:

- *if there exists $V \in \binom{A}{s}$ with d common neighbors, then $\gamma(G) < (1+\varepsilon)d^c$;*
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where $s = \binom{k}{2}, d = k^{O(k^3)}$.

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Theorem

*Assuming **ETH**, there is no $f(\gamma(G)) \cdot |G|^{O(1)}$-time algorithm which on every input graph G outputs a dominating set of size at most $\sqrt[4+\varepsilon]{\log(\gamma(G))} \cdot \gamma(G)$.*

Previous inapproximability results of dominating set

Results of **Polynomial-time** inapproximability:

Ratio	Assumptions	Ref
$c \log n$	$\mathbf{P} \neq \mathbf{NP}$	Raz and Safra 97
$(1-\varepsilon) \ln n$	$\mathbf{NP} \not\subseteq \mathrm{DTIME}(n^{O(\log \log n)})$	Feige 98
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Parameterized inapproximability of **independent** dominating set problem:

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Remark

Independent dominating set problem is not monotone.

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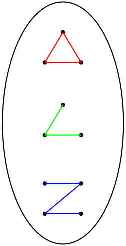
Example ($k = 3, s = 3$)



G

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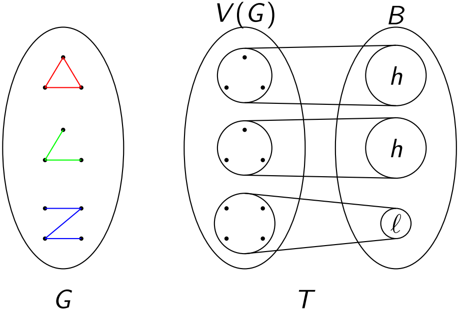

key idea: construct a bipartite graph $T = (V(G) \dot\cup B, E(T))$ satifying:

- T1 $\forall V \in \binom{V(G)}{k+1}$, $|\Gamma(V)| \leq \ell$;
- T2 for a random $V \in \binom{V(G)}{k}$, with high probability $|\Gamma(V)| \geq h$;

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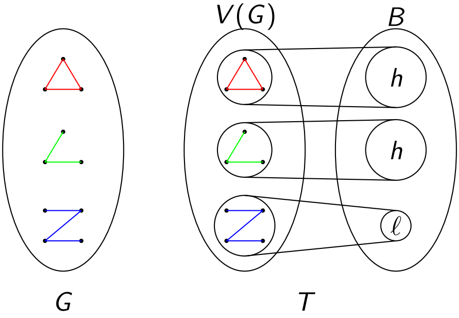

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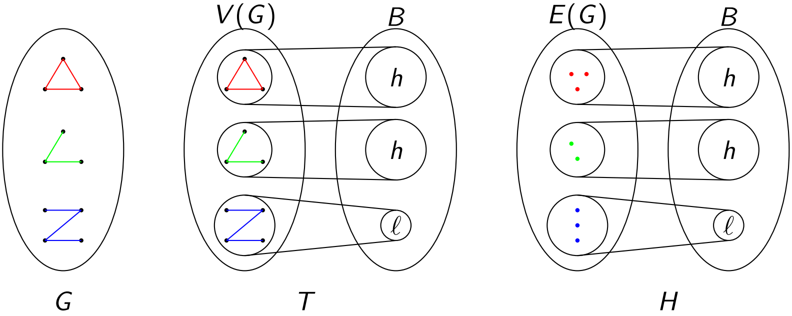

Probabilistic construction of T

Bipartite Random Graph: $T = (A \dot\cup B, E)$

- $|A| = |B| = n$
- $u \in A$ and $v \in B$ is joined by an edge with probability $p = n^{-1/(k+1)}$

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Derandomizing the reduction

Define bipartite graph $T = (A \dot\cup B, E) = ((V_1 \dot\cup V_2 \dot\cup \cdots \dot\cup V_n) \dot\cup B, E)$ satifying:

- T1 every $k+1$ vertices in A has at most ℓ common neighbors;
- T2' for every k distinct indices $i_1, \cdots, i_k$, there exist

$$v_{i_1} \in V_{i_1}, \cdots, v_{i_k} \in V_{i_k}$$

s.t. $v_1, \cdots, v_k$ have at least h common neighbors.

Derandomizing the reduction

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Remark

The reduction can be adapted to T satisfying T1 and T2'.

Lemma

For $\ell = (k+1)! < h = n^{\Theta(1/k)}$, we can construct T satisfying T1 and T2' in polynomial time.

Summary

- We give an fpt gap-producing reduction from k-Clique to Max-k-Subset-Intersection.
- Under **ETH**, we can rule out moderate exponential approximation algorithms for Max-k-Subset-Intersection.
- Inapproximability of other natural parameterized problem.
  - k-Dominating-Set: no constant fpt-approximation

Summary

- We give an fpt gap-producing reduction from k-CLIQUE to MAX-k-SUBSET-INTERSECTION.
- Under **ETH**, we can rule out moderate exponential approximation algorithms for MAX-k-SUBSET-INTERSECTION.
- Inapproximability of other natural parameterized problem.
  - k-DOMINATING-SET: no constant fpt-approximation

Open questions

- Does k-CLIQUE have constant fpt-approximation?
- Does k-DOMINATING-SET have fpt-approximation with ratio $\rho(k)$?

Thank You!