(S)ETH and A Survey of Consequences

Mohan Paturi

Simons Institute, August 2015

Outline

1. Exact Algorithms and Complexity
2. Exponential-Time Hypothesis
3. Explanatory Value of **ETH** and **SETH**
4. Probabilistic Polynomial Time Algorithms
5. Open Problems

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- What are the obstructions that limit the improvements?
- What principles explain the exact complexities of **NP**-complete problems?

Exact Complexity — Parametrization of **NP** Problems

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- Any **NP**-complete problem can be reduced to Circuit Sat preserving the complexity parameter exactly.

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- Also known as moderately exponential-time or nontrivial exponential-time algorithms

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- k-SAT, number of variables as the complexity parameter —
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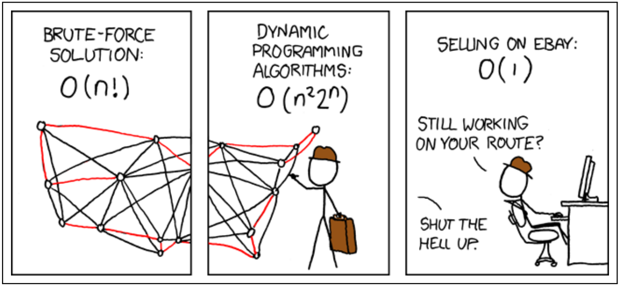

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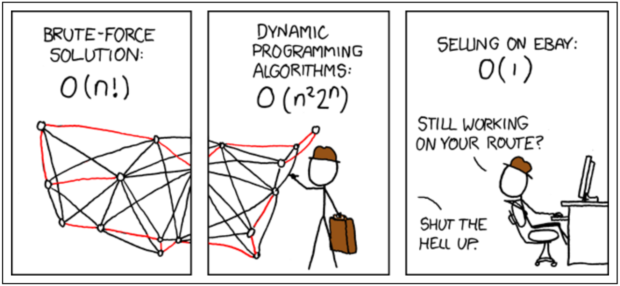

  - undirected HAMILTONIAN PATH — $2^{0.67n}$

Improved Algorithms for COLORABILITY

- COLORABILITY, number of vertices as the complexity parameter — dynamic programming, Möbius inversion
  Björklund, Husfeldt, Kaski, Koivisto (2006-2008), $\cdots$, Byskov (2004), Eppstein (2003), Lawler (1976)
    - COLORABILITY — 2^n in exponential space

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  - Depth-2 Threshold Circuit Satisfiability for circuits with cn

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  Lower bounding the exponent for 3-SAT under suitable complexity assumptions?
- Is progress on different problems connected?
  Do improved algorithms for 3-SAT imply improved algorithms for 3-Colorability or vice versa?
  If Colorability has a 2^{cn} algorithm, can we prove CNFSAT has a 2^{dn} algorithm for some $c, d < 1$?

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- A priori, it is not clear that we can expect a common principle to govern the exact complexities.

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- Complexity parameter increases polynomially, thus preventing any useful conclusion about 3-SAT.

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Lemma (Sparsification Lemma, IPZ 1997)

$\exists$ algorithm A $\forall k \geq 2, \epsilon \in (0,1], \phi \in k\text{-}\mathrm{CNF}$ with n variables, $A_{k,\epsilon}(\phi)$ outputs $\phi_1, \ldots, \phi_s \in k\text{-}\mathrm{CNF}$ in $2^{\epsilon n}$ time such that

1. $s \leq 2^{\epsilon n}$; $\mathbf{Sol}(\phi) = \bigcup_i \mathbf{Sol}(\phi_i)$, *where* $\mathbf{Sol}(\phi)$ *is the set of satisfying assignments of* ϕ
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- Key ideas: subexponential time reductions, sparsification

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- **SNP** includes k-SAT and k-COLORABILITY for $k \geq 3$. $\exists S \forall (y_1, \ldots, y_k) \forall (s_1, \ldots, s_k) [R_{(s_1, \ldots, s_k)}(y_1, \ldots, y_k) \implies \wedge_{1 \leq i \leq k} S_{s_i}(y_i)$, where $s_i \in \{+, -\}$ and S is a subset of $[n]$.

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- **SNP** includes k-SAT and k-COLORABILITY for $k \geq 3$. $\exists S \forall (y_1, \ldots, y_k) \forall (s_1, \ldots, s_k) [R_{(s_1,\ldots,s_k)}(y_1, \ldots, y_k) \implies \wedge_{1 \leq i \leq k} S_{s_i}(y_i)$, where $s_i \in \{+, -\}$ and S is a subset of $[n]$.
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Theorem (Impagliazzo, P, and Zane (1977))

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- Furthermore, **SNP**-hard problems such as HAMILTONIAN PATH cannot have a subexponential time algorithm.

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  - ideally, by providing lower bounds that match the upper bounds from the best known algorithms.
- **ETH** will be useful if it helps factor out the essential difficulty of dealing with exponential time algorithms for **NP**-complete problems.

Lower Bounds based on **ETH** — I

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- Lower bounds for $W[1]$-complete problems: There is no $f(k)n^{o(k)}$ algorithm for Clique or Independent Set. Chen, Chor, Fellows, Huang, Juedes, Kanj, and Xia (2005, 2006)

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- Lower bounds for $W[2]$-complete problems: There is no $f(k)n^{o(k)}$ algorithm for DOMINATING SET. Fellows (2011), Lokshtanov (2009)

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- Provides evidence to the observation that heuristics for k-SAT perform worse as k increases.
- Proof Sketch: Trade clause width up to reduce the number of variables: reduce k-SAT to k'-CNF for $k' \gg k$ such that the resultant formula has fewer variables.

