Recovering Short Generators of Principal Ideals in Cyclotomic Rings

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Short Generators of Ideals in Cryptography

A few recent lattice-related cryptoschemes [SV10, GGH13, LSS14, CGS14] share this KeyGen:

- sk Choose a "*short*" g in some ring R (e.g., $R = \mathbb{Z}[X]/(X^n + 1)$)
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Not obvious *a priori* that g is even uniquely defined. But any short enough element in $\mathcal{I}$ suffices to break system.

Cost of the Two Steps

1. Principal Ideal Problem (find some generator h)
   - ⋆ Subexponential $2^{\tilde{O}(n^{2/3})}$-time classical algorithm [BF14, Bia14].
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This Work: Main Theorem

In cryptographic setting, SGP can be solved in *classical polynomial time*, for any prime-power cyclotomic number ring $R = \mathbb{Z}[\zeta_{p^k}]$.

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1. Devising hard distributions of lattice problems is very tricky: exploitable structure abounds!
2. Worst-case hardness protects us from weak instances.

Agenda

1. Introduction
2. Log-Unit Lattice
3. Attack and Proof Outline

(Logarithmic) Embedding

Let $K \cong \mathbb{Q}[X]/f(X)$ be a number field of degree n and let $\sigma_i \colon K \mapsto \mathbb{C}$ be its n complex embeddings. The *canonical embedding* is

$$\begin{aligned} \sigma \colon K &\to \mathbb{C}^n \\ x &\mapsto (\sigma_1(x), \ldots, \sigma_n(x)). \end{aligned}$$

The *logarithmic embedding* is

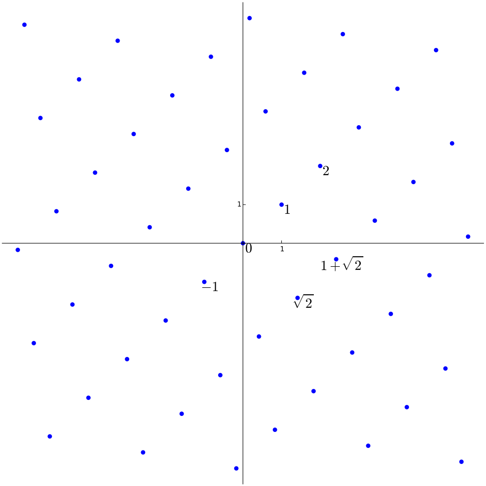
$$\begin{aligned} \mathrm{Log} \colon K \setminus \{0\} &\to \mathbb{R}^n \\ x &\mapsto (\log|\sigma_1(x)|, \ldots, \log|\sigma_n(x)|). \end{aligned}$$

It is a group homomorphism from $(K \setminus \{0\}, \times)$ to $(\mathbb{R}^n, +)$.

Example: Power-of-2 Cyclotomics

- $K \cong \mathbb{Q}[X]/(X^n + 1)$ for $n = 2^k$.
- $\sigma_i(X) = \omega^{2i-1}$, where $\omega = \exp(\pi\sqrt{-1}/n)$.
- $\mathrm{Log}(X^j) = \vec{0}$ and $\mathrm{Log}(1 - X) =$ [whiteboard]

Example: Embedding $\sigma(\mathbb{Z}[\sqrt{2}]) \subset \mathbb{R}^2$



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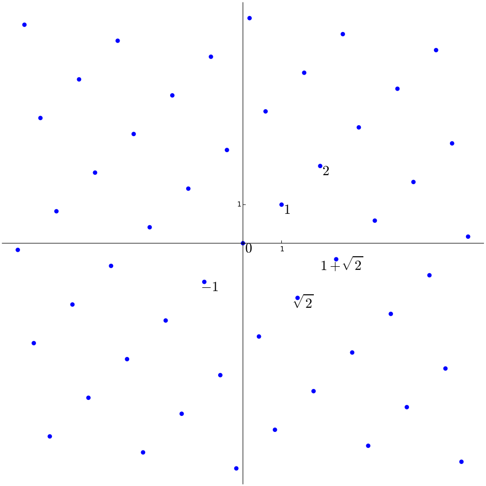

