# Multilinear Maps over the Integers
# From Design to Security

Tancrède Lepoint    CryptoExperts

The Mathematics of Modern Cryptography Workshop, July 10th 2015

# Timeline: The Hype Cycle of Multilinear Maps

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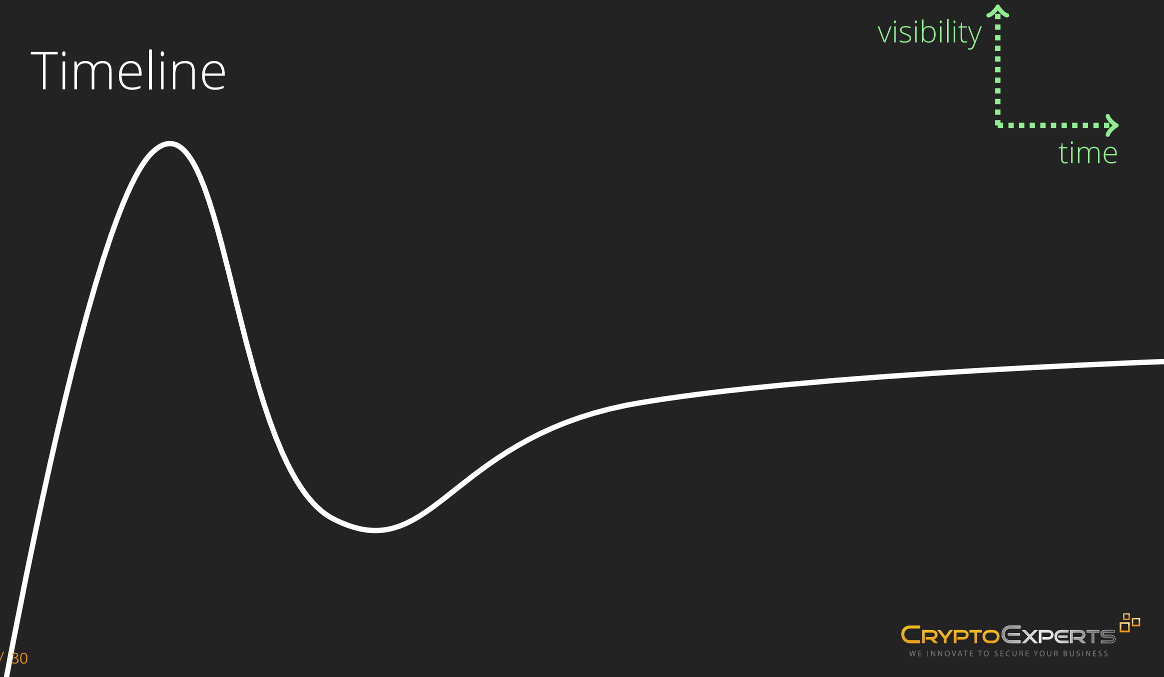




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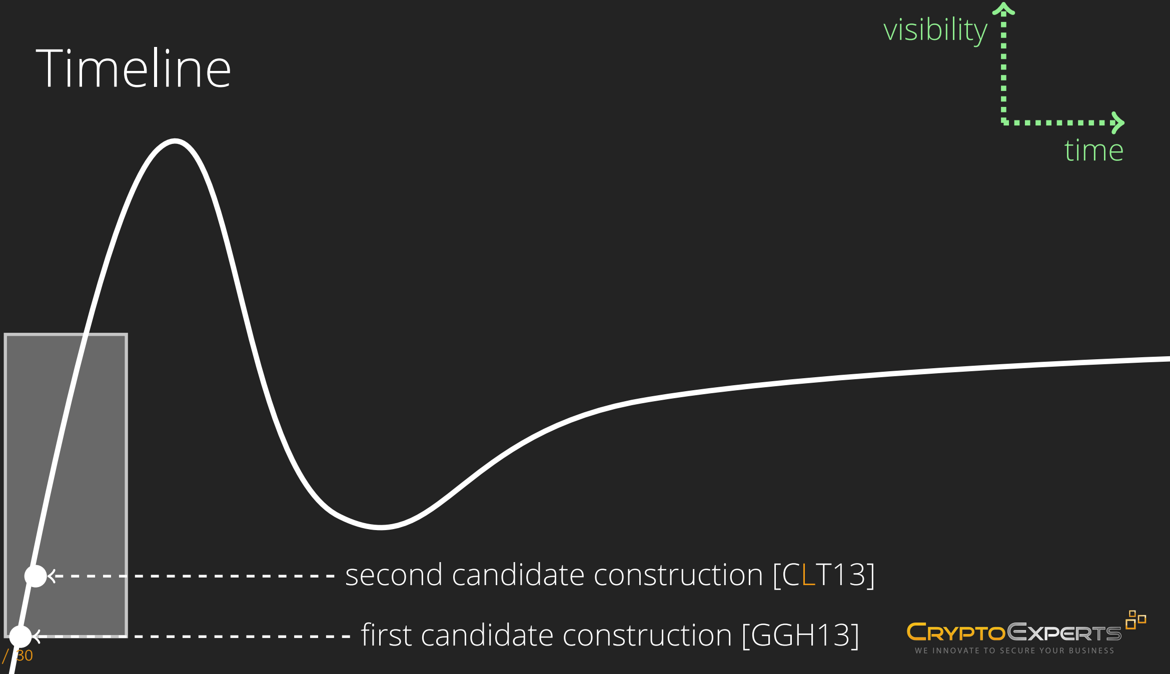


