Superlinear Lower Bounds for Multipass Graph Processing

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Joint work with **Venkat Guruswami** (CMU)

Streaming Algorithms for Graphs

Model:

- Input: large stream of edges
- Goal: minimize the amount of space
  and processing time per edge
- Allowed: randomization and small error probability

Algorithm $\longleftarrow$ $\mathbf{(5,4)\ (1,2)\ (4,3)\ (2,5)\ (3,1)\ \ldots}$

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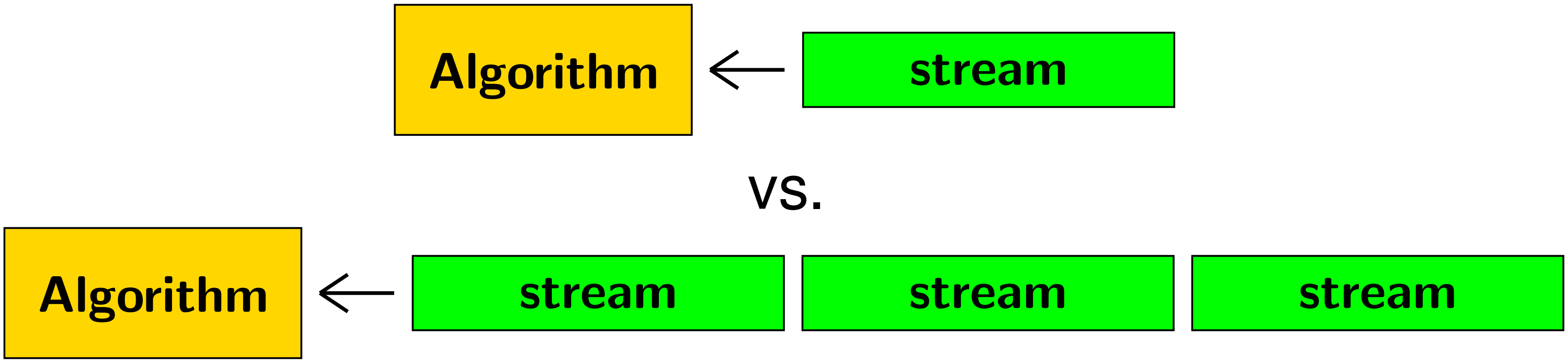
Algorithm $\longleftarrow$ $\mathbf{(5,4)\ (1,2)\ (4,3)\ (2,5)\ (3,1)\ \ldots}$

- Worst-case ordering of edges (as opposed to random)
  - The adversary knows the algorithm
    but not its random bits

One Pass vs. Multiple Passes



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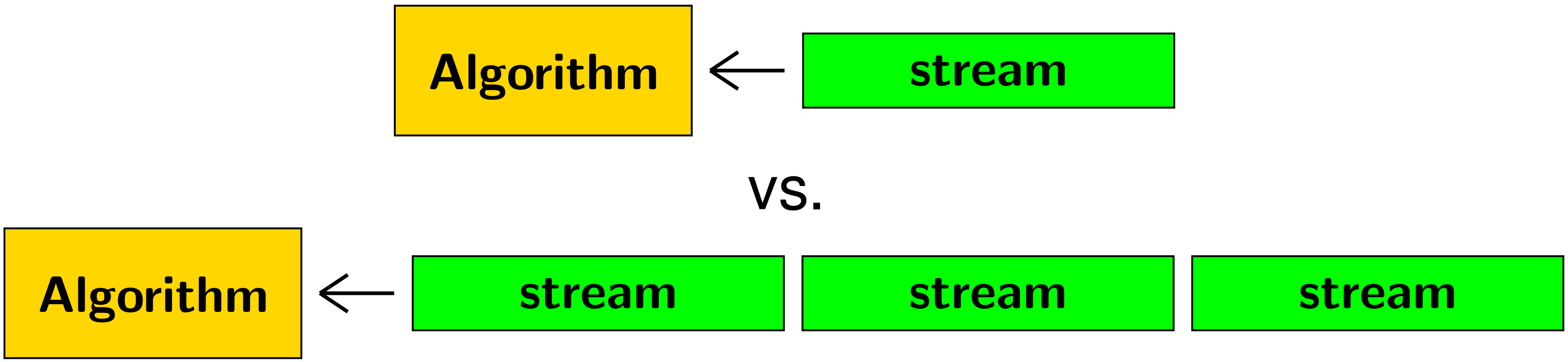

Do multiple passes make sense?

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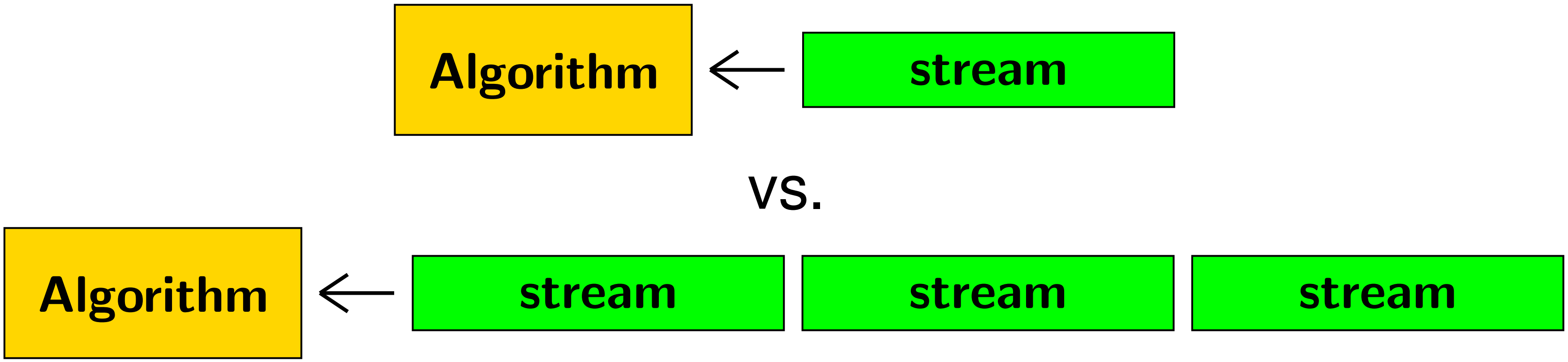
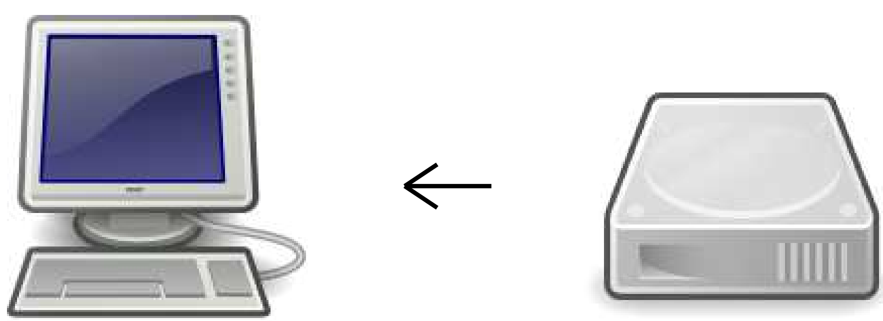


Do multiple passes make sense?

YES:

- Data on a large external storage device
- Sequential access often maximizes throughput



Graph Streaming

“Sweet-spot” for graph streaming: Semi-streaming model
[Muthukrishnan 2003]

- Allow $n \cdot \mathrm{poly}(\log n)$ space
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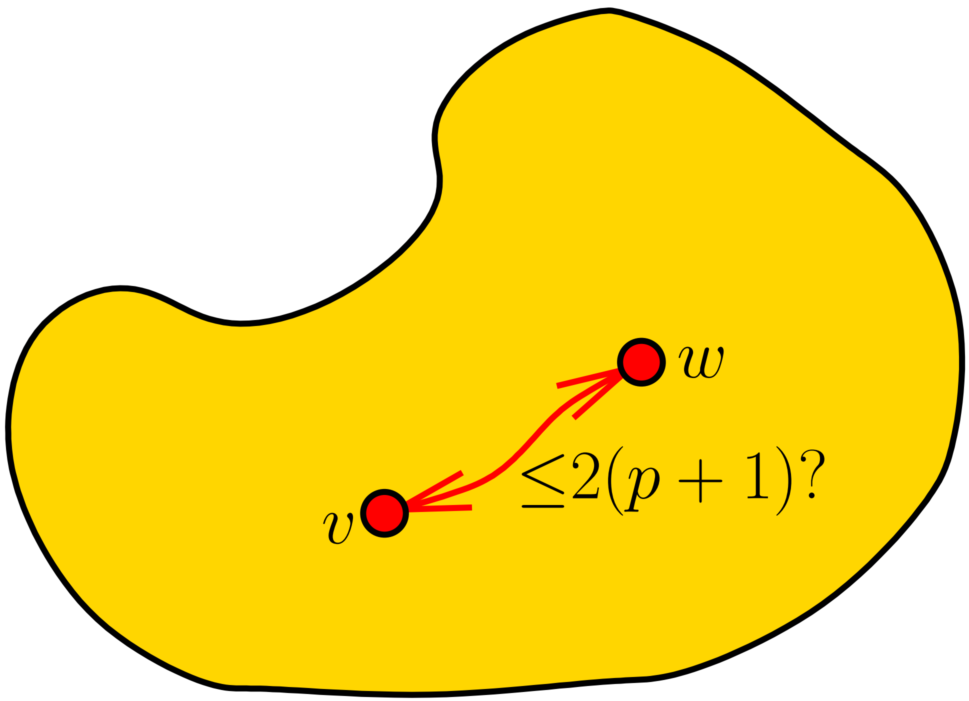
General challenge: Which graph-theoretic problems admit $n \cdot \mathrm{poly}(\log n)$ space streaming algorithms in one or a few passes?

This Work: Rule out such algorithms for some basic graph problems

Our Results

Undirected graphs:

- **Problem 1:** Are v and w at distance at most $2(p+1)$?



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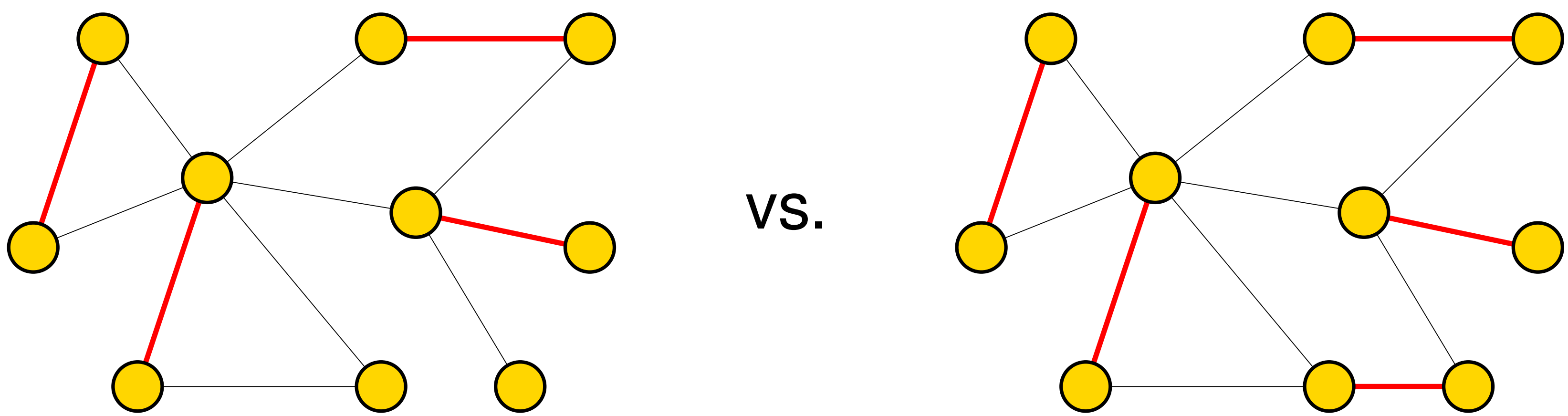
Undirected graphs:

- **Problem 1:** Are v and w at distance at most $2(p+1)$?
- **Problem 2:** Is there a perfect matching?