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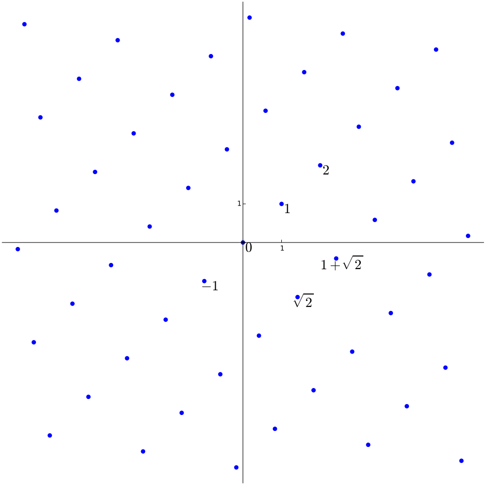

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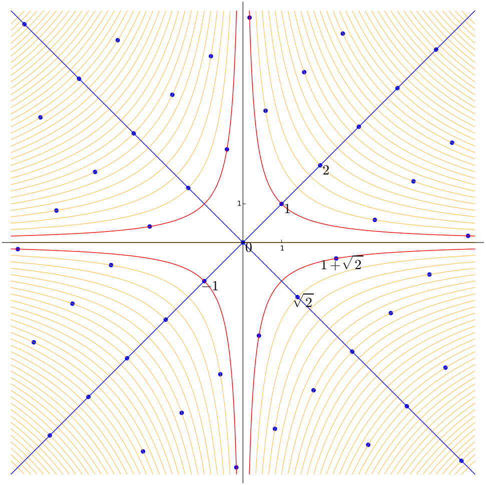

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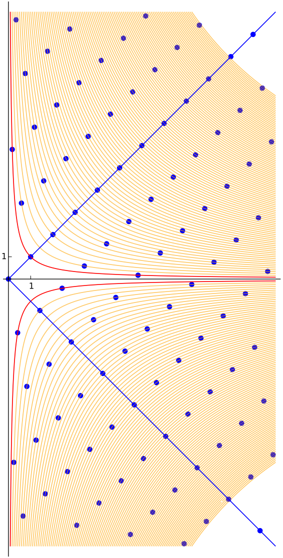
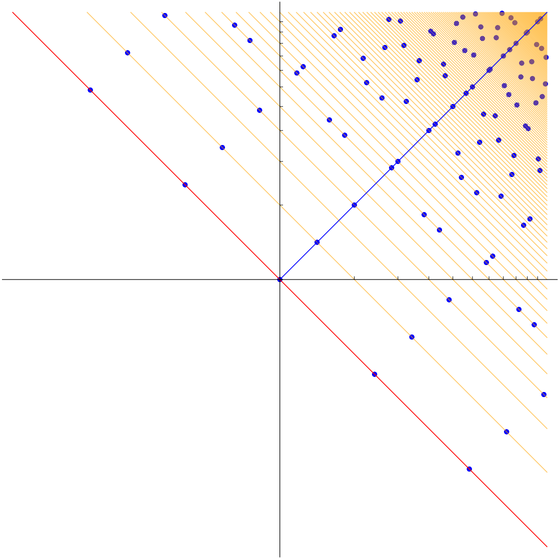
- "Orthogonal" elements
- Units (algebraic norm 1)
- "Isonorms"

Example: Logarithmic Embedding $\operatorname{Log} \mathbb{Z}[\sqrt{2}]$

$\Lambda = \{\bullet\} \cap \diagdown$ is a rank-1 lattice of $\mathbb{R}^2$, orthogonal to $(1, 1)$