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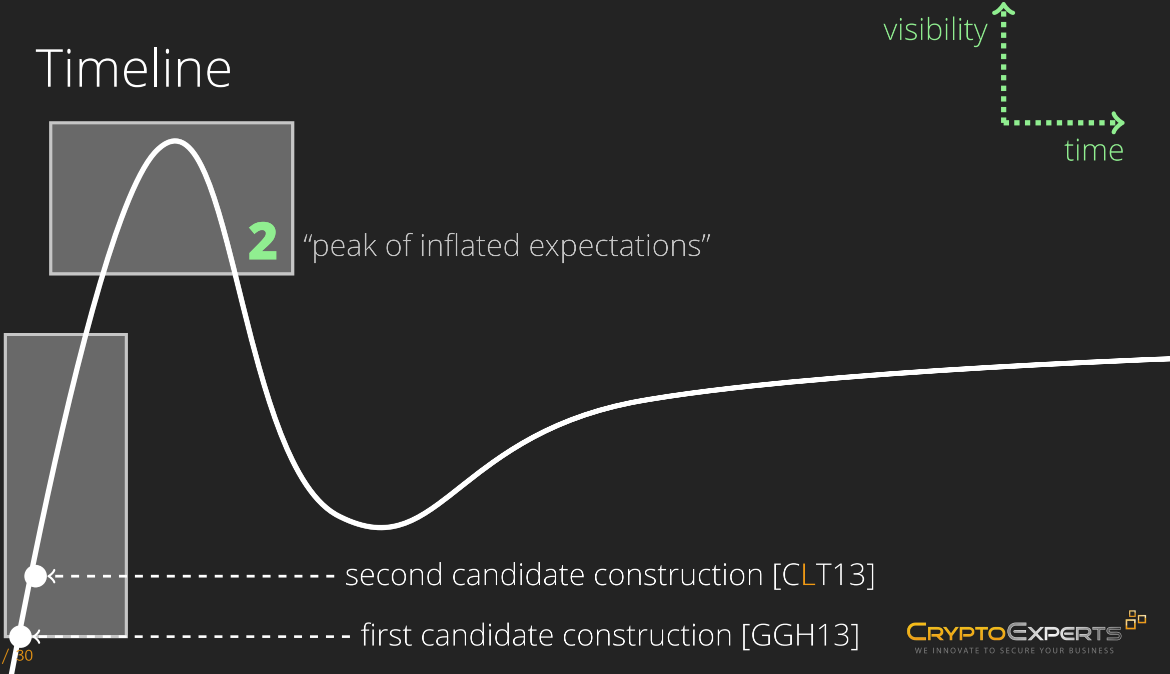