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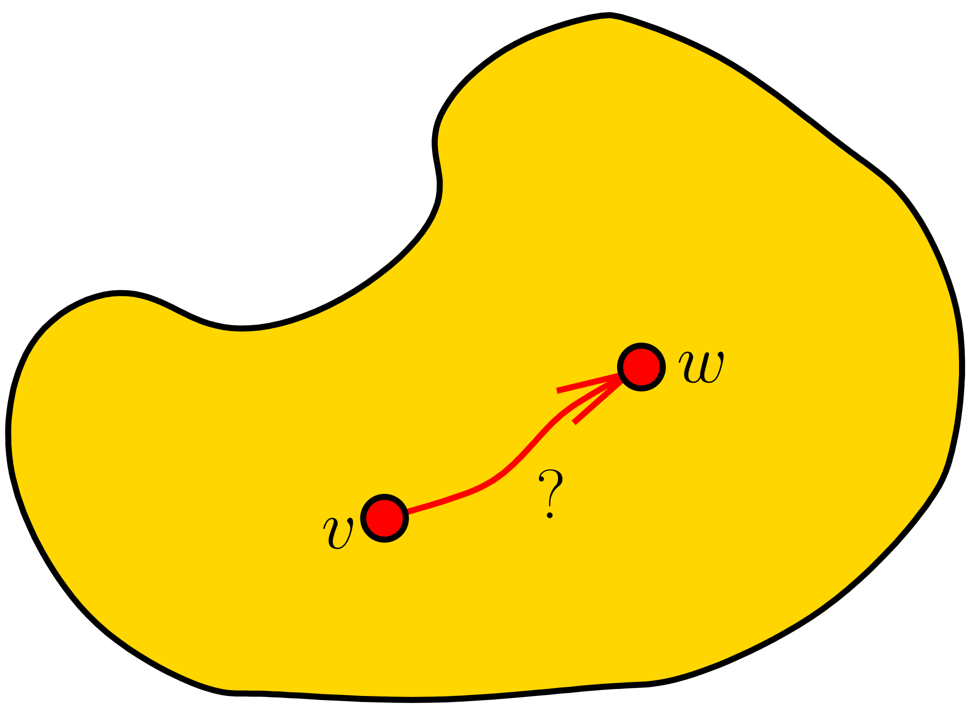

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Solving these graph problems in p passes requires

$$\Omega\left(\frac{n^{1+1/(2p+2)}}{p^{20}\log^{3/2} n}\right) = \frac{n^{1+\Omega(1/p)}}{p^{O(1)}}$$

bits of space (n = #vertices)

Comparison to Previous Results

- Known to require $\Omega(n^2)$ bits in one pass
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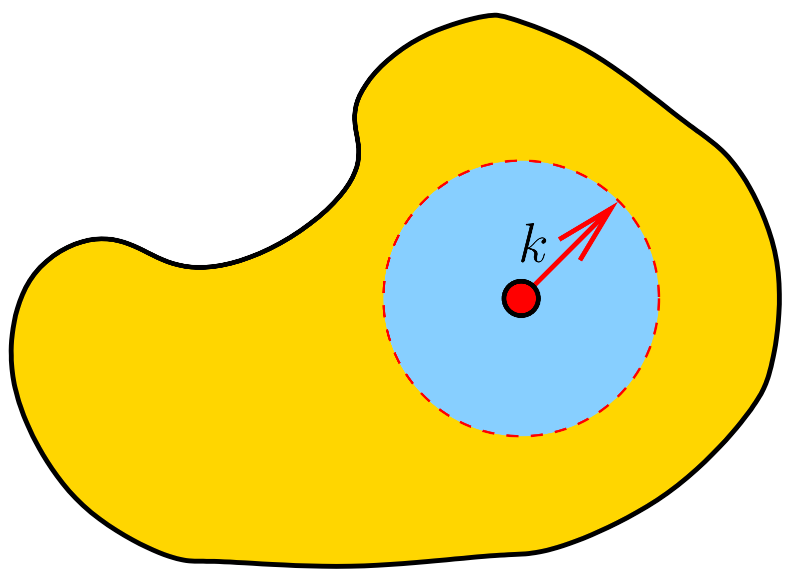
- Easy to prove $\Omega(n/p)$ for p passes via set disjointness
- We want $n^{1+\Omega(1)}$ lower bounds
- Main challenge: embed hard problems into
  the "space of edges"
  not just vertices

Related Results: Shortest Path(s)

Feigenbaum, Kannan, McGregor, Suri, Zhang (2005):

Computing the first k layers of BFS tree in $<k/2$ passes requires $\Omega(n^{1+1/k}/k^{O(1)}(\log n)^{1/k})$ space



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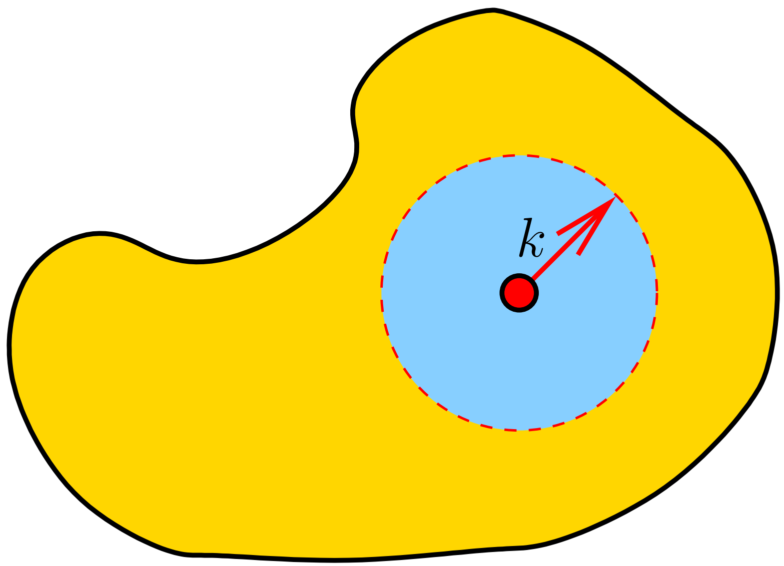

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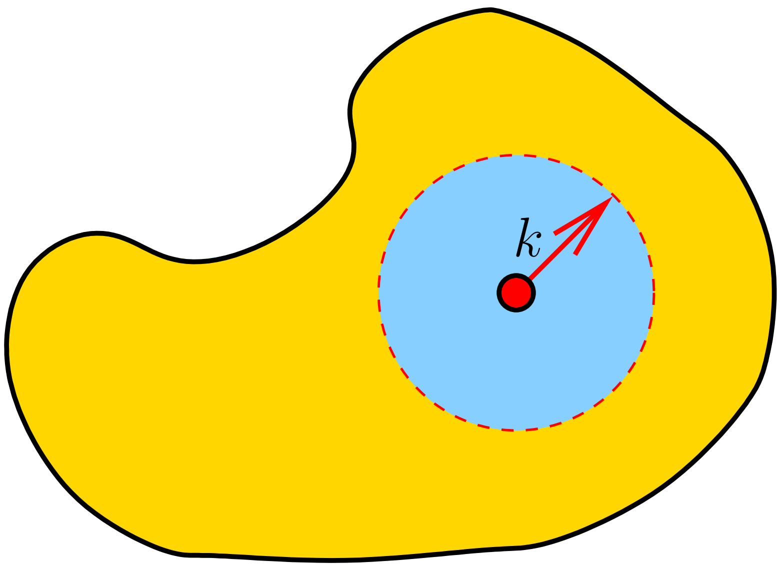
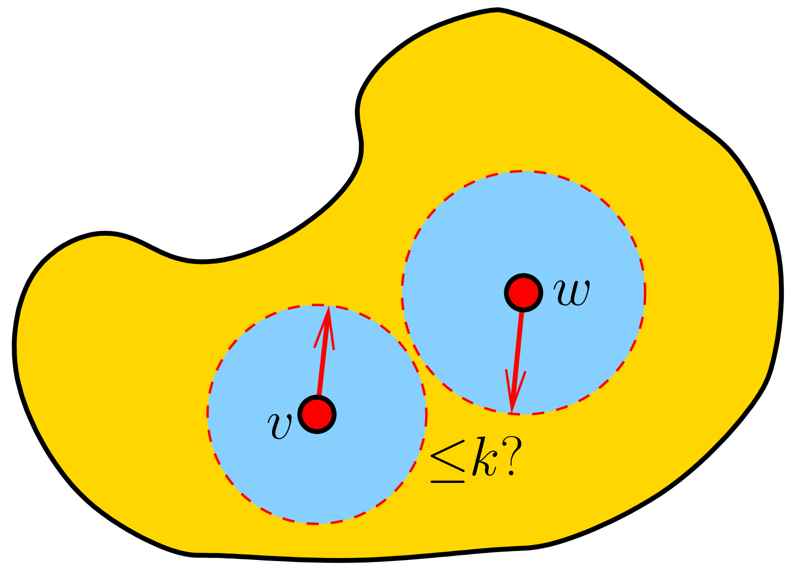
Our problem: Fewer passes suffice



[FKMSZ'05]



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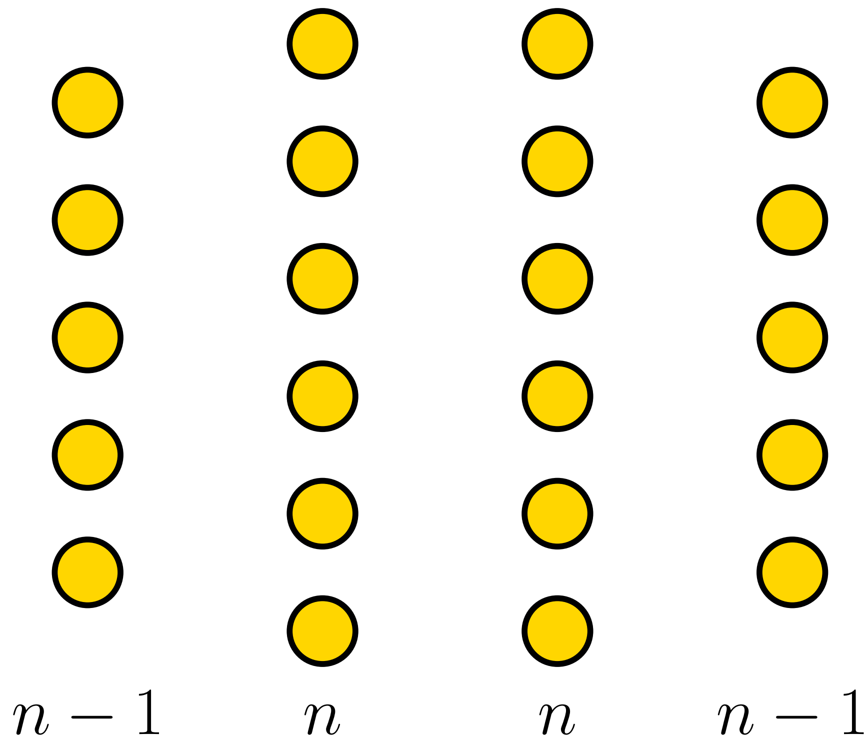
Warmup:
One-Pass Lower Bound
[Feigenbaum et al. 2004]

