Ideal Lattices

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Berkeley, 07/07/2015

Lattice-based cryptography: elegant but impractical

- Lattice-based cryptography is fascinating: simple, (presumably) post-quantum, expressive
- But it is very **slow**

Recall the SIS hash function:

$$\begin{array}{ccc} \{0,1\}^m & \to & \mathbb{Z}_q^n \\ \mathbf{x} & \mapsto & \mathbf{x}^T \cdot \mathbf{A} \end{array}$$

- Need $m = \Omega(n \log q)$ to compress
- q is $n^{O(1)}$, $\mathbf{A}$ is uniform in $\mathbb{Z}_q^{m \times n}$

$\Rightarrow \widetilde{O}(n^2)$ space and cost

- Example parameters: $n \approx 2^6$, $m \approx n \cdot 2^4$, $\log_2 q \approx 2^3$

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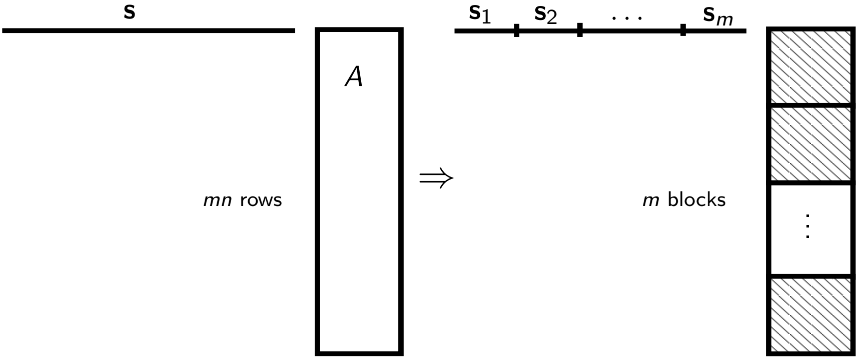
Speeding up linear algebra



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- Structured matrices ⇒ much less space
- Structured matrices ≡ polynomials ≡ fast algorithms
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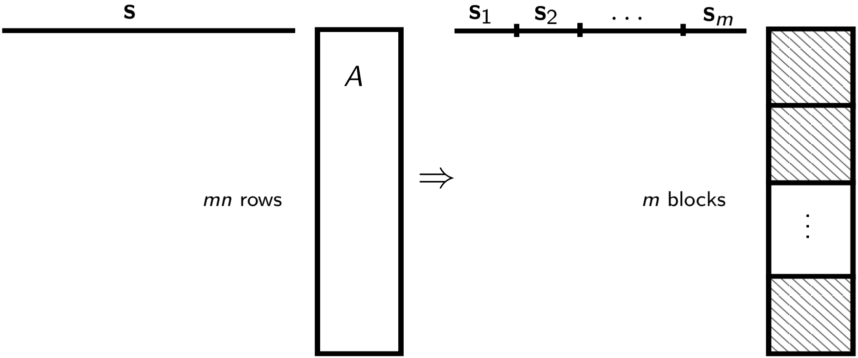

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Structured lattices in crypto: historical perspective

- [NTRU'96,'98,'01]: Encryption and signature, heuristic security

- [Micciancio03]: One-way hash function with cyclic lattices
- [LyMi06,PeRo06]: **Ring-SIS**, collision-resistant hashing
- [Lyu08,Lyu12,DDLL13]: Schnorr-like Ring-SIS signature

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Goals of this talk

- Introduce Ring-SIS and Ring-LWE
- Describe the lattices that lurk behind

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- Introduce Ring-SIS and Ring-LWE
- Describe the lattices that lurk behind

1- **Ideal lattices**

2- Ring-SIS

3- Ring-LWE

4- Other lattices from algebraic number theory

Some algebra

Number field

Let $\zeta \in \mathbb{C}$ algebraic with minimum polynomial $P \in \mathbb{Q}[X]$. Let

$$K \ := \ \sum_{i=0}^{n-1} \mathbb{Q} \cdot \zeta^i \ \subseteq \ \mathbb{C}$$

with $n = \deg P$. This is a field, and $K \cong \mathbb{Q}[X]/P$.

Ring of integers of K

The ring of integers $R = \mathcal{O}_K$ is the set of $\sum y_i \cdot \zeta^i \in K$ that are roots of monic polynomials with integer coefficients.

$$\mathbb{Z}[X]/P \ \cong \ \sum_{i=0}^{n-1} \mathbb{Z} \cdot \zeta^i \ \subseteq \ R.$$

In general, the inclusion is strict.
But there always exist $(\zeta_i)_i$ such that $R = \sum_i \mathbb{Z} \cdot \zeta_i$.