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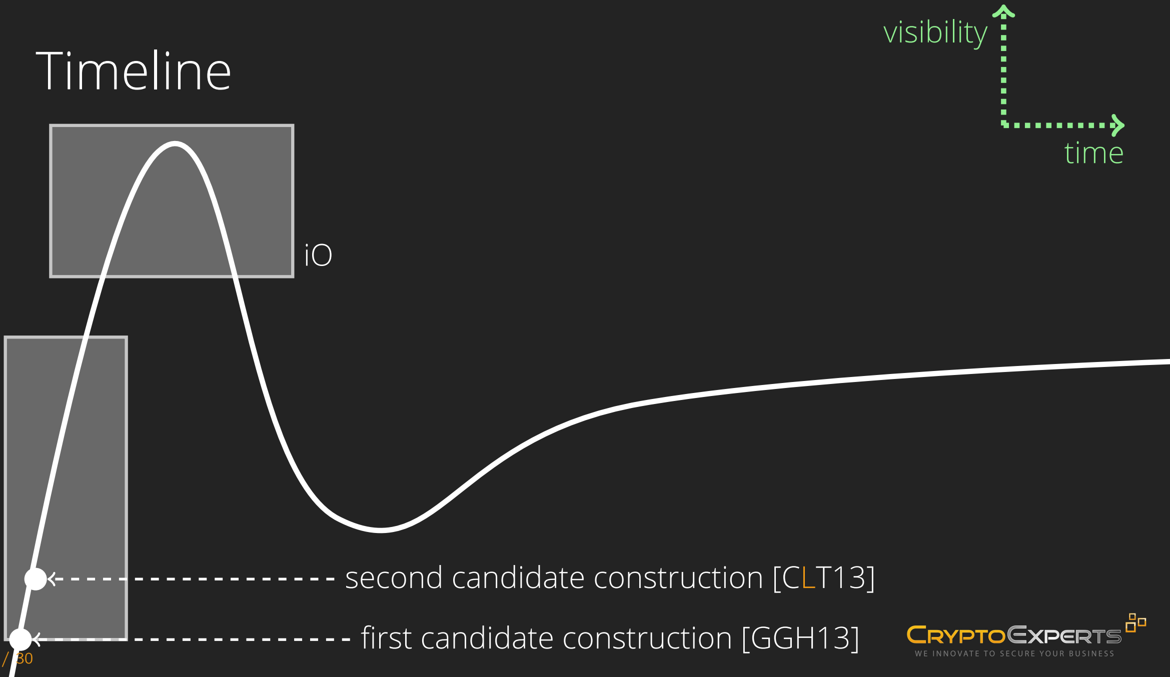
visibility
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second candidate construction [CLT13]
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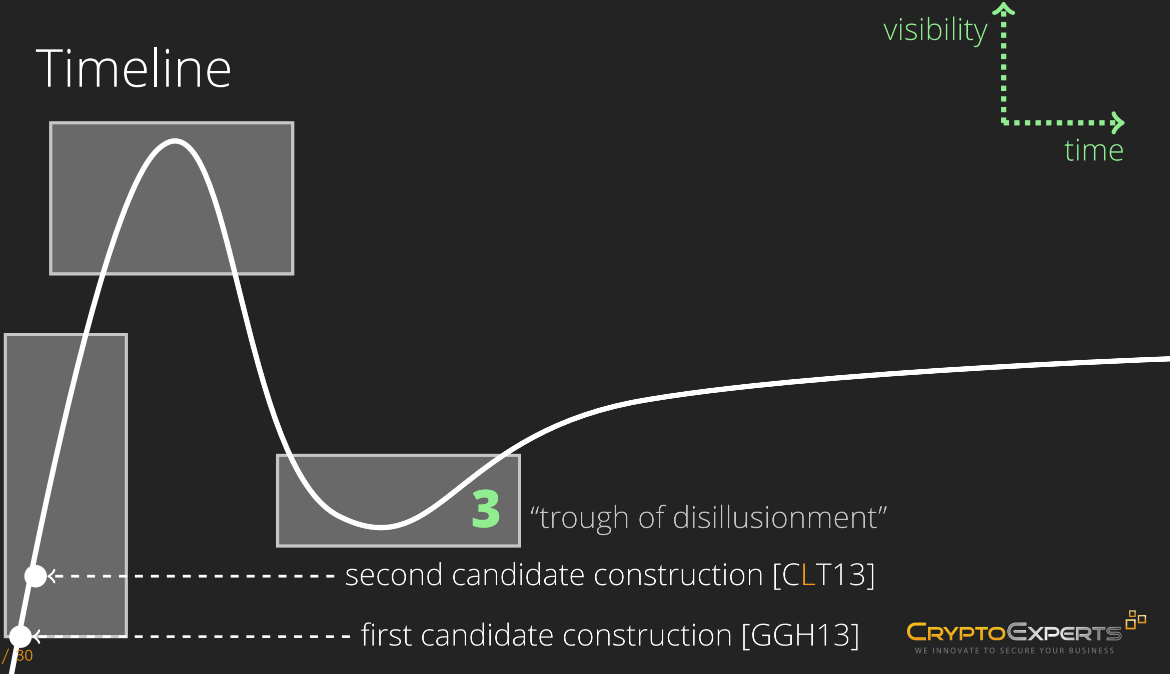
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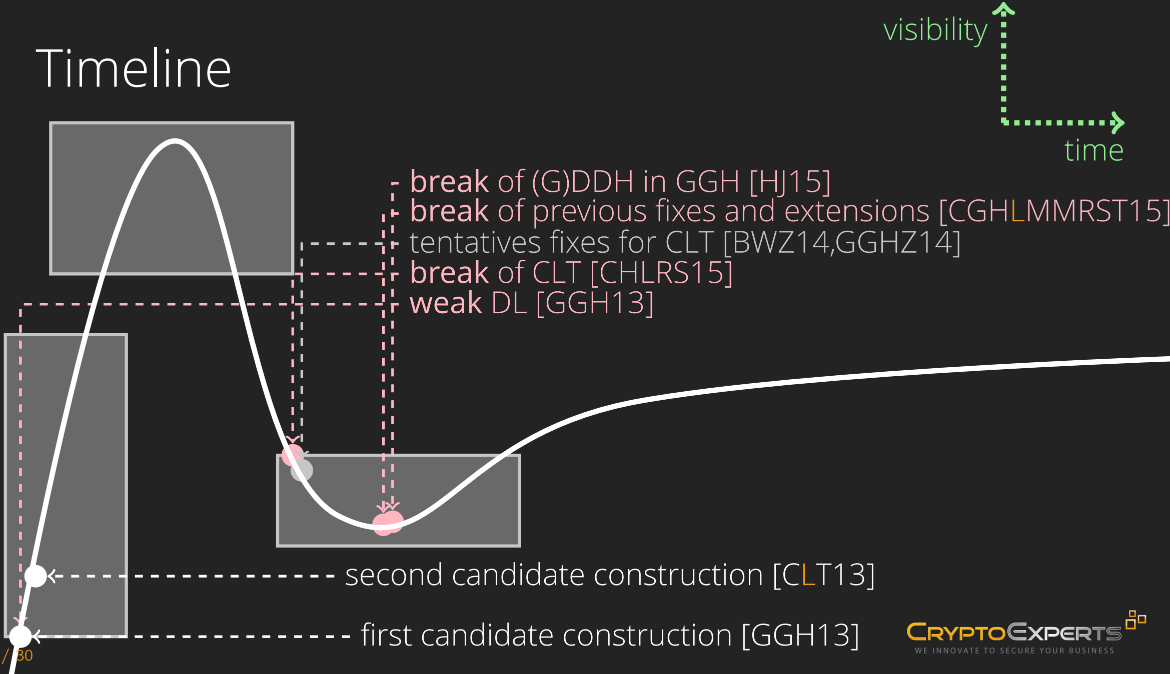
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break of (G)DDH in GGH [HJ15]
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tentatives fixes for CLT [BWZ14,GGHZ14]
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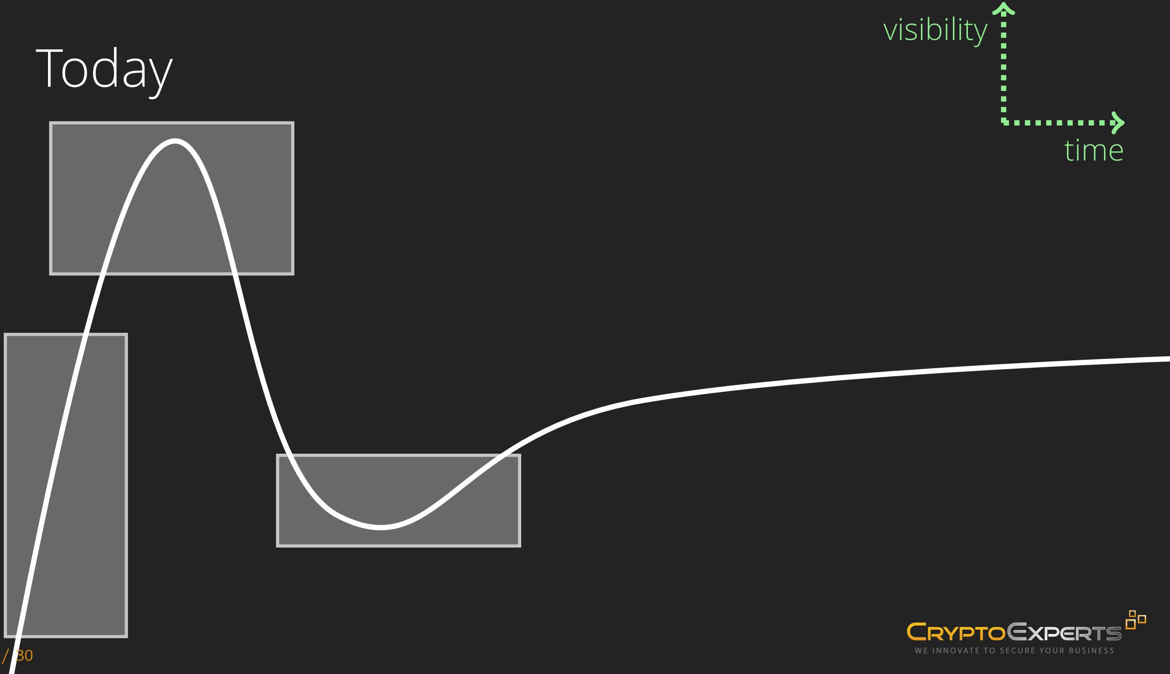