Construction for Perfect Matching



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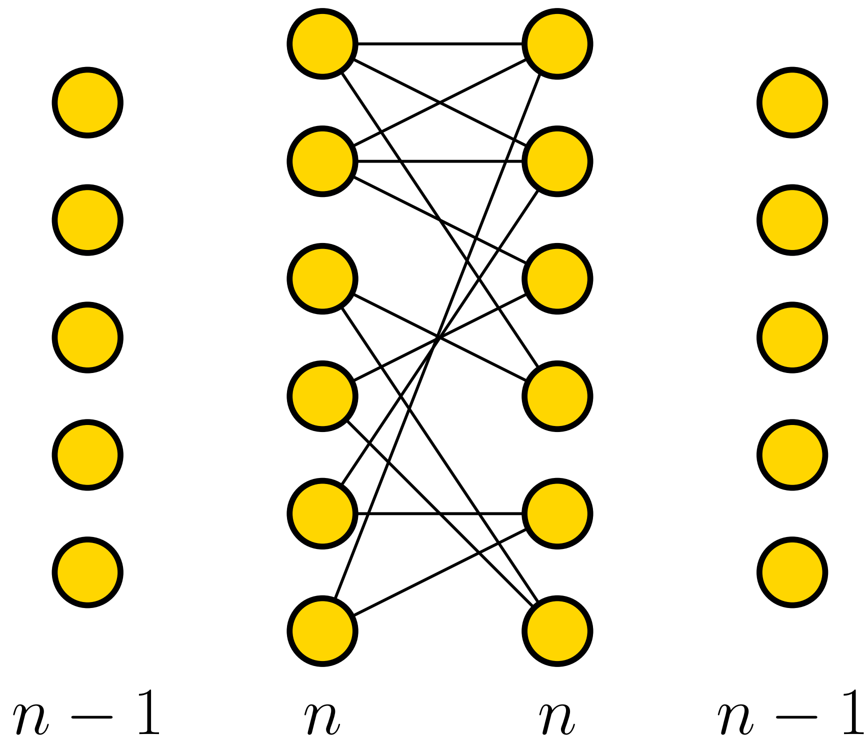

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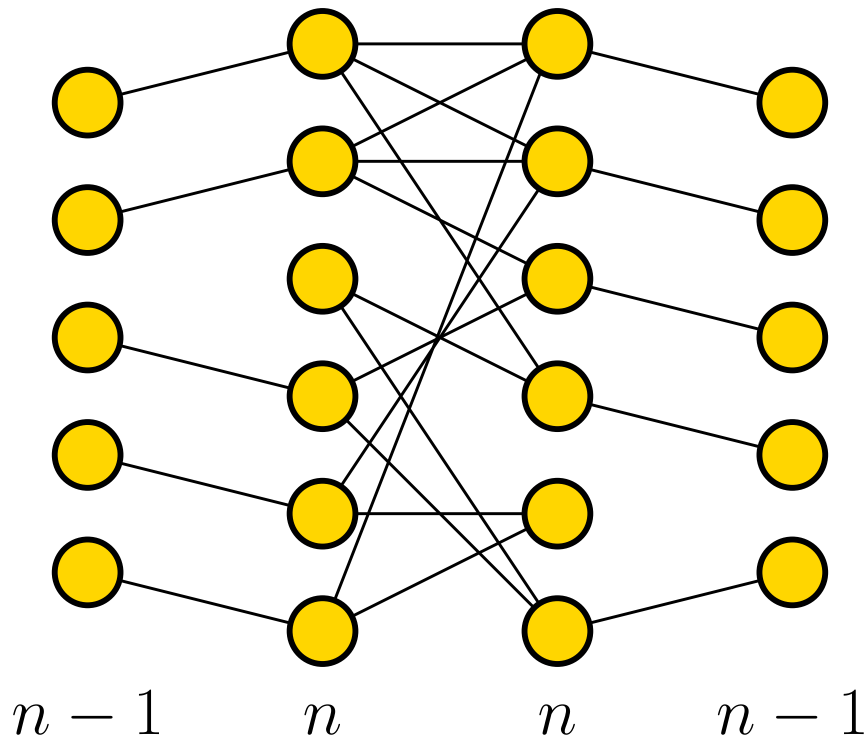

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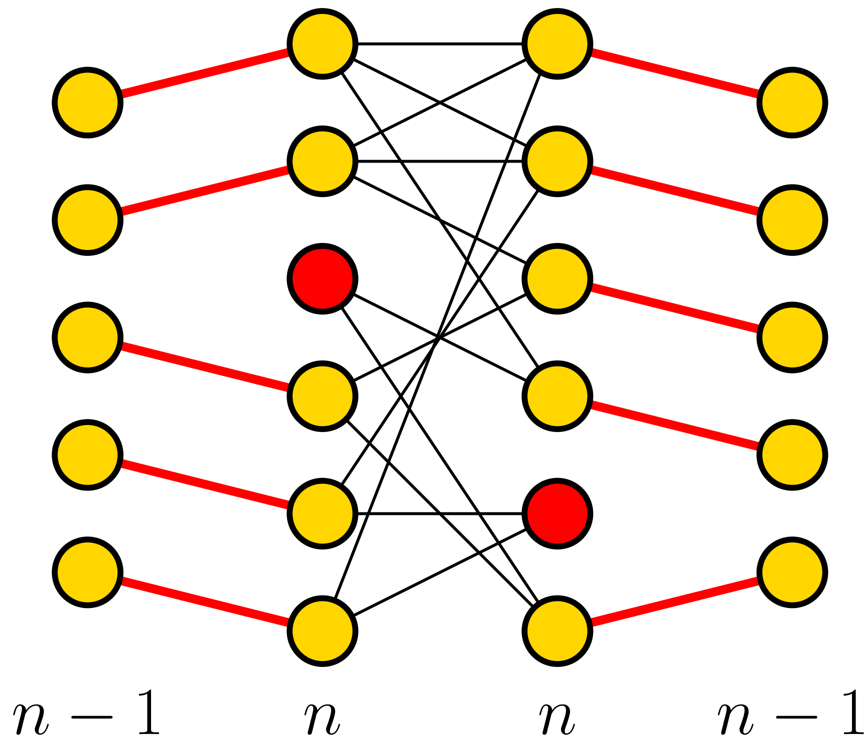

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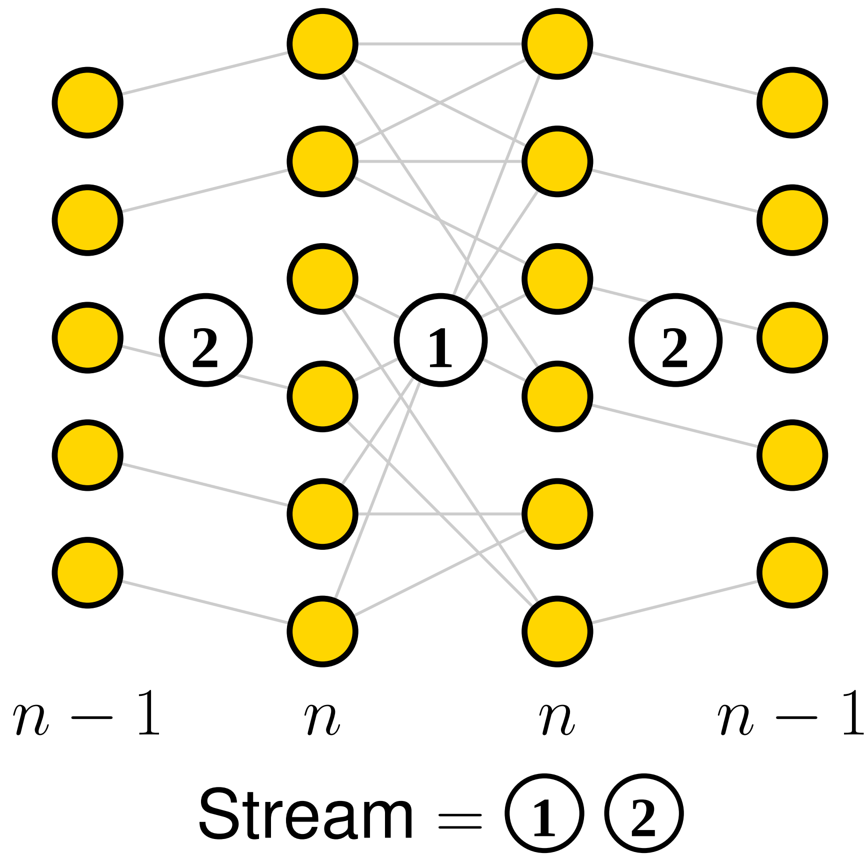

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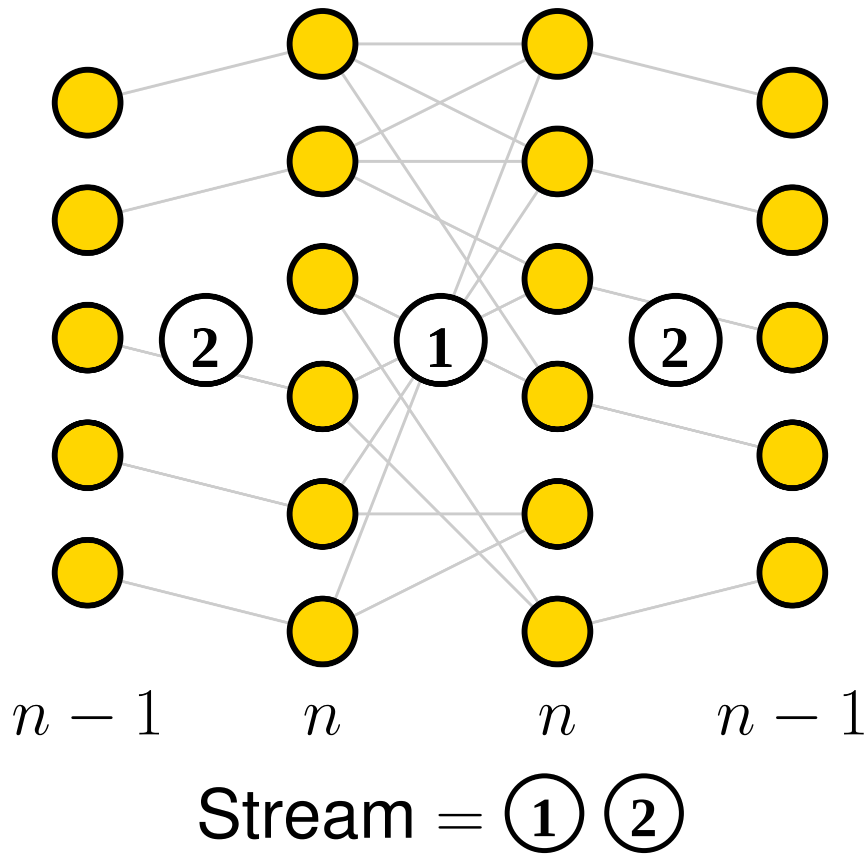

Lower bound of $\Omega(n^2)$ via indexing

Alice	$\Rightarrow$	Bob
$A[1 \ldots n^2]$		x

Bob's task: output $A[x]$

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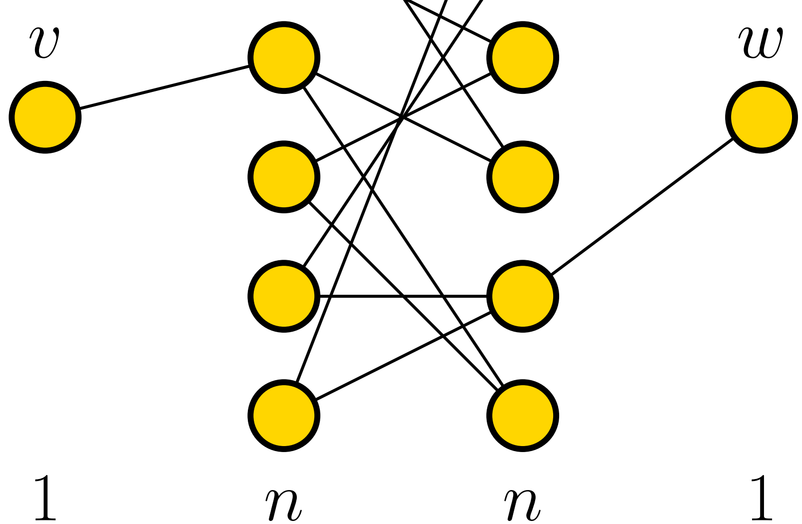