Complexity of CSP Problems

- **ETH** implies that $(d, 2)$-CSP requires d^{cn} time where c is an absolute constant. The constant c depends on s_3. Traxler 2008
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- Greater expressiveness of k'-CNF and $(d', 2)$-CSP has been exploited.

SETH — Strong Exponential Time Hypothesis

- Earlier result regarding the increasing complexity of k-SAT tempts one to hypothesize
  SETH — Strong Exponential Time Hypothesis: $s_\infty = 1$

SETH and Its Equivalent Statements

Theorem

The following statements are equivalent:

- $\forall \varepsilon < 1$, $\exists k$, k-SAT, *the satisfiability problems for n-variable* k-CNF *formuals, cannot be computed in time* $O(2^{\varepsilon n})$ *time.*

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— Cygan, Dell, Lokshtanov, Marx, Nederlof, Okamoto, P, Saurabh, Wahlstrom, 2012

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SETH *determines the exact complexities of the following problems in* **P**.

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- What is the best success probability achievable in OPP or OP$(T(n, m))$?

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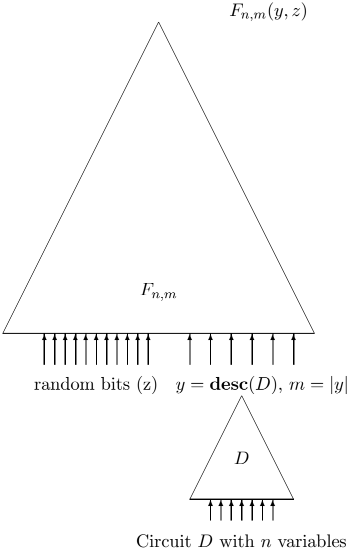
- For what problems, does this quantity decrease/stay the same over a certain range of time?
- We present (weak) evidence that for CIRCUIT SAT, $(\log t + \log 1/p)/n$ may not significantly decrease with increasing time.

Circuit Family for deciding CIRCUIT SAT



Probabilistic Circuit for **CircuitSat**

Success Probability for CIRCUIT SAT with OPP algorithms

Theorem (P, Pudlák 2010)

If CIRCUIT SAT *can be decided with probabilistic circuits of size m^k for some k with success probability $2^{-\delta n}$ for $\delta < 1$, then there exists a $\mu < 1$ depending on k and δ such that* CIRCUIT SAT(n, m) *can be decided by deterministic circuits of size $2^{O(n^{\mu} \lg^{1-\mu} m)}$.*

Results: Quasilinear Size Circuits

Theorem

If $\textsc{Circuit Sat}$ *can be decided with probabilistic circuits of size* $\tilde{O}(m)$ *with success probability* $2^{-\delta n}$ *for* $\delta < 1$, *then* $\textsc{Circuit Sat}(n, m)$ *can be decided by deterministic circuits of size* $O(\texttt{poly}(m) n^{O(\lg \lg m)})$.

Results: Quasilinear Size Circuits

Theorem

If CIRCUIT SAT *can be decided with probabilistic circuits of size* $\tilde{O}(m)$ *with success probability* $2^{-\delta n}$ *for* $\delta < 1$, *then* CIRCUIT SAT(n, m) *can be decided by deterministic circuits of size* $O(\mathtt{poly}(m) n^{O(\lg \lg m)})$.

- The consequence is very close to the statement $\mathbf{NP} \subseteq \mathbf{P}/\mathtt{poly}$.

Success Probability for Circuit Sat with Subexponential Size Circuits

Theorem (P, Pudlák 2010)

If Circuit Sat *can be decided with probabilistic circuits of size* $2^{o(n)}\tilde{O}(m)$ *with success probability* $2^{-\delta n}$ *for* $\delta < 1$*, then* Circuit Sat(n, m) *can be decided by deterministic circuits of size* $2^{o(n)}\mathtt{poly}(m)$.

Exponential Amplification Lemma

Lemma (P and Pudlák 2010)

Exponential Amplification Lemma: *Let $\mathcal{F}$ be an f-bounded family for some $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ that decides* Circuit Sat *with success probability $2^{-\delta n}$ for $0 < \delta < 1$. Then there exists a g-bounded circuit family $\mathcal{G}$ that decides* Circuit Sat *with success probability at least $2^{-\delta^2 n}$ where $g(n, m) = O(f(\lceil \delta n \rceil + 5, \tilde{O}(f(n, m))))$.*

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- $\mathcal{F} : (f(n, m), \delta n) \to \mathcal{G} : (g(n, m), \delta^2 n)$

Drucker's Recent Result

Theorem (Drucker, 2013)

For any $\mu < 1$, if there is an OPP algorithm which takes the description of a 3-$\textsc{SAT}$ *formula of length m as input and decides its satisfiability with success probability at least $2^{-m^{\mu}}$, then* $\mathbf{NP} \subseteq \mathbf{coNP}/\mathtt{poly}$

Other Connections

- Hardest instances
- Satisfiability and circuit lower bounds
- $\cdots$

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- Assuming **SETH**, can we prove a 2^n lower bound on COLORABILITY?
- Are there better non-**OPP** algorithms for k-SAT or CIRCUIT SAT?
- Does there exist a c^{-n} success probability **OPP** algorithm for HAMILTONIAN PATH?

Thank You