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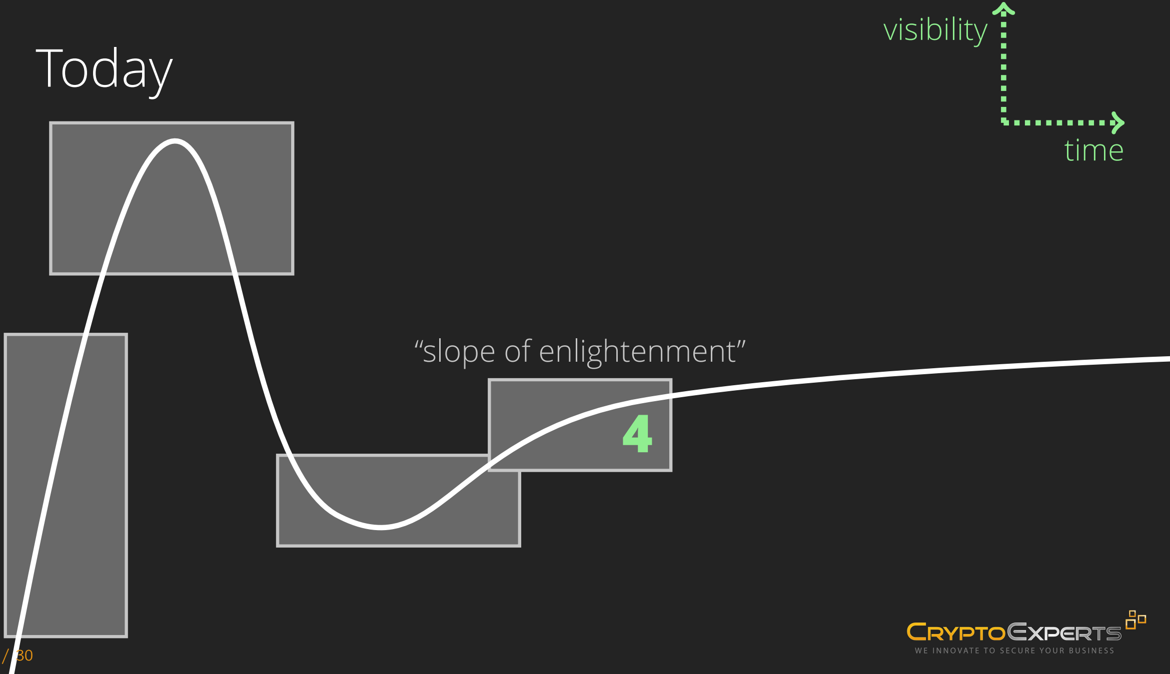