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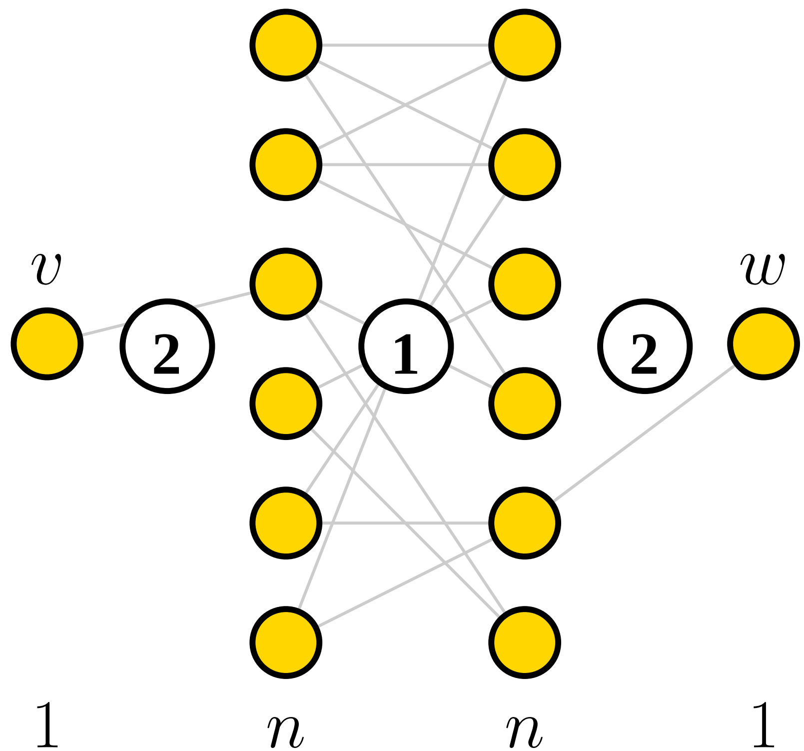

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Stream Ordering

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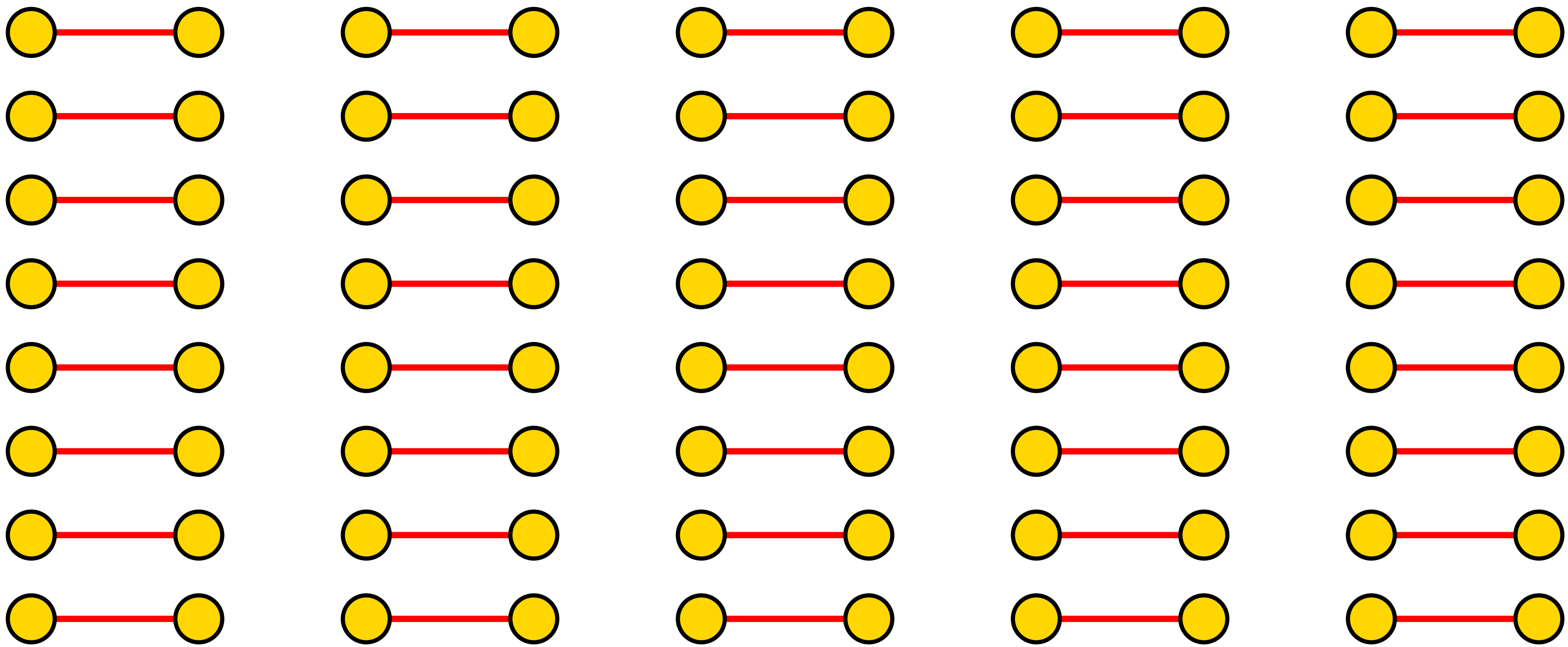
- Graphs $=$ vertices $+$ relations between them
- Solving most graph problems requires exploration
- To prove lower bounds, create obstacles for exploration
- One possibility: present edges in order opposite
  to what is suitable for exploration

Hard Instance for Multiple Passes

Construction for Perfect Matching

Is there a perfect matching?



$\Theta(1)$ columns

Each column $\Theta(n)$ rows

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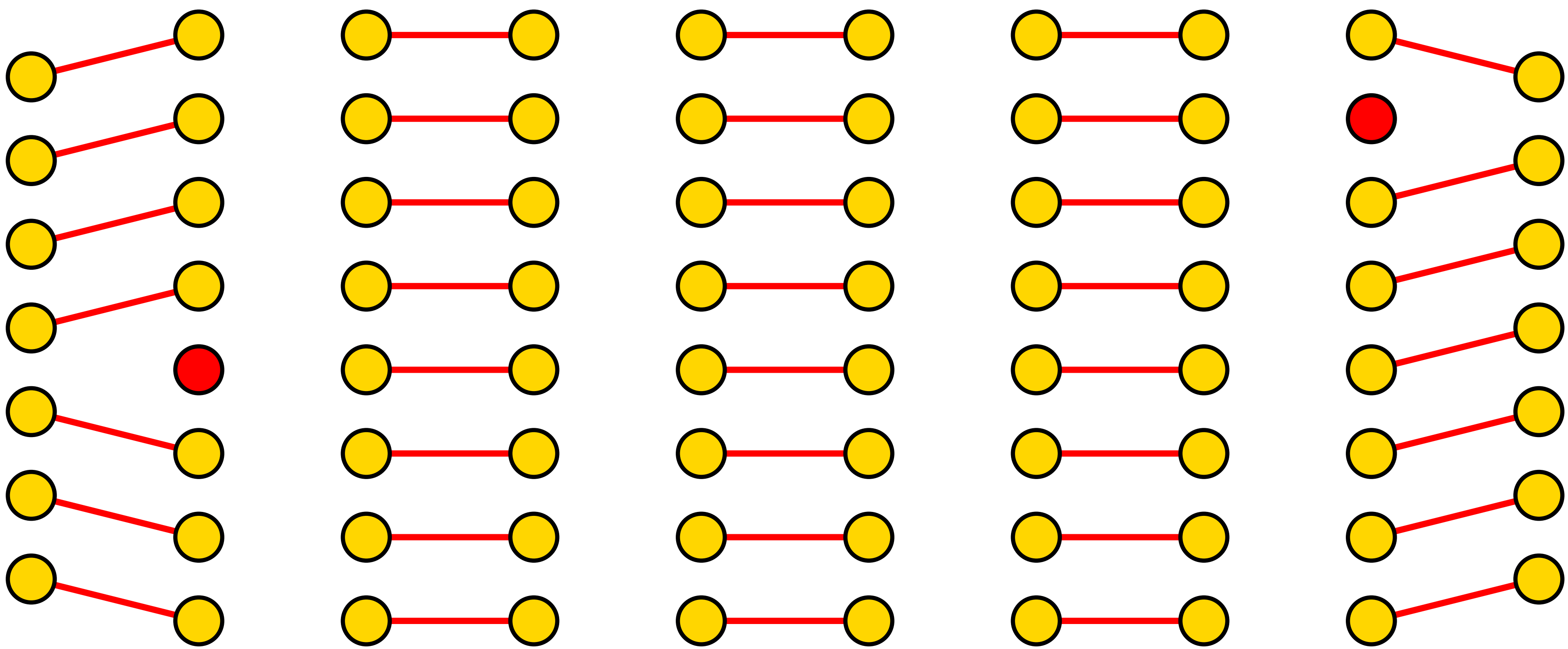

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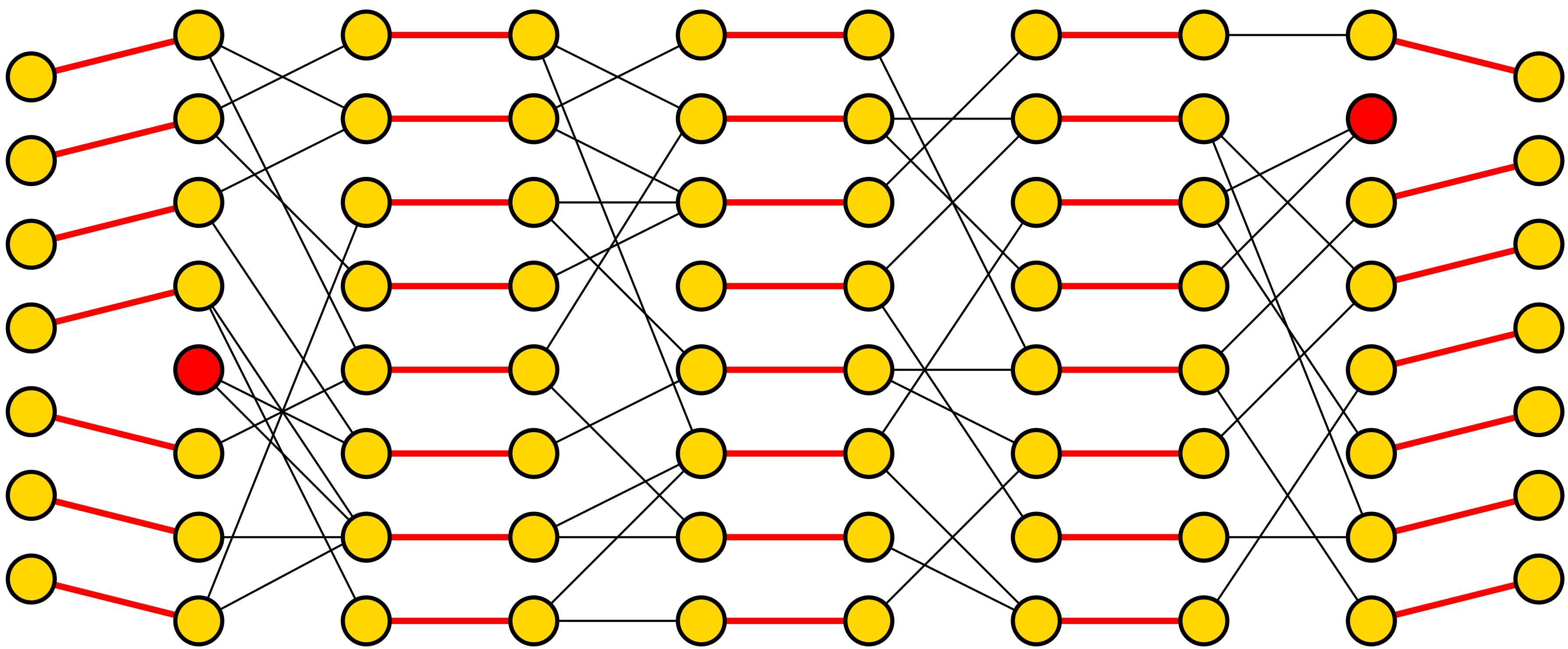