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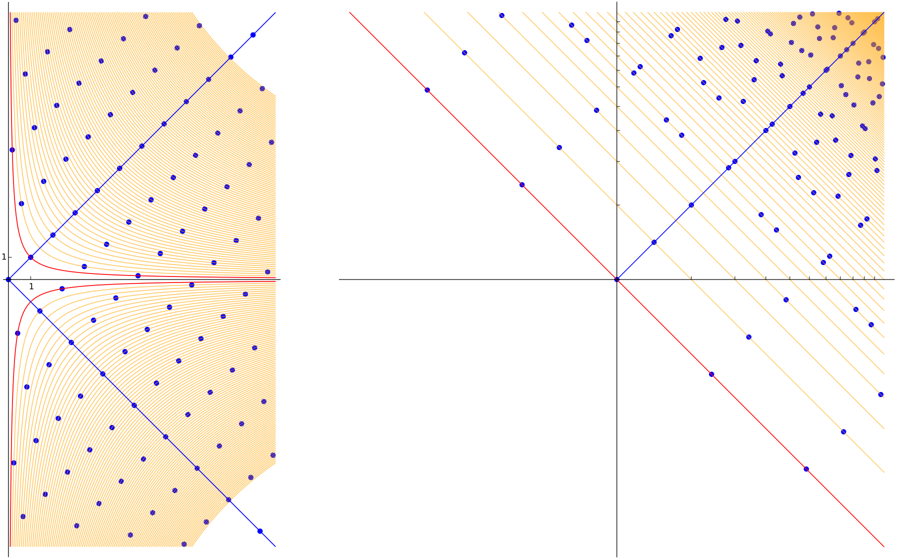
$\{\bullet\} \cap$ ╲ are finite # shifted copies of Λ



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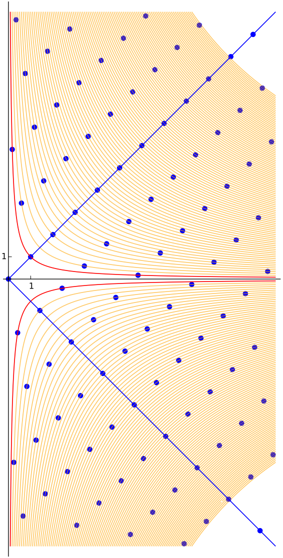
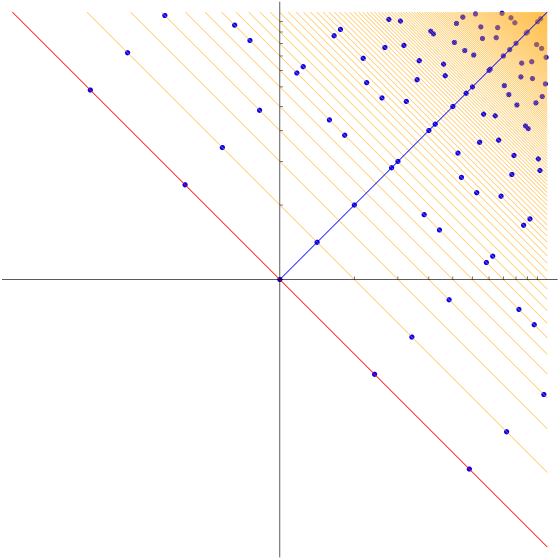
Some $\{\bullet\} \cap$ ╲ may be empty (e.g., no elements of norm 3)





Unit Group and the Log-Unit Lattice

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Dirichlet's Unit Theorem:

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Short Generators via CVP

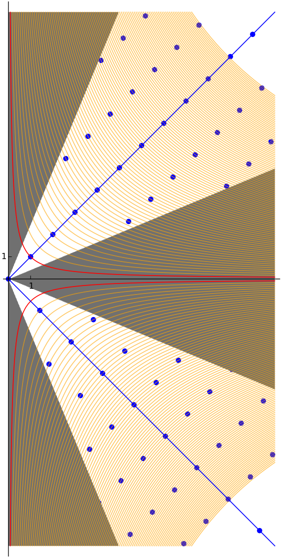
Elements $g, h \in R$ generate the same ideal if and only if $g = h \cdot u$ for some unit $u \in R^\times$, i.e.,