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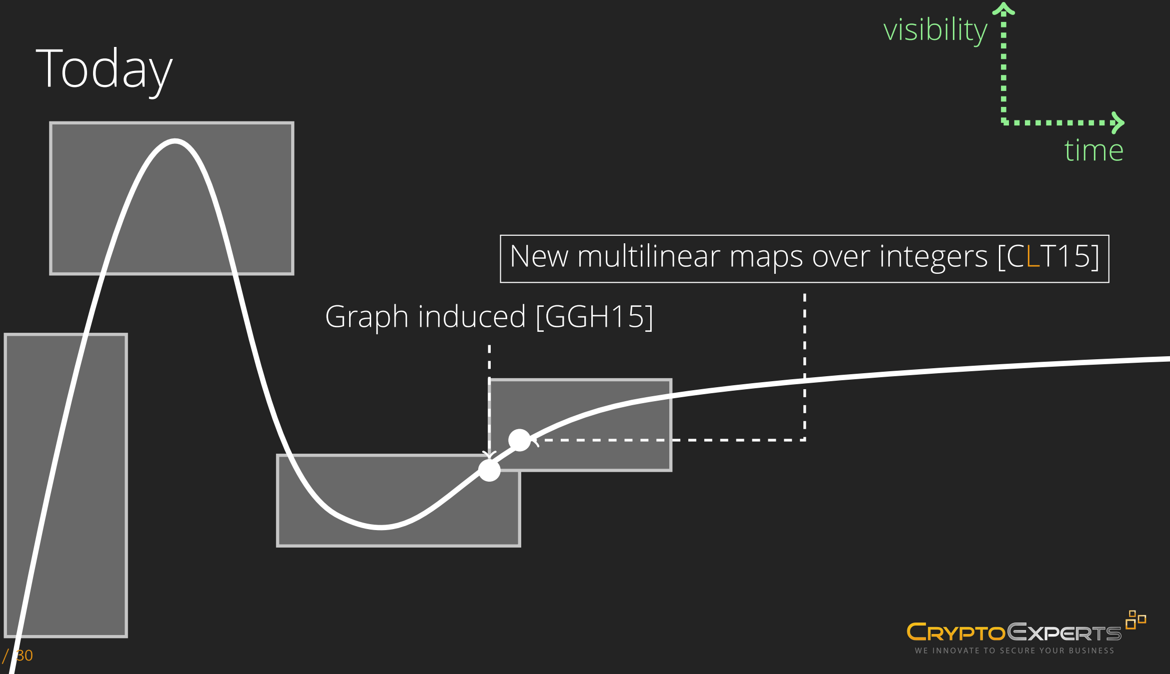

visibility
time
New multilinear maps over integers [CLT15]
Graph induced [GGH15]

# The CLT Scheme

## Multilinear maps over the integers

[CoronL.Tibouchi'13'15]

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Useful for **many** applications...

# SWHE vs. MMAPs

## Computation over encrypted data

We want to compute homomorphically over encrypted data

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| | |
|---|---|
| MMAPS | can test if it is zero, at level $k$ (and hard to compute at degree $> k$) |
| SHWE | no information on $a$ from the result, except with secret key |

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# Starting from Homomorphic Encryption

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for $q \leftarrow [0, q_0)$ and $r \leftarrow \chi$ "small"

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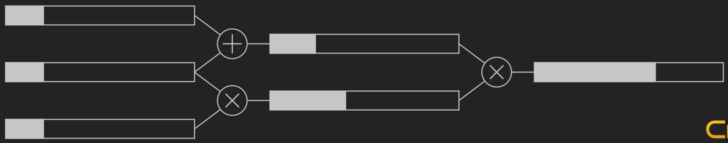

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# Adding Sharp Levels

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$$[\vec{m}]_j = c/z^j \bmod x_0 = \frac{\mathrm{CRT}_{q,p_1,\ldots,p_n}(q', r_1 \cdot g_1 + m_1, \ldots, r_n \cdot g_n + m_n)}{z^j} \bmod x_0$$