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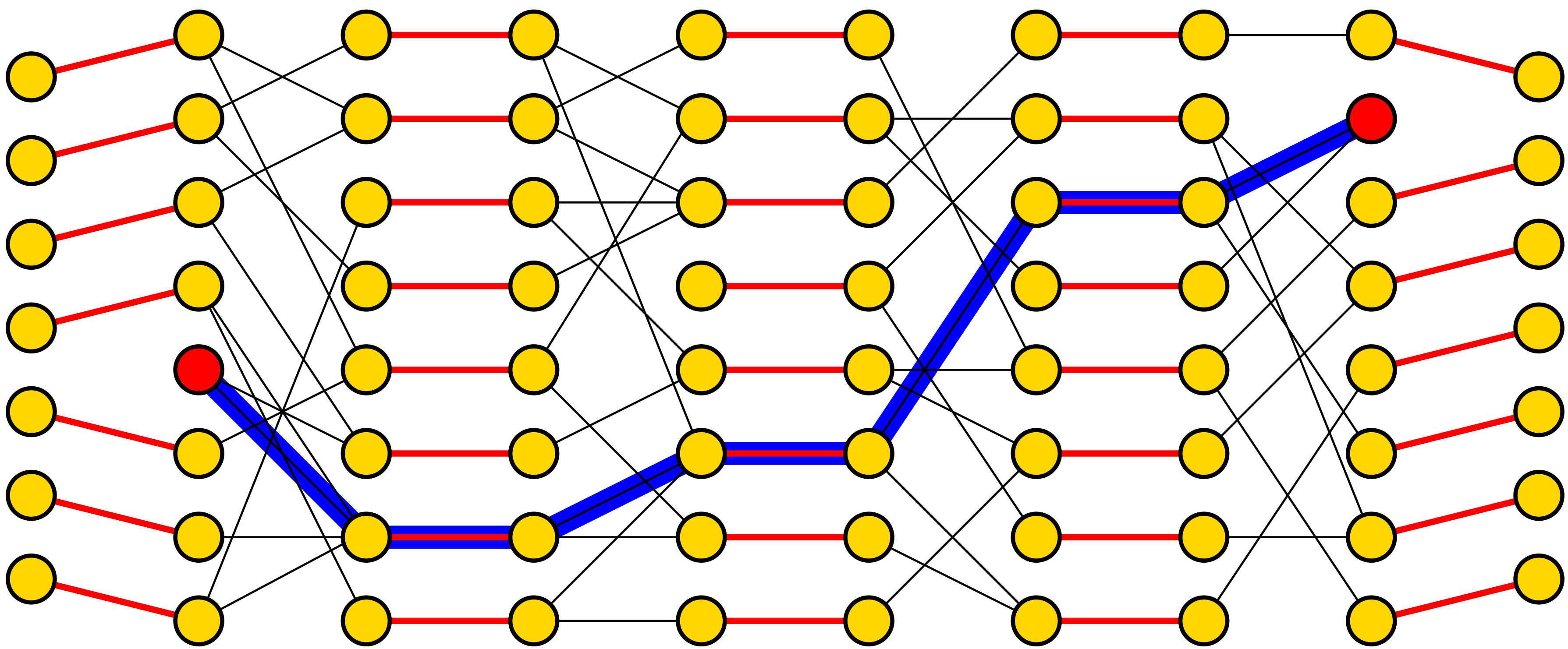

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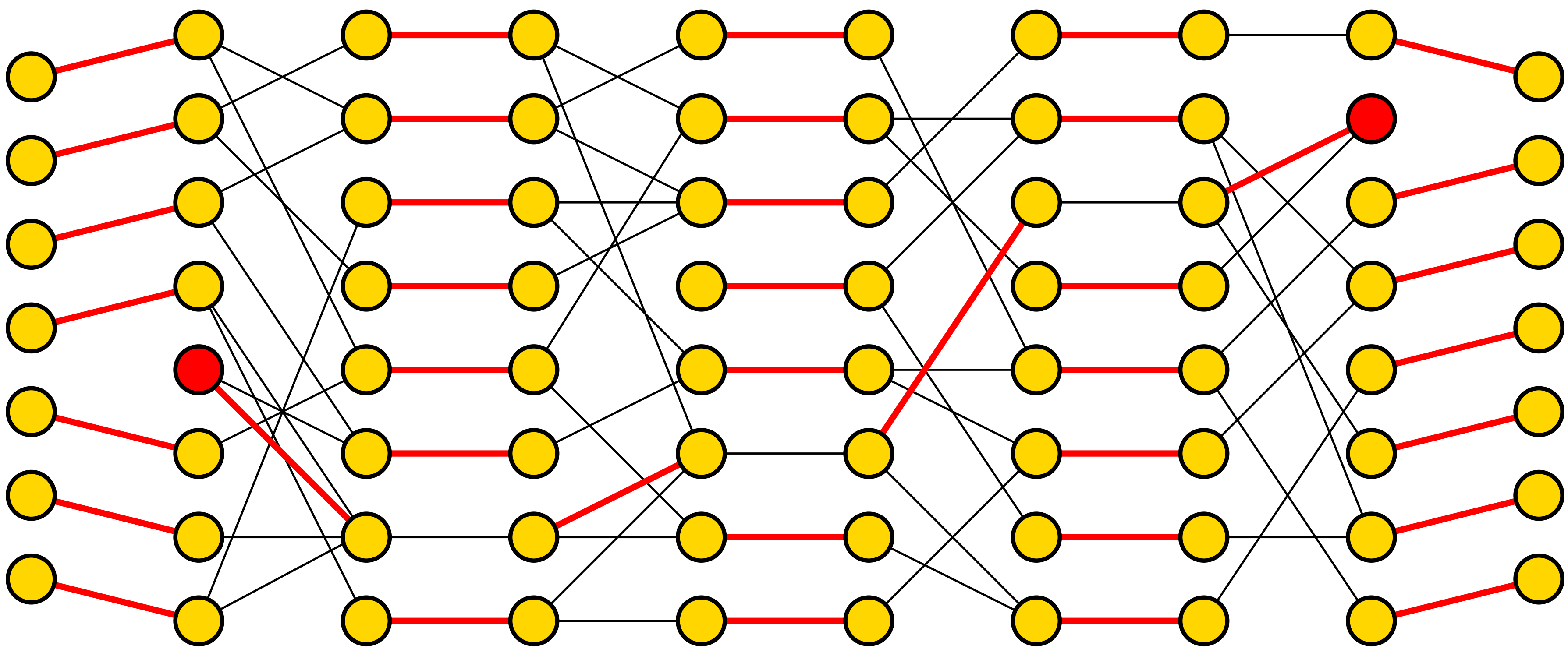

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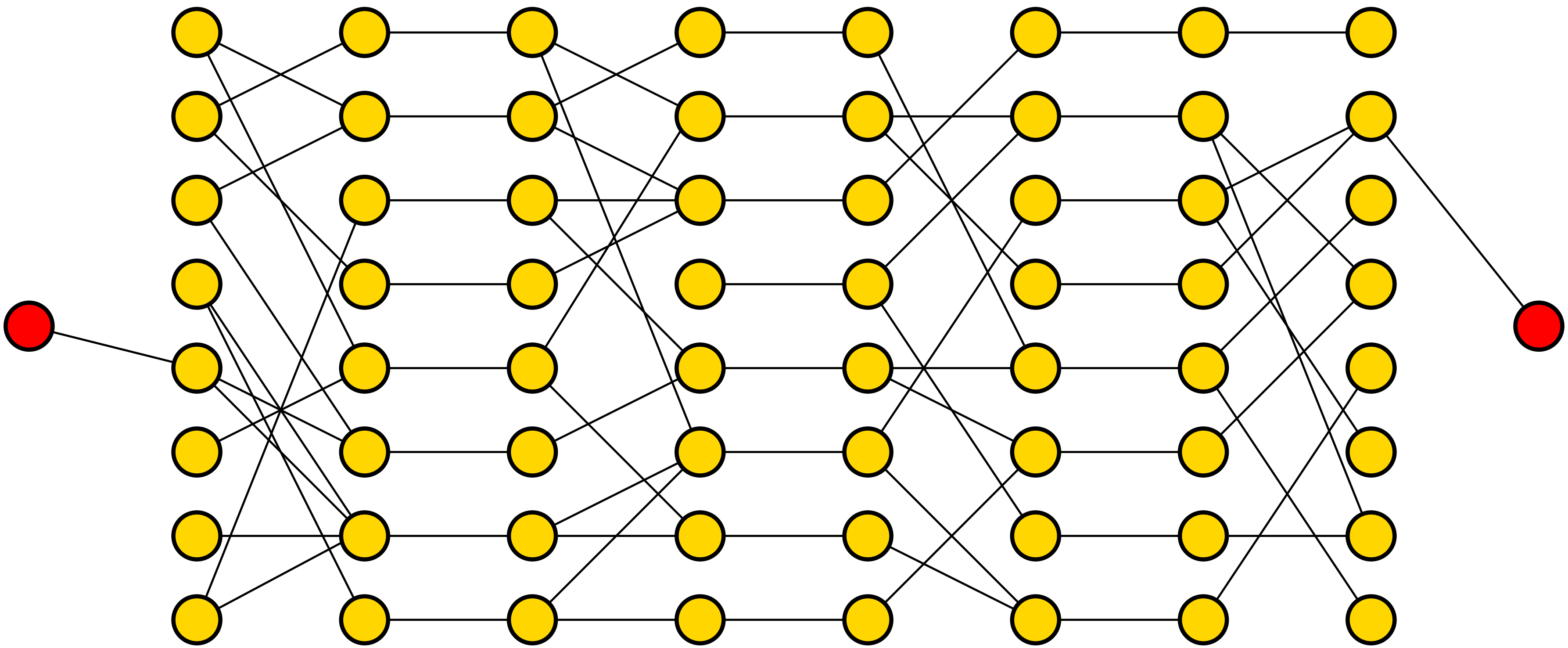
Is there a path of length 9 between red nodes?