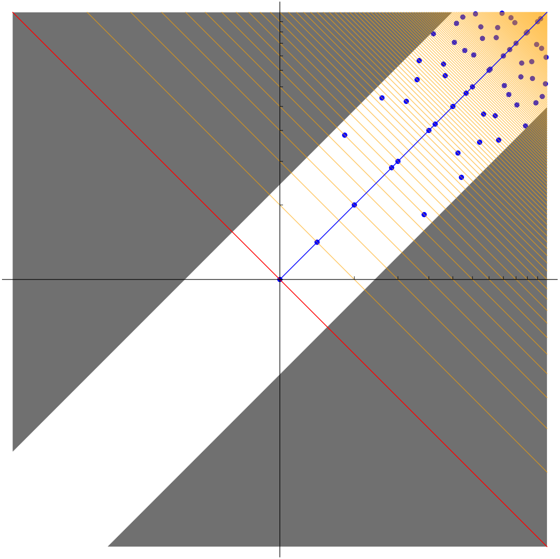
$$\operatorname{Log} g = \operatorname{Log} h + \operatorname{Log} u \in \operatorname{Log} h + \Lambda.$$

In particular, g is a "smallest" generator iff $\operatorname{Log} g$ is a "shortest" element of $\operatorname{Log} h + \Lambda$.

Decoding $\Lambda = \operatorname{Log} \mathbb{Z}[\sqrt{2}]^\times$

Decoding mod Λ into various fundamental domains.





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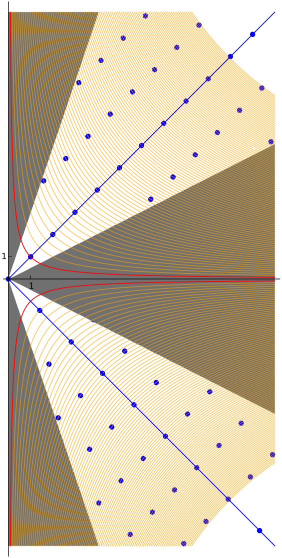

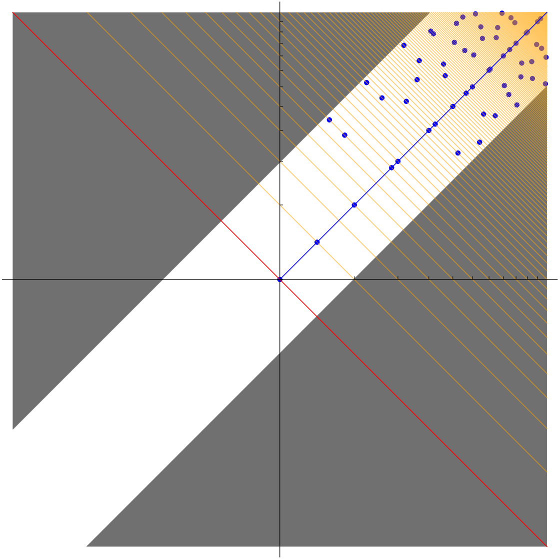

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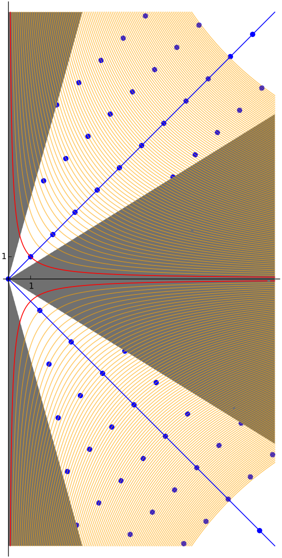