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Operations over $\mathbb{Z}_{x_0}$:

| | | |
|---|---|---|
| Addition | $[\vec{m}]_j + [\vec{m}']_j$ | $\simeq [\vec{m} + \vec{m}']_j$ |
| Multiplication | $[\vec{m}]_{j_1} \times [\vec{m}']_{j_2}$ | $\simeq [\vec{m} \cdot \vec{m}']_{j_1+j_2}$ |

# Main Ingredient: **Testing for Zero**

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How to test whether two degree-$k$ encodings are equal?

$$[\vec{m}]_k \simeq [\vec{\ell}]_k \text{ (i.e. } \vec{m} = \vec{\ell}) \iff [\vec{m} - \vec{\ell}]_k \simeq [\vec{0}]_k$$

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Idea of [GGH13]: multiply by an element which will cancel $z^k$ and when the $r_i$'s are small ($r_i g_i \ll p_i$), yield something small compared to $x_0$.

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# Simplifications for Zero-Testing

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$$[\vec{0}]_k = \sum_i g_i r_i \cdot ({p_i^*}^{-1}/z^k \bmod p_i) \cdot p_i^* + (\prod p_j) \cdot q'' \bmod x_0$$

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In the following $x_0 = \prod p_j$, and

$$[\vec{m}]_j = c/z^j \bmod x_0 = \frac{\mathrm{CRT}_{p_1,\ldots,p_n}(r_1 \cdot g_1 + m_1, \ldots, r_n \cdot g_n + m_n)}{z^j} \bmod x_0$$

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# Zero-Testing Procedure

Multiply by the public element (where $h_i \ll p_i$)

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We have (we prove equivalence whp when many $p_{zt}$'s are given)

$$\vec{m} = \vec{0} \quad \Rightarrow \quad \left| [\vec{m}]_k \cdot p_{zt} \bmod x_0 \right| \ll x_0$$

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# Hardness Assumptions

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**GDDH:** Given $(k+1)$ elements $[\vec{m}_i]_1$ and $[\vec{m}']_k$, determine whether $\vec{m}' \simeq \prod_{i=1}^{k+1} \vec{m}_i$.

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Asymptotic parameters obtained from numerous attacks
- orthogonal lattice attack on encodings
- GCD attack on zero-testing
- hidden subset sum attack on zero-testing
- attacks on the inverse zero-testing matrix
- brute-force on the noises, ...

# But… Zeroizing Attack

## Eurocrypt 2015 best paper

[CHLRS15]

**Cryptanalysis of the Multilinear Map over the Integers**

Jung Hee Cheon[1], Kyoohyung Han[1], Changmin Lee[1], Hansol Ryu[1], Damien Stehlé[2]

[1] Seoul National University (SNU), Republic of Korea
[2] ENS de Lyon, Laboratoire LIP (U. Lyon, CNRS, ENSL, INRIA, UCBL), France.

**Abstract.** We describe a polynomial-time cryptanalysis of the (approximate) multilinear map of Coron, Lepoint and Tibouchi (CLT). The attack relies on an adaptation of the so-called *zeroizing* attack against the Garg, Gentry and Halevi (GGH) candidate multilinear map. Zeroiz-

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## Exploiting the (bi)linearity of the zero-testing procedure

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$$[\vec{0}]_{k-2} \cdot [\vec{b}]_1 \cdot [\vec{c}]_1 \cdot p_{zt} \;= \sum_i r_i \cdot \hat{b}_i \cdot \hat{c}_i \cdot (h_i \cdot p_i^*) \in \mathbb{Z}$$

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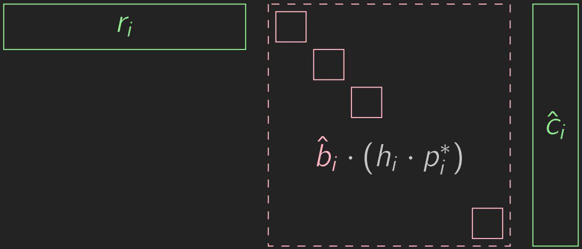

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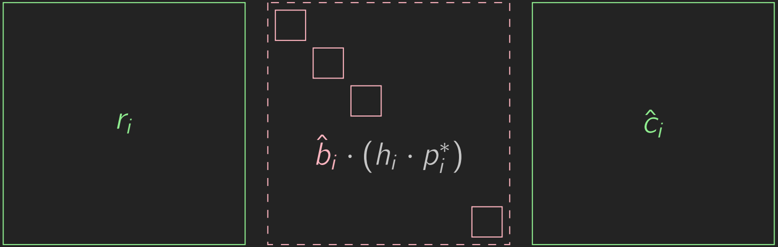

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## Inversion over $\mathbb{Q}$

Let's do it with many $[\vec{0}]_{k-2}$, $[\vec{c}]_1$ and two targets $[\vec{b}]_1$, $[\vec{b}']_1$