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Construction for Perfect Matching

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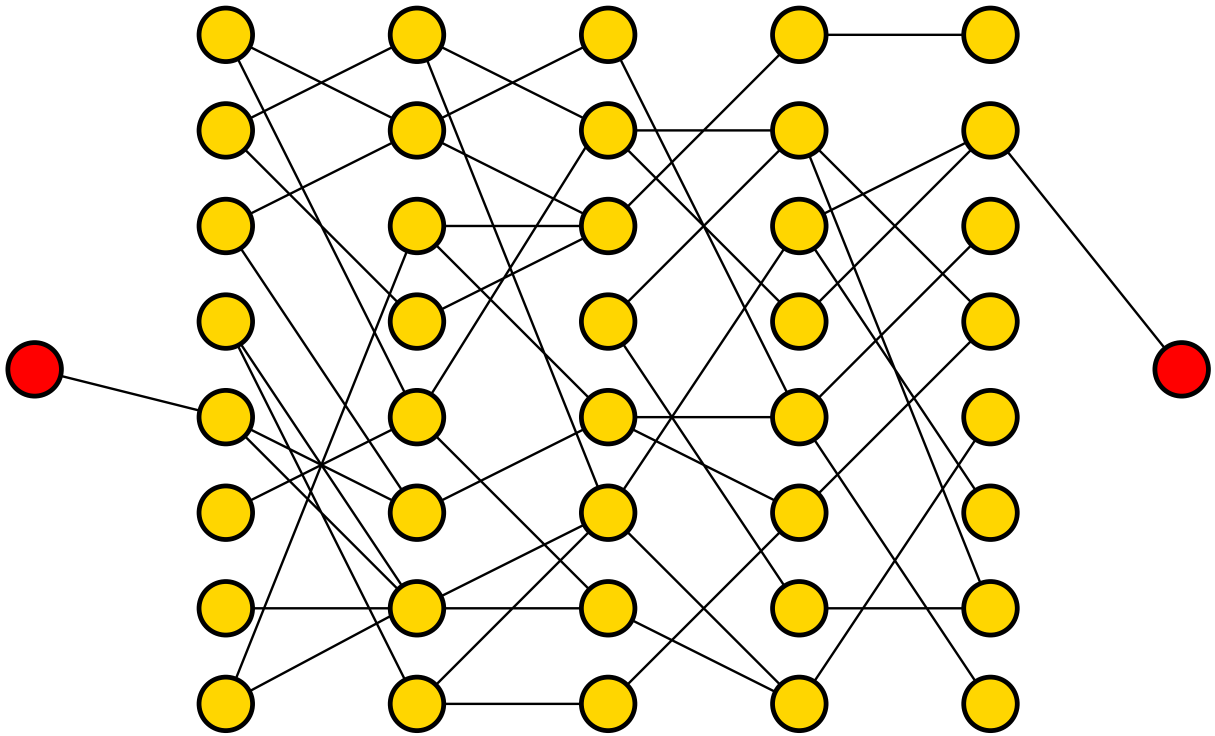

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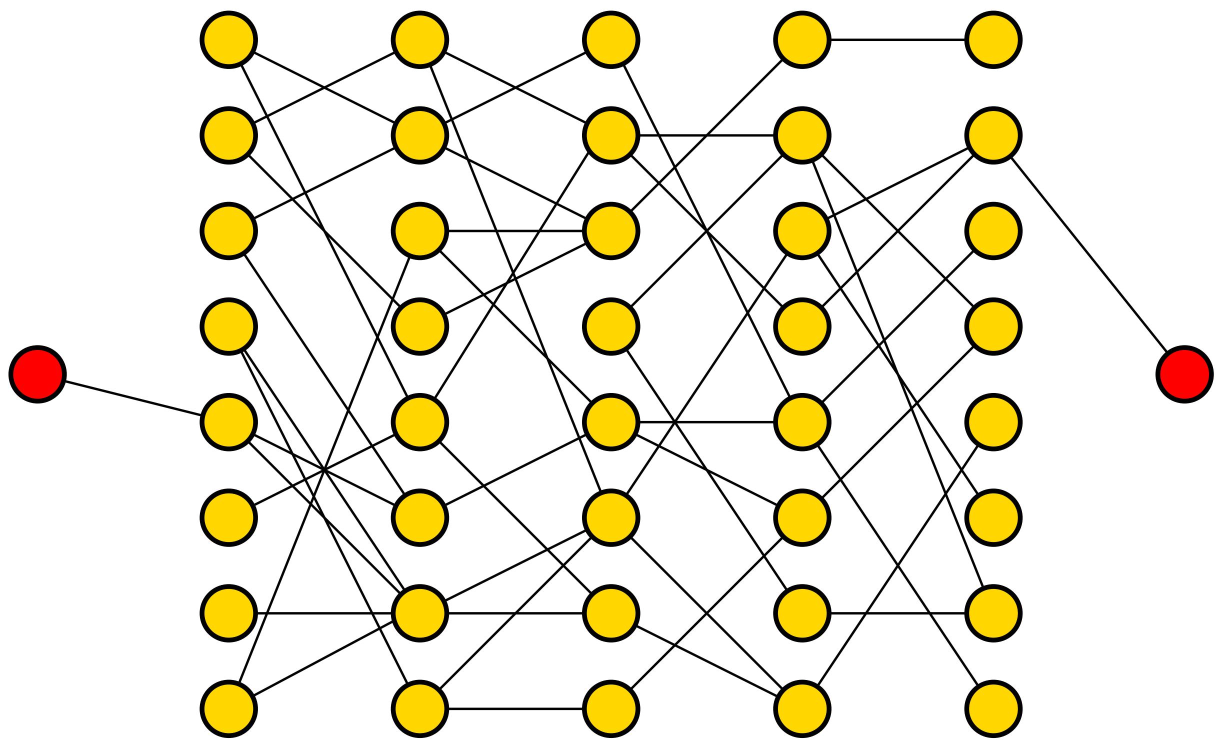

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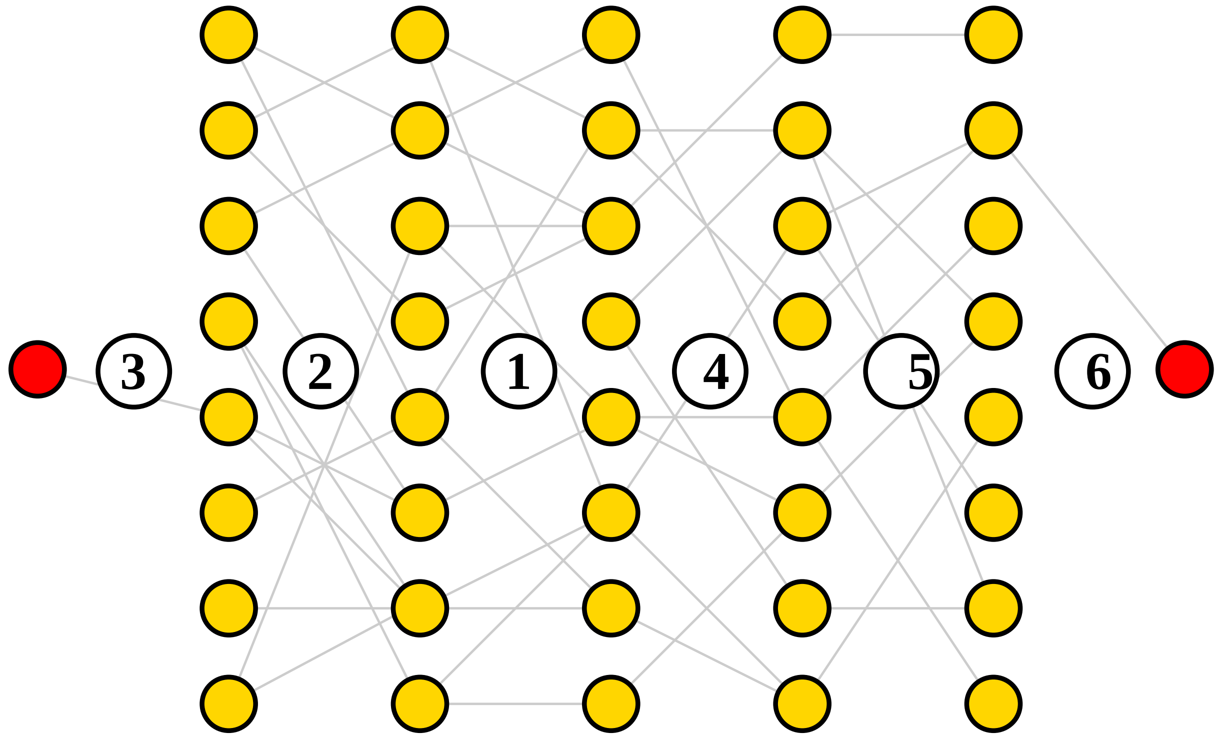

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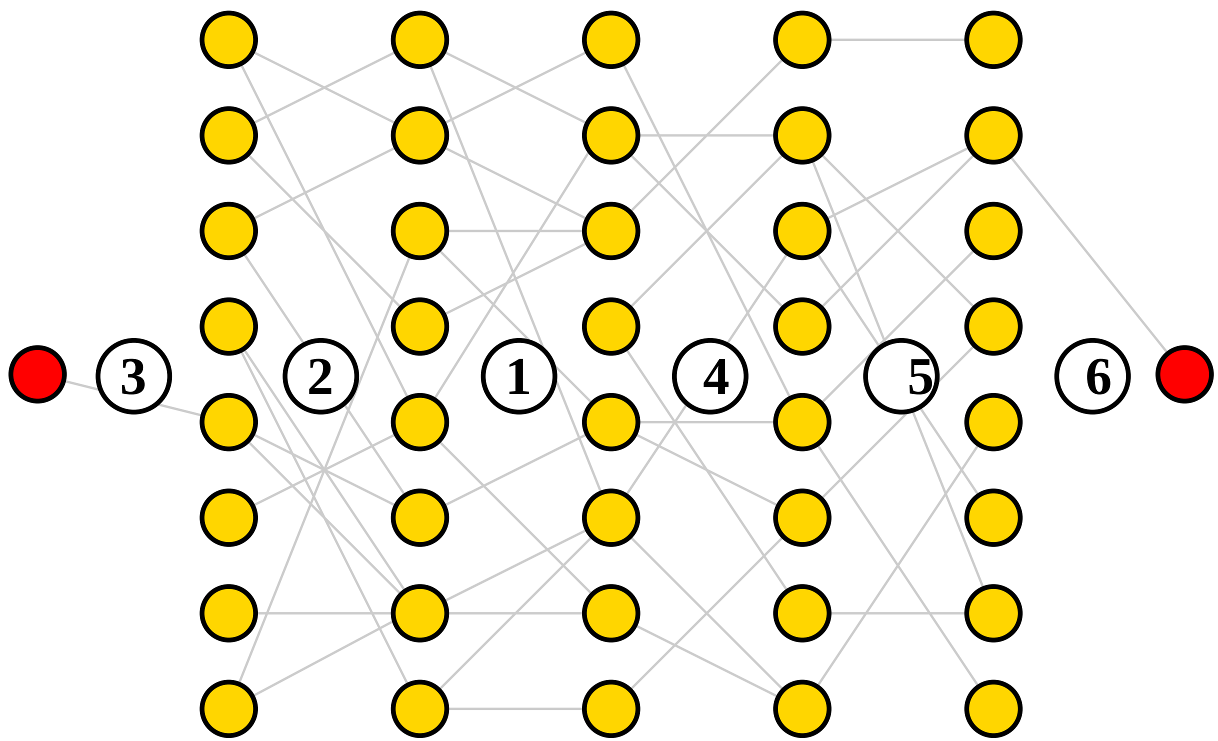