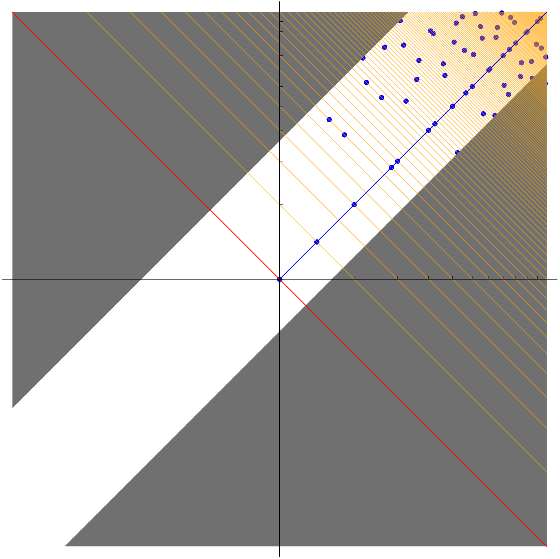

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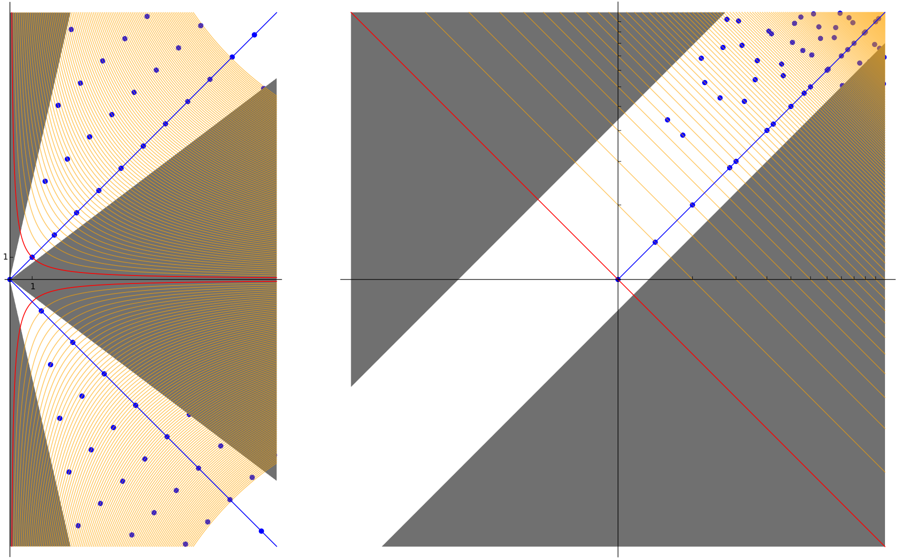

Round-Off Decoding

The simplest algorithm to solve CVP/BDD:

$\textsc{Round}(\mathbf{B}, \mathbf{t})$ for $\mathbf{B}$ a basis of Λ

- Return $\mathbf{B} \cdot \mathsf{frac}(\mathbf{B}^{-1} \cdot \mathbf{t})$.

Used as a decoding algorithm, its correctness is characterized by the error $\mathbf{e}$ and the *dual basis* $\mathbf{B}^\vee = \mathbf{B}^{-T}$.

Fact

Suppose $\mathbf{h} = \mathbf{u} + \mathbf{g}$ for some $\mathbf{u} \in \Lambda$. If $\langle \mathbf{b}_j^\vee, \mathbf{g} \rangle \in [-\frac{1}{2}, \frac{1}{2})$ for all j, then

$$\textsc{Round}(\mathbf{B}, \mathbf{h}) = \mathbf{g}.$$

Recovering the Short Generator: Proof Outline

1. Construct a basis $\mathbf{B}$ of the log-unit lattice $\Lambda = \operatorname{Log} R^{\times}$.
   - ⋆ For $K = \mathbb{Q}(\zeta_m)$, $m = p^k$, a canonical (almost[1]-)basis is given by

$$\mathbf{b}_j = \operatorname{Log} \frac{1-\zeta^j}{1-\zeta}, \quad 2 \le j < m/2,\ j \text{ coprime with } m.$$

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3. Prove that $\mathbf{g} = \operatorname{Log} g$ is sufficiently small when g generated as in cryptosystem, so that $\langle \mathbf{b}_j^\vee, \mathbf{g} \rangle \in [-\frac{1}{2}, \frac{1}{2})$.

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Technical Contributions

2. Show $\|\mathbf{b}_j^\vee\| = \tilde{O}(1/\sqrt{m})$ using Gauss sums and Dirichlet L-series.
3. Bound $\langle \mathbf{b}_j^\vee, \mathbf{g} \rangle$ using theory of subexponential random variables.

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