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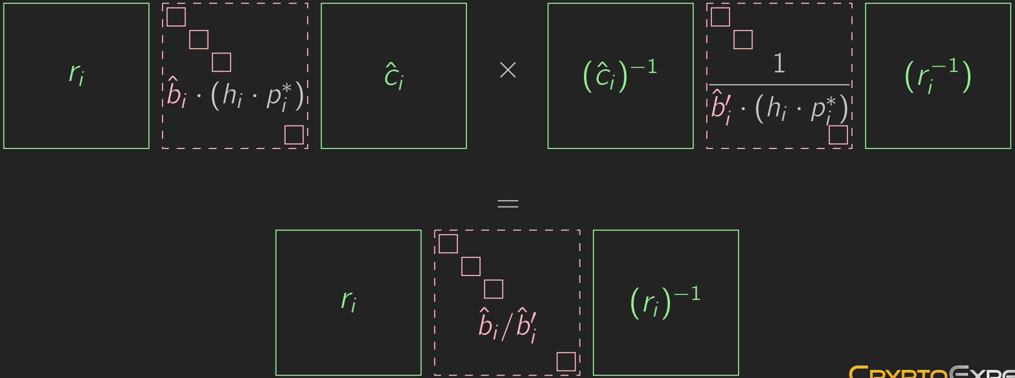

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## Computing eigenvalues

Consider the target encodings

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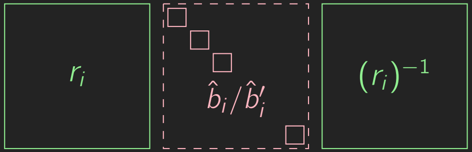

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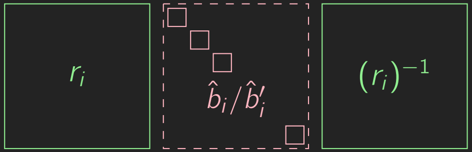


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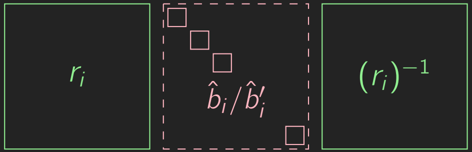


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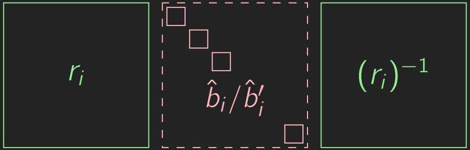


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$$p_i \mid (\beta'_i \cdot [\vec{b}]_1 - \beta_i \cdot [\vec{b}']_1)$$

Compute

$$p_i = \gcd(\beta'_i \cdot [\vec{b}]_1 - \beta_i \cdot [\vec{b}']_1, x_0)$$

# Generalizing the Zeroizing Attack on CLT13

## Zeroizing without low-level zeroes [CGHLMMRST15]

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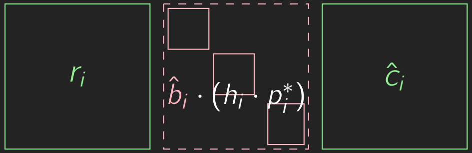
Can be diagonal per block. Instead of computing eigenvalues use **characteristic polynomial**.

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# Thwarting Cheon et al. Attack?

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In [CoronL.Tibouchi15], we revisit the **zero-testing procedure** itself



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In a nutshell:

- the zero-testing is done modulo a new prime modulus $N$;
- $x_0$ is no longer public.

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# Inherent randomness in current encodings

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Current form of encodings

$$[\vec{m}]_k = \mathrm{CRT}_{p_i}(m_i + g_i r_i)/z^k \bmod x_0$$

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with $u_i = (g_i {p_i^*}^{-1} z^{-k} \bmod p_i) p_i^*$.

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with $u_i = (g_i p_i^{*-1} z^{-k} \bmod p_i) p_i^*$.

The element $a$ is highly non-linear in the $r_i$'s

The element $a$ is different from the random $q'$ we had before when adapting DGHV ($\vec{m} = \vec{0} \leftrightarrow a$ is small)

# New Zero-Test Parameter

Pick a random, large prime $N \gg x_0$. We want to generate a new zero-test value $\alpha_{zt}$ such that

$$\left|[\vec{m}]_k \cdot \alpha_{zt} \bmod N\right| \ll N \iff \vec{m} = 0$$

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In particular, we have

$$\begin{aligned}&[\vec{m}]_k \cdot \alpha_{zt} \bmod N \\ &= \sum_i (m_i g_i^{-1} + r_i \bmod p_i) \cdot (u_i \cdot \alpha_{zt}) + a \cdot x_0 \cdot \alpha_{zt} \bmod N\end{aligned}$$

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so we want $|\alpha_{zt} \cdot u_i \bmod N| \ll N$ and $|\alpha_{zt} \cdot x_0 \bmod N| \ll N$

# How To Generate $\alpha_{zt}$?

Given $N$, the generation of $\alpha_{zt} \in \mathbb{Z}_N$ such that for all $i$, $|u_i \alpha_{zt} \bmod N|$ and $|x_0 \alpha_{zt} \bmod N|$ are small is not obvious.