In general, finding a $\mathbb{Z}$-basis of R from P is expensive

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Cyclotomic fields

Cyclotomic polynomial

Φ_m is the unique irreducible polynomial dividing $X^m - 1$ which is not dividing any $X^k - 1$ for $k < m$.

$$\Phi_m(X) \;=\; \prod_{k:gcd(k,m)=1}(X - e^{\frac{2ik\pi}{m}}).$$

- If m is a power of 2, then $\Phi_m = 1 + X^{m/2}$
- If m is prime, then $\Phi_m = \frac{X^m - 1}{X - 1}$

Cyclotomic field

The mth cyclotomic field is $K(e^{\frac{2i\pi}{m}}) \;\cong\; \mathbb{Q}[X]/\Phi_m$.

Why cyclotomic fields?

- More is known, and they tend to be simpler to deal with
- E.g.: $R \;=\; \sum_{i=0}^{n-1} \mathbb{Z} \cdot \zeta^i \;\cong\; \mathbb{Z}[x]/\Phi_m$

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Ideals

Ideal of $\mathcal{O}_K$

$I \subseteq R$ is an (integral) ideal if $\forall a, b \in I$, $\forall r \in R$:

$$a + b \in I \quad \text{and} \quad r \cdot a \in I.$$

If $I \neq \{0\}$, then R/I is a finite ring and we let $\mathcal{N}(I) = |R/I|$.

Principal ideal

If $g \in R$, then $(g) = g \cdot R$ is an ideal, called principal ideal.

- For large n, most ideals are not principal.
- Every ideal is of the form $\sum_{i \leq n} g_i \cdot \mathbb{Z}$ for some $g_i \in R$.
- Every ideal is generated by 2 elements:

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Number fields and geometry

We have $K \subseteq \mathbb{C}$... this is geometrically boring

Polynomial embedding σ_P

Using $K \cong \mathbb{Q}[X]/P$, we can identify elements of K with polynomials of degree $< n$, and hence with elements of $\mathbb{Q}^n$.

Canonical embedding σ_C

Let $(\zeta_i)_i$ be the roots of P. For $g \in \mathbb{Q}[X]/P$, we define

$$\forall i \leq n: \quad \sigma_i(g) = g(\zeta_i) \in \mathbb{C}$$

$\sigma_C := (\sigma_i)_i$ sends K to a $\mathbb{Q}$-vector subspace of $\mathbb{C}^n$ of dimension n.

This is multi-evaluation!

- Easy to compute
- $+$ and $\times$ in K are mapped to componentwise $+$ and $\times$ in $\mathbb{C}^n$

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σ_P versus σ_C

- Multiplication is (mathematically) simpler for σ_C
- Products make norms grow less for σ_C:
  - $\frac{\|\sigma_P(g_1 \cdot g_2)\|}{\|\sigma_P(g_1)\| \cdot \|\sigma_P(g_2)\|}$ can be very large even if P is small,
  - $\frac{\|\sigma_C(g_1 \cdot g_2)\|}{\|\sigma_C(g_1)\| \cdot \|\sigma_C(g_2)\|} \leq 1$
- For the power-of-2 cyclotomic field of degree n:

$$\forall g \in K: \quad \|\sigma_P(g)\| = \frac{1}{\sqrt{n}} \cdot \|\sigma_C(g)\|$$

Ideal lattices

Ideal lattice

Let K a number field and σ an add-homomorphism from K to $\mathbb{R}^n$.
Then $I \subseteq R$ ideal $\Rightarrow$ $\sigma(I) \subseteq \mathbb{R}^n$ lattice.

By default, one uses σ_C to look at the geometry of ideals

Ideal-SVP

Let $(K_i)_i$ be a sequence a number fields of growing degrees n_i.
An Ideal-SVP instance is an ideal I of R_i.
One has to find $b \in I \setminus \{0\}$ minimizing $\|\sigma_C(b)\|$.

This is SVP restricted to ideals of $(R_i)_i$.

E.g., we can study SVP for ideals of power-of-2 cyclotomic fields.

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Are ideal lattice problems any easier than lattice problems?

Property 1. $b \in I$ small $\Rightarrow$ $\zeta^i \cdot b$ small, for all i.