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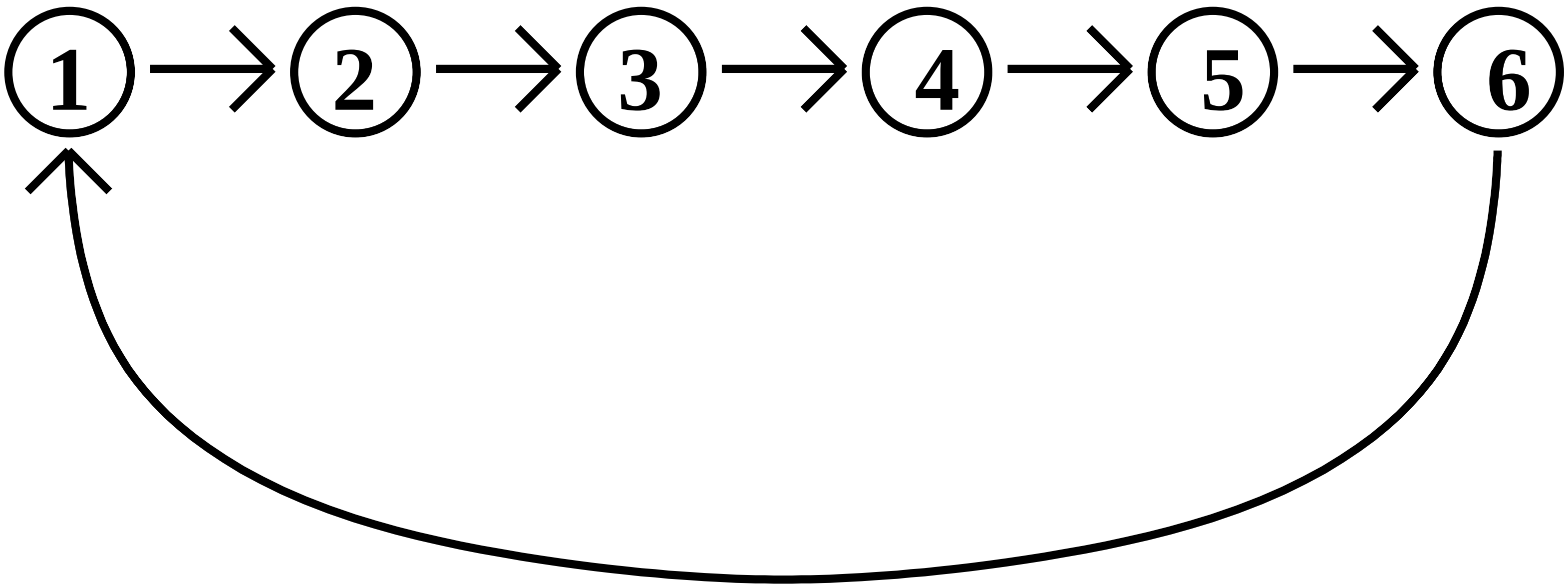
Streaming and Communication Protocols

- Assign each layer to one player



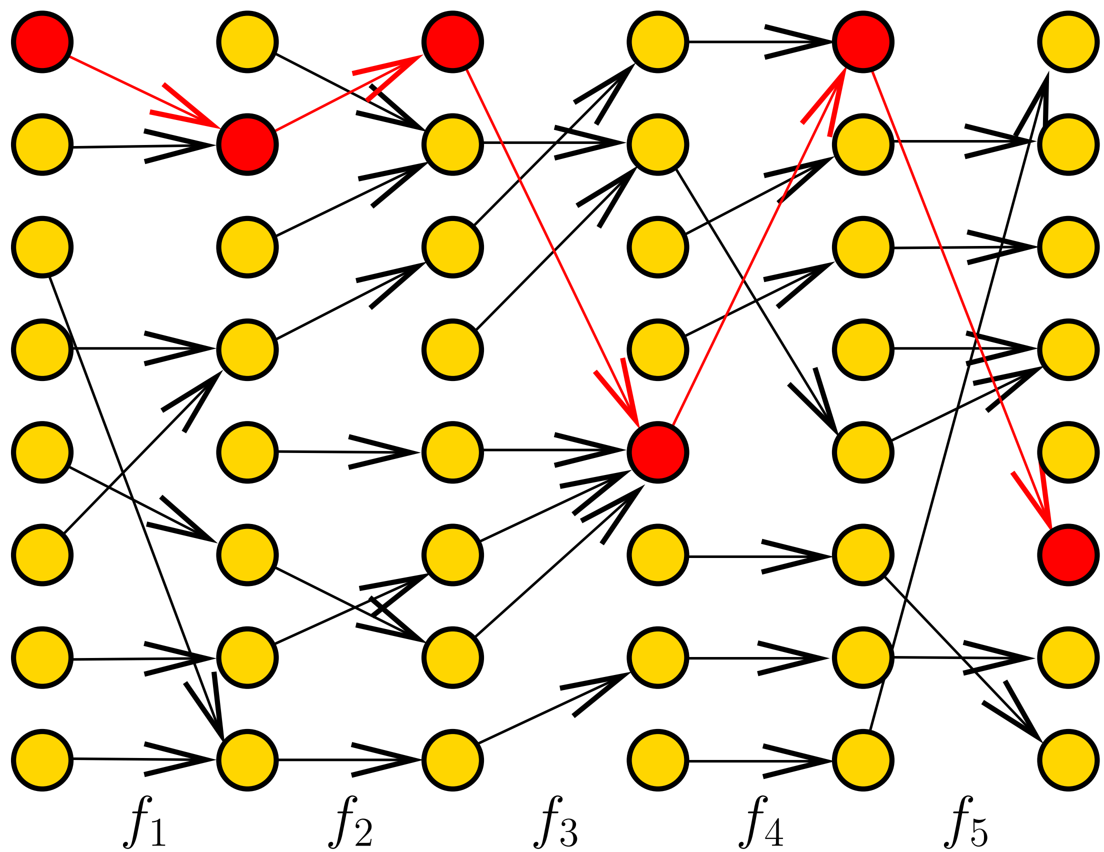
- Small-space streaming algorithm
  $\Rightarrow$ efficient communication protocol
- Goal: prove communication lower bound

The Proof

Important Problem: Pointer Chasing

Definition:

- Input: p functions $f_i : [n] \to [n]$
- Goal: Compute $f_p(f_{p-1}(\dots f_2(f_1(1))\dots))$



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Two-player version:

- What players have:

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$f_2, f_4, f_6, \dots$	$f_1, f_3, f_5, \dots$

- Alice speaks first

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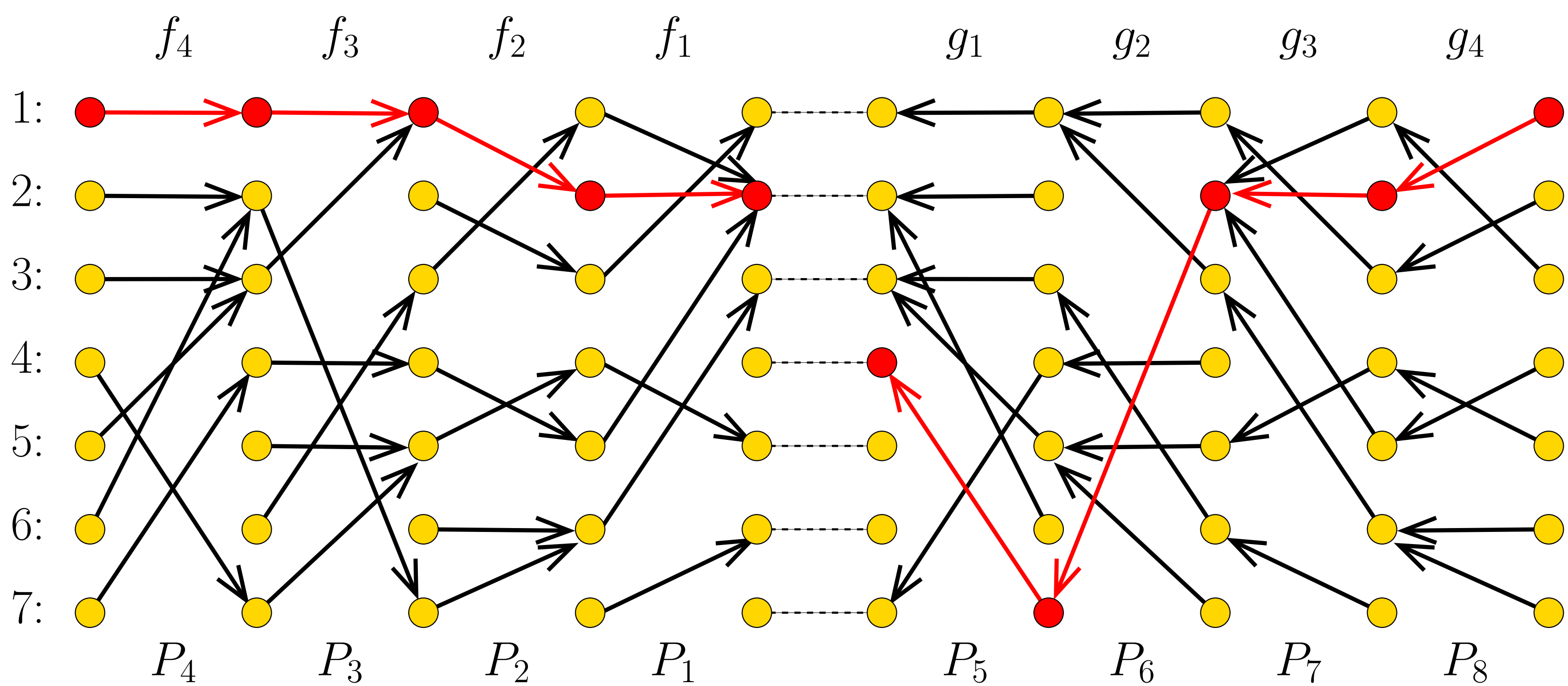
Our problem:

- Only need to check if BFS trees intersect
- Seems hard to infer full tree from this

Proof Overview

Problem BBB (Basic Building Block):

- $2p$ players with two instances of pointer chasing



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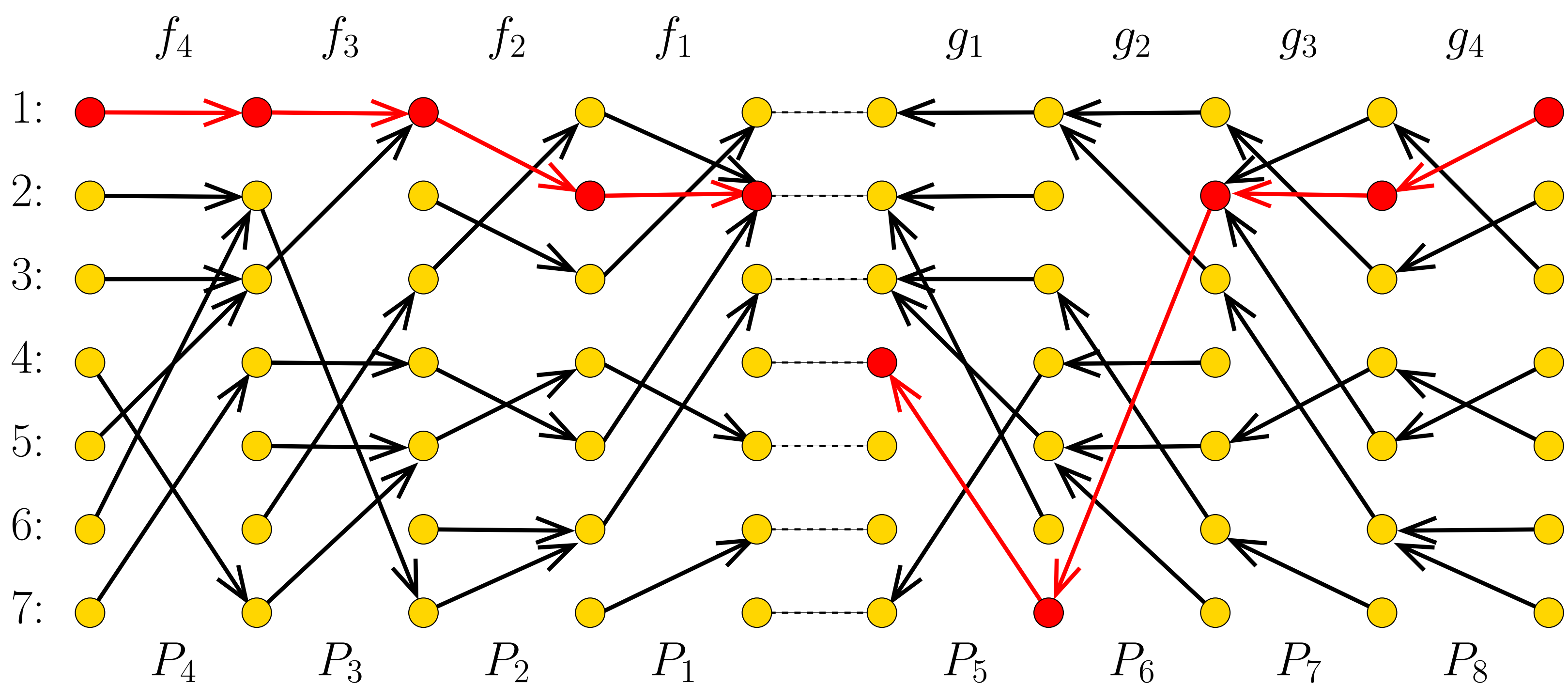

