Learning and Testing Structured Distributions

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This Talk

Algorithmic Framework for Distribution Estimation:
Leads to *fast & robust* estimators
for a wide variety of statistical models.

[Chan-D-Servedio-Sun, STOC'14]
[Chan-D-Servedio-Sun, NIPS'14]
[Acharya-D-Hegde-Li-Schmidt, PODS'15]

[Acharya-D-Li-Schmidt, manuscript'15]

Key Idea:

Exploit piecewise polynomial approximation
for structured model estimation

This Talk

A family of optimal estimators for hypothesis testing
for a wide variety of structured models.

"Given samples from a statistical model does it satisfy a given property?"

[Daskalakis-D-Servedio-Valiant-Valiant, SODA'13]
[Chan-D-Valiant-Valiant, SODA'14]
[D-Kane-Nikishkin, SODA'15]

[D-Kane-Nikishkin, manuscript'15]
[Canonne-D-Gouleakis-Rubinfeld, manuscript'15]

Main Message of the Talk

We can algorithmically exploit the underlying structure to perform statistical estimation efficiently.

Outline

- Learning via Piecewise Polynomial Approximation
  - Introduction
  - Framework Overview
  - Statistical Efficiency
  - Computational Efficiency
  - Empirical Results
- Applications to other Inference Tasks
- Future Directions and Concluding Remarks

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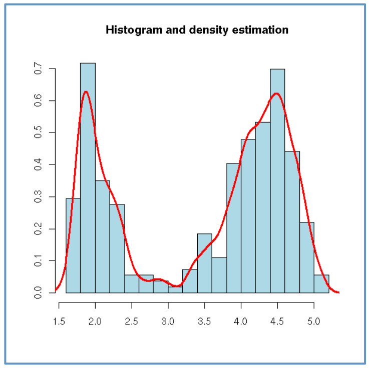
Distribution Learning (Density Estimation)

Given samples (observations) from an unknown probability distribution (model), construct an accurate estimate of the distribution.

- Classical Problem in Statistics
- Introduced by Karl Pearson (1891).
- Last fifteen years (TCS): computational aspects

Distribution Learning: History