(For σ_P and power-of-2 cyclotomics, these are the famous negacyclic shifts)

Property 2. λ_1 approximately known. For power-of-2 cyclotomics

$$\sqrt{n} \cdot \mathcal{N}(I)^{1/n} \leq \lambda_1(I) \leq n \cdot \mathcal{N}(I)^{1/n}$$

- RHS. Minkowski's theorem $(\det I = \sqrt{n}^n \cdot \mathcal{N}(I))$.
- LHS. Take b reaching λ_1. Then
  - $(b) \subseteq I$
  - $(b \cdot \zeta^i)_i$ is a basis of (b), made of vectors of norms $\|b\|$

  $\Rightarrow \mathcal{N}(I) \leq \mathcal{N}((b)) = \sqrt{n}^{-n} \cdot \det(b) \leq \sqrt{n}^{-n} \|b\|^n$

Apart from these two properties, no other known weakness for lattice problems restricted to ideal lattices, in the worst case.

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... but no proof that no other structural weakness exists.

Some problems become easy for some families of ideal lattices, at least for cyclotomic fields.

Gentry-Szydlo — see Alice's talk

If $I = (g)$ and we are given $B^t B$ for the basis B of I corresponding to the $\zeta^i \cdot g$'s, then we may recover g in polynomial time.

SPIP — see Chris' talk

If $I = (g)$ with g "exceptionally" small, then we may recover g in subexponential time.

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Roadmap

1- Ideal lattices

2- **Ring-SIS**

3- Ring-LWE

4- Other lattices from algebraic number theory

Two rings

$$R \cong \mathbb{Z}[x]/(x^n+1) \quad \text{and} \quad R_q = \mathbb{Z}_q[x]/(x^n+1) = R/qR$$

If $f \in R$ is known to have small coeffs, then $(f \bmod q)$ reveals f

Multiplication in R_q and linear algebra:

$$[a_0\ a_1\ \ldots\ a_{n-1}] \cdot \begin{bmatrix} b_0 & b_1 & \ldots & b_{n-1} \\ -b_{n-1} & b_0 & \ldots & b_{n-2} \\ \vdots & & \vdots & \\ -b_1 & -b_2 & \ldots & b_0 \end{bmatrix} = [c_0\ c_1\ \ldots\ c_{n-1}],$$

with $c(x) = a(x) \cdot b(x) \bmod (x^n+1)$

- Quasi-linear time multiplication
- It's even practical, for $q = 1 \bmod 2n$ (number-theory transform)

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The Ring-SIS problem

SIS

Given $\mathbf{a}_i, \ldots, \mathbf{a}_m \leftarrow U(\mathbb{Z}_q^n)$, find $\mathbf{s} \in \mathbb{Z}^m$ s.t.

$$0 < \|\mathbf{s}\| \leq \beta \;\text{ and }\; \sum s_i \cdot \mathbf{a}_i = \mathbf{0} \mod q$$

Ring-SIS

Given $a_1, \ldots, a_m \leftarrow U(R_q)$, find $s_1, \ldots, s_m \in R$ s.t.

$$0 < \|\sigma_C(\mathbf{s})\| \leq \beta \;\text{ and }\; \sum s_i \cdot a_i = 0 \mod q$$

- Here $\sigma_C(\mathbf{s}) = (\ \sigma_C(s_1)| \ldots |\sigma_C(s_m)\ ) \in \mathbb{C}^{nm}$
- The m of Ring-SIS should be taken n times smaller than that of SIS, for fair comparison
- Ring-SIS leads to fast signatures

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Ring-SIS and ideal lattices

Worst-case to average-case reduction [LyMi06,PeRo06,PeRo07]

Any ppt **Ring-SIS** algorithm succeeding with non-negligible probability leads to a ppt **Ideal-SVP**$_\gamma$ algorithm, with $\gamma, q \gg \sqrt{n}\beta$

- This result is for $R = \mathbb{Z}[x]/(x^n + 1)$ with n a power of 2
- It extends to any sequence of rings of integers R_n of degree n number field K_n, assuming that:
  - R_n is known,
  - $|\det \sigma_C(R_n)| \leq n^{O(n)}$.

A weak variant of Ring-SIS

Ring-SIS

Given $a_1, \ldots, a_m \leftarrow U(R_q)$, find $s_1, \ldots, s_m \in R$ s.t.

$$0 < \|\sigma_C(\mathbf{s})\| \leq \beta \quad \text{and} \quad \sum s_i a_i = 0 \bmod q$$

Take $R = \mathbb{Z}[X]/(X^n - 1)$.

- We have $X^n - 1 = (X-1) \cdot Q(X)$ for $Q(X) = 1 + \ldots + X^{n-1}$
- By the CRT: $R \cong \mathbb{Z}[X]/(X-1) \times \mathbb{Z}[X]/Q(X)$

We can solve mod $X - 1$ and mod $Q(X)$, and CRT-reconstruct.