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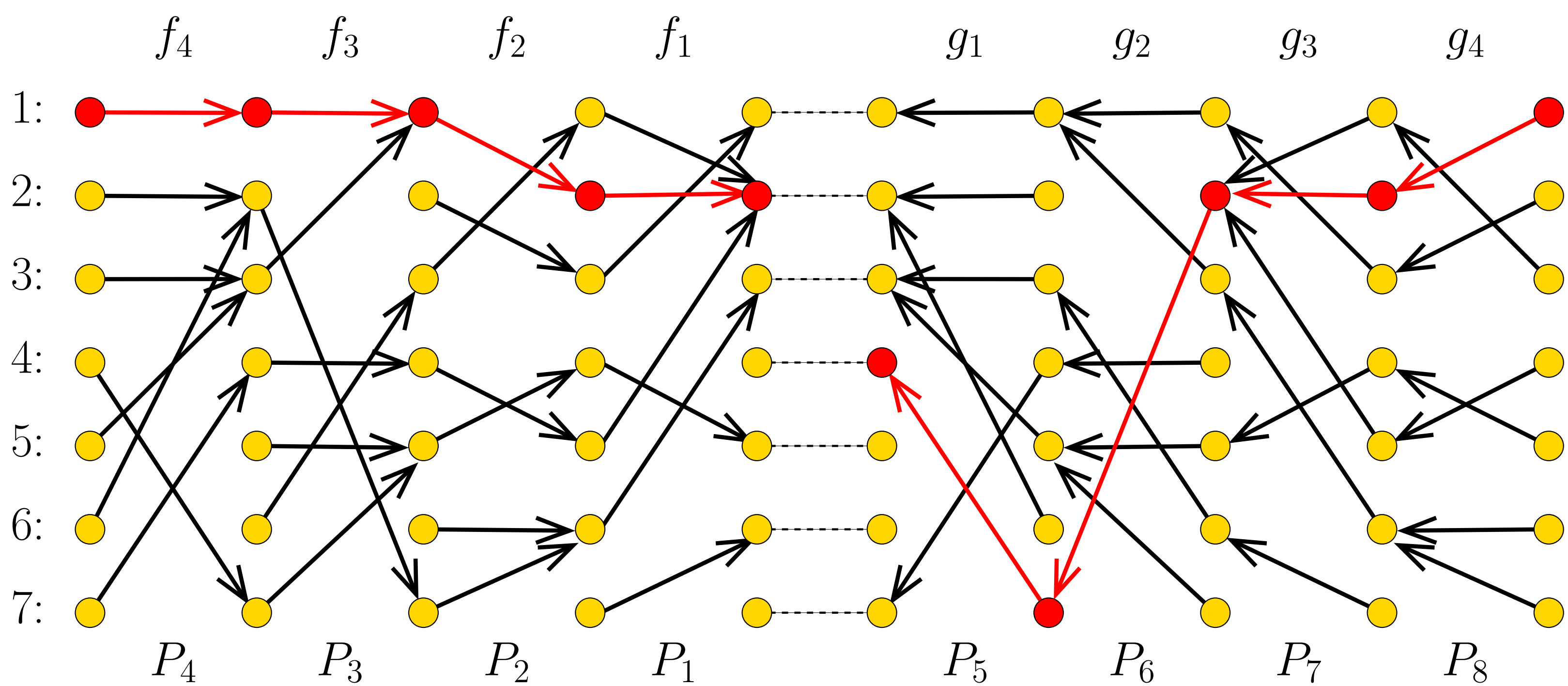

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   - Implies: $\mathrm{CC}_{1/10}(\bigvee_{i=1}^{k} \text{BBB}) \gtrsim \Omega(kn)$
3. $\mathrm{CC}_{1/20}(\text{BFS tree intersection}) \gtrsim \mathrm{CC}_{1/10}(\bigvee_{i=1}^{k} \text{BBB})$ for $k = n^{O(1/p)}$

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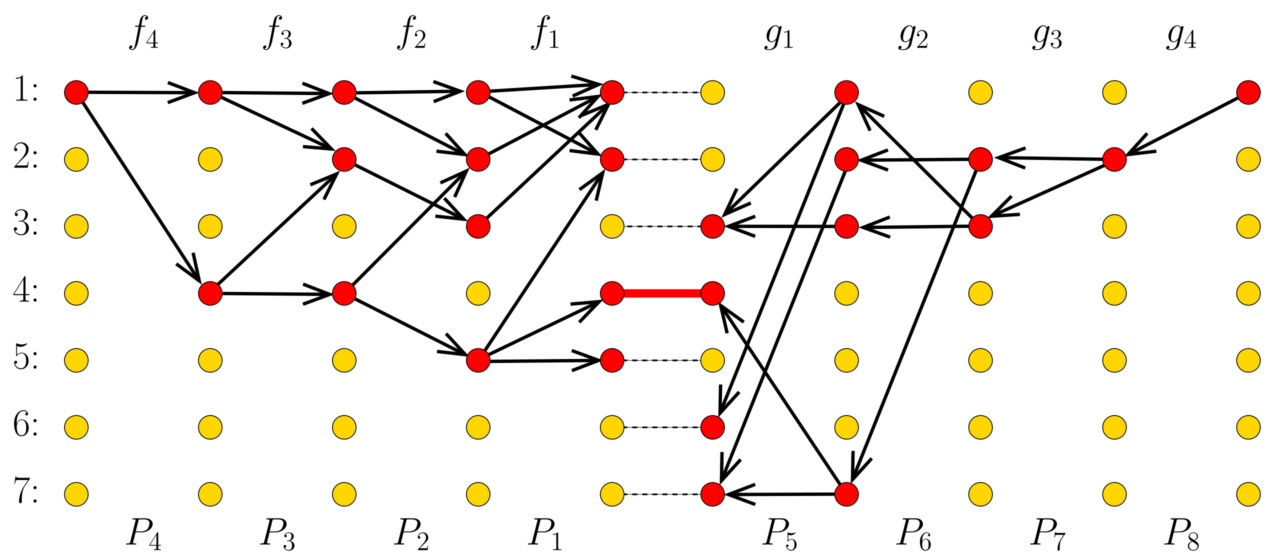

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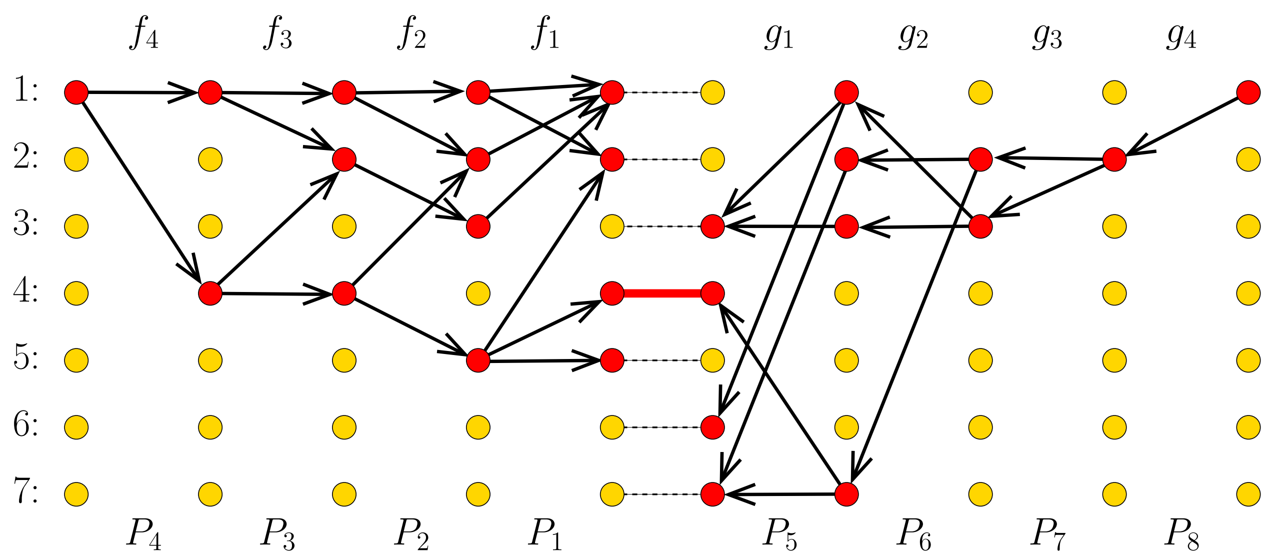

Gives instance of BFS tree intersection, but pointers from two different instances may intersect

Step 3

$$\mathrm{CC}_{1/20}(\text{BFS tree intersection}) \gtrsim \mathrm{CC}_{1/10}(\bigvee_{i=1}^{k} \text{BBB})$$

for $k = n^{\Theta(1/p)}$

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- Randomly relabel intermediate results of functions and stack them on top of each other
  - If pair of pointer chasing instances gives the same element, BFS trees intersect
  - $k^p \ll n$ and random scrambling $\Longrightarrow$ If no pair gives the same element (and no $\Theta(\log n)$-to-1 mapping), BFS trees unlikely to intersect

Step 2

Statement:

$\mathrm{IC}_{\mu^k,1/(2n^2)}(\bigvee_{i=1}^{k} \text{BBB}) \gtrsim k \cdot \mathrm{IC}_{\mu,1/n^2}(\text{BBB}) \approx \Omega(kn)$ for $k \ll n$

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  information cost $= \sum_{i=1}^{k}$ information cost on instance i

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- Information cost won't decrease significantly on
  $\bigvee$(other instances) $=$ true

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- communication complexity for pointer chasing is $\Omega(n)$ for uniform distribution [Nisan, Wigderson 1993], [Guha, McGregor 2007]

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Obstacles:

1. Need a proof for information complexity
2. Equality of pointer chasing instances
   - Need to account for impact of $\Theta(\log n)$-to-1 maps

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- Use [Jain, Radhakrishnan, Sen 2003]?
- $\Pi =$ constant-round protocol revealing information IC with error ϵ:

  There is a protocol Π' with total communication $\sim \mathrm{IC}/\delta^2$ that errs with probability $\epsilon + \delta$

i.e., "small information $\Rightarrow$ small communication"

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- Won't suffice for us: $\delta = o(1/n)$

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Our solution (part 1):

- Use techniques of [JRS] to produce a protocol Π'
  - Π' is deterministic
  - errs with twice the probability
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- Note: prob. of long message $\gg$ prob. of answer YES

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  - by induction, $H(f_{\text{next}}(x_{\text{current}})) = \log n - o(1)$
    - for $1 - o(1)$ fraction of internal nodes
    - for $1 - o(1)$ fraction of leaves

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  - with this entropy, prob. of correct solution is $o(1)$

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  (impact of rare long messages small)

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- Probability $\Omega(1/n)$ for $\frac{3}{4}n$ elements

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  $$n/4 \cdot \Omega(1/n)^2 = \Omega(1/n)$$
- protocol errs with probability $\Omega(1/n)$

Main result:

Shortest Path, Perfect Matching, and Directed Connectivity require $\sim n^{1+\Omega(1/p)}$ space in p passes

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- Simpler proof?
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Open Questions:

- Simpler proof?
- Improve lower bounds from $\sim\Omega(n^{1+1/(2p)})$ to $\sim\Omega(n^{1+1/p})$?
- Better bounds for maximum matching?
  - Is looking for a few augmenting paths harder?
  - Can the techniques be used for approximate matchings?

Questions?