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The problem amounts to finding a relatively short vector in a lattice

$$\begin{pmatrix} 1 & u_1 & \cdots & u_n & x_0 \\ & N & & & \\ & & \ddots & & \\ & & & N & \\ & & & & N \end{pmatrix}$$

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Use LLL? (we can tolerate an exponential approx. factor over SVP), but typically $n \geq 10^5$

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First note that $p_j^{-1} u_i \bmod N$ is small for all $i \neq j$
Only $p_j^{-1} u_j \bmod N$ is not a priori small

Let us find $\alpha_j$ such that $\alpha_j \cdot p_j^{-1} u_j \bmod N$ is small
As before it amounts to finding a short vector in

$$\begin{pmatrix} \lceil N/B \rceil & p_j^{-1} u_j \\ & N \end{pmatrix}$$

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$$\binom{\lceil N/B \rceil \quad p_j^{-1} u_j}{N}$$

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$$\begin{pmatrix} \lceil N/B \rceil & p_j^{-1} u_j \\ & N \end{pmatrix}$$

We chose $B$ such that LLL finds a short vector

$$(\alpha_j \cdot \lceil N/B \rceil, \beta_j)$$

where $|\alpha_j| \leq \sqrt{p_j}$ and $|\beta_j = \alpha_j \cdot p_j^{-1} u_j \bmod N| \leq N/\sqrt{p_j}$.

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$$\alpha_{zt} = \sum_j h_j \cdot \alpha_j \cdot p_j^{-1} \bmod N$$

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# How To Generate $\alpha_{zt}$? Using the structure of the $u_i$'s

New zero-testing element (sizes to keep in mind $N \approx x_0 \cdot p_j, \quad \alpha_j \approx \sqrt{p_j}$):

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When applied on an encoding $[\vec{m}]_k$:

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# An Important Caveat

## Cannot work directly modulo $x_0$

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$x_0$ cannot be made public, contrary to [CLT13]

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However, define $v_0 = x_0 \cdot \alpha_{zt} \bmod N$, and

$$
\begin{aligned}
&([\vec{0}]_k \cdot \alpha_{zt} \bmod N) \bmod v_0 \\
&= \Big(\sum_i r_i \cdot \big(h_i\beta_i + \sum_{j \neq i} h_j\alpha_j \cdot u_i/p_j\big) + a \cdot v_0 \in \mathbb{Z}\Big) \bmod v_0 \\
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We can apply Cheon et al. attack modulo $v_0$

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Making $x_0$ secret is somewhat inconvenient:
when we add or multiply encodings, we cannot reduce them modulo $x_0$ anymore to keep them of the same size

Solution (taken from [DGHV10]): publish a ladder of encodings of 0 of increasing size

- encodings
$$X_i^{(j)} = (\mathrm{CRT}_{p_i}(r_i g_i)/z^j \bmod x_0) + q_i \cdot x_0$$
with $q_i \leftarrow [0, 2^i)$ for $i = 1, \ldots, \log(x_0)$
- do the operation over $\mathbb{Z}$, and remove $X_i^{(j)}$ for decreasing $i$'s

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Consider $u = [\vec{0}]_{k-2} \cdot [\vec{b}]_1 \cdot [\vec{c}]_1$

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Consider $u = [\vec{0}]_{k-2} \cdot [\vec{b}]_1 \cdot [\vec{c}]_1$

Apply the ladder to reduce its size to the size of $x_0$:

$$u' = u + \sum s_i X_i^{(k)}$$

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Write $u'$ over $\mathbb{Z}$:

$$u' = \sum_i (r_i \cdot \hat{b}_i \cdot \hat{c}_i + s_i \cdot r_{X,i,k}) \cdot u_i - a \cdot x_0$$

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All $s_i$'s and $a$ come up in the way of Cheon et al. attack



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# Proof-of-concept Implementation

https://github.com/tlepoint/new-multilinear-maps

| Instantiation | $\lambda$ | $\kappa$ | $n$ | $\eta$ | $\Delta$ | $\rho$ | $\gamma = n \cdot \eta$ | pp size |
|---|---|---|---|---|---|---|---|---|
| Small | 52 | 6 | 540 | 1679 | 23 | 52 | $0.9 \cdot 10^6$ | 27 MB |
| Medium | 62 | 6 | 2085 | 1989 | 45 | 62 | $4.14 \cdot 10^6$ | 175 MB |
| Large | 72 | 6 | 8250 | 2306 | 90 | 72 | $19.0 \cdot 10^6$ | 1.2 GB |
| Extra | 80 | 6 | 25305 | 2619 | 159 | 85 | $66.3 \cdot 10^6$ | 6.1 GB |

| Setup | Publish | KeyGen |
|---|---|---|
| 5.9 s | 0.10 s | 0.17 s |
| 36 s | 0.33 s | 1.06 s |
| 583 s | 2.05 s | 6.17 s |
| 4528 s | 7.8 s | 23.9 s |

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Open problems for CLT15:

- Analyze the reparation
- Improve the efficiency
- Adapt the technique to [GGH13]?

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# Thank You

## Questions & Discussion




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# Discussion

1. Design
   - public encoding space / inversion
2. Attacks
3. Assumptions
   - what sort of assumptions can be made?
   - base multilinear maps on well-known problems
4. Applications
   - something that look different from obfuscation
   - what can you do with a small number of levels?
   - relation between 2-multilinear maps / pairings in applications