- Mod Q: Choose $s_i = 0$ for all i
- Mod $X - 1$: fix $s_1 = 1$ for all i

With probability $1/q$, we have $\sum s_i a_i = 0 \bmod (q, X-1)$.

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Roadmap

1- Ideal lattices

2- Ring-SIS

3- **Ring-LWE**

4- Other lattices from algebraic number theory

Challenge distributions

LWE challenge distribution $A_{\mathbf{s},\phi}$

For $\mathbf{s} \in \mathbb{Z}_q^n$ secret and ϕ a small (error) distribution over $\mathbb{Z}$, a sample from $A_{\mathbf{s},\phi}$ is of the form

$$(\mathbf{a}, \langle \mathbf{a}, \mathbf{s} \rangle + e) \in \mathbb{Z}_q^{n+1} \text{ with } \mathbf{a} \leftarrow U(\mathbb{Z}_q^n),\ e \leftarrow \phi$$

For a cost $\widetilde{O}(n)$, we give out **one $\mathbb{Z}_q$-hint** on $\mathbf{s}$

Ring-LWE challenge distribution $A^R_{s,\phi}$

For $s \in R_q$ secret and ϕ a small (error) distribution over R, a sample from $A^R_{s,\phi}$ is of the form:

$$(a, a \cdot s + e) \in R_q^2 \text{ with } a \leftarrow U(R_q),\ e \leftarrow \phi$$

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Ring-LWE challenge distribution $A_{s,\phi}^R$

For $s \in R_q$ secret and ϕ a small (error) distribution over R, a sample from $A_{s,\phi}^R$ is of the form:

$$(a, a \cdot s + e) \in R_q^2 \quad \text{with} \quad a \leftarrow U(R_q),\ e \leftarrow \phi$$

For a cost $\widetilde{O}(n)$, we give out n **($\mathbb{Z}_q$)-hints** on s.

The Ring-LWE problem, search version

Search Ring-LWE

Set ϕ and take $s \in R_q$. The goal is to find s, given arbitrarily many samples $(a, a \cdot s + e)$ from $A^R_{s,\phi}$.

Hardness of search Ring-LWE [LyPeRe10]

Let Φ be the set of distributions ϕ s.t. for all i, $\sigma_i(\phi)$ is an independent 1-dim Gaussian with standard deviation $\approx \alpha q$.

Any ppt **search Ring-LWE** algorithm for all $\phi \in \Phi$ leads to a **quantum** ppt algorithm for **Ideal-SVP**$_\gamma$, with $\gamma, q \geq n^{O(1)}/\alpha$.

- Same assumptions on $(R_n)_n$ as for Ring-SIS
- Note that we have a distribution ensemble
- We do not know how to get a classical reduction for small q

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Search to decision reduction

Decision Ring-LWE

Sample ϕ and $s \leftarrow U(R_q)$. With non-negligible probability over ϕ and s, we have to distinguish between $A^R_{s,\phi}$ and $U(R_q^2)$

Decision Ring-LWE is more suited for cryptographic design

Hardness of decision Ring-LWE [LyPeRe10]

Let ϕ sampled s.t. for all i, $\sigma_i(\phi)$ is an independent Gaussian with standard deviation $\approx \alpha q$.
Let R be the ring of integers of the cyclotomic field of order m, and set $q = 1 \bmod m$ prime.
Then **search Ring-LWE** reduces to **decision Ring-LWE**.

The random choice of ϕ is not very important

Why these algebraic/arithmetic conditions?

"Let R be the ring of integers of the cyclotomic field of order m, and choose $q = 1 \bmod m$ prime."

With this q:

- $\Phi_m(X)$ splits into n distinct linear factors mod q.
- By the CRT: $R_q \cong (\mathbb{Z}_q)^n$, as rings.

Field automorphisms:

- $\tau_k : X \mapsto X^k$ for any k coprime with m
- τ_k behaves nicely with Ring-LWE samples:

$$\tau_k(as + e) = \tau_k(a)\tau_k(s) + \tau_k(e), \quad \text{with } \tau_k(e) \text{ small}$$

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Conditions on q

The choice of q seems necessary for reducing search Ring-LWE to decision Ring-LWE. However...

Modulus switching for Ring-LWE [LaSt14]

Let $q \approx q'$. Then Ring-LWE(q) reduces to Ring-LWE(q').

Arithmetic properties of q, q' play no role

Proof idea: $(a, b) \in (R_q)^2 \mapsto (\lfloor \frac{q'}{q} a \rceil, \lfloor \frac{q'}{q} b \rceil) \in (R_{q'})^2$.

- Use Gaussian rounding to ensure uniformity of $\lfloor \frac{q'}{q} a \rceil$
- Use a small secret s, to prevent noise blow-up

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Weak variant Ring-LWE

Take Ring-LWE with $R = \mathbb{Z}[X]/(X^n - 1)$.

- Get samples $(a_i, b_i)_{i \leq m}$ for some m
- Use the weak Ring-SIS variant solver, to find $x_1, \ldots, x_m \in R$ small and not all zero, such that $\sum_i x_i a_i = 0 \bmod q$
- If $b_i \approx a_i \cdot s_i$ for all i, then $\sum_i x_i b_i \bmod q$ is small
- If b_i is uniform, then $\sum_i x_i b_i \bmod (q, X-1)$ is uniform

More on weak variants of Ring-LWE in Kristin's talk!

Roadmap

1- Ideal lattices

2- Ring-SIS

3- Ring-LWE

4- **Other lattices from algebraic number theory**

Ring-SIS/Ring-LWE lattices

Ring-SIS

Given $a_1, \ldots, a_m \leftarrow U(R_q)$, find $s_1, \ldots, s_m \in R$ s.t.

$$0 < \|\sigma_C(\mathbf{s})\| \leq \beta \quad \text{and} \quad \sum s_i \cdot a_i = 0 \bmod q$$

Ring-SIS is about finding **s** small and non-zero in

$$M(a_1, \ldots, a_m) = \{\mathbf{x} \in R^m : \textstyle\sum_i x_i \cdot a_i = 0 \bmod q\}.$$

This set is a rank m module over R.

- We don't know how to express Ring-SIS as an ideal lattice problem
- We could imagine that ideal lattice problems turn out to be easy, while Ring-SIS remains hard

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Module lattices

Module lattices

A module lattice in K^m is a set of the form

$$M = \sum_{j \leq k} I_j \cdot \mathbf{b}_j,$$

where the I_j's are ideals and the $\mathbf{b}_j$'s are K-linearly independent

- Ideal lattices: $k = 1$
- Euclidean lattices: $R = \mathbb{Z}$

Reductions from Ideal-SVP to Ring-SIS/Ring-LWE can be extended to reductions from Module-SVP to Module-SIS/Module-LWE

Module-SIS [LaSt14]

Given $\mathbf{a}_1, \ldots, \mathbf{a}_m \leftarrow U(R_q^k)$, find $s_1, \ldots, s_m \in R$ s.t.

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Log unit lattice – More in Chris' talk

Units

Units u are invertible elements in R. We have: $\prod_i \sigma_i(u) = 1$

Dirichlet's theorem: $R^\times \cong \langle g \rangle \times \mathbb{Z}^d$

Every unit u is of the form

$$g_0^k \cdot u_1^{k_1} \cdot \ldots \cdot u_{d-1}^{k_d}, \quad k_i \in \mathbb{Z},$$

where $\langle g \rangle \subset \mathbb{C}$ is finite, the $\langle u_i \rangle$'s are independent and infinite, and $d = n/2 - 1$ in the case of cyclotomic fields

The log-unit lattice is $\left\{ \begin{pmatrix} \log|\sigma_1(u)| \\ \vdots \\ \log|\sigma_n(u)| \end{pmatrix} : u \in R^\times \right\} \subseteq \mathbb{R}^n$.

It is related to the multiplicative structure of R

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Open problems

More hardness guarantees?

- Reduction from lattice problems to ideal lattice problems?
- Or to Ring-LWE/Ring-SIS?
- Classical reduction from ideal lattice problems to Ring-LWE?

More constructions?

- Adapting to Ring-SIS/Ring-LWE all SIS/LWE constructions, with the expected efficiency gain?
- A multilinear map, **provably** secure under the assumption that lattice problems for ideal lattices are hard in the worst case?

More attacks? Can we better exploit the multiplicative structure?

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Very partial bibliography

Books:

- P. Samuel: Algebraic theory of numbers
- H. Cohen: A course in computational algebraic theory
- H. Cohen: Advanced topics in computational number theory
- L. C. Washington: Introduction to cyclotomic fields

Selection of articles:

- C. Peikert and A. Rosen: Lattices that Admit Logarithmic Worst-Case to Average-Case Connection Factors
- V. Lybashevsky, C. Peikert and O. Regev: On Ideal Lattices and Learning with Errors Over Rings

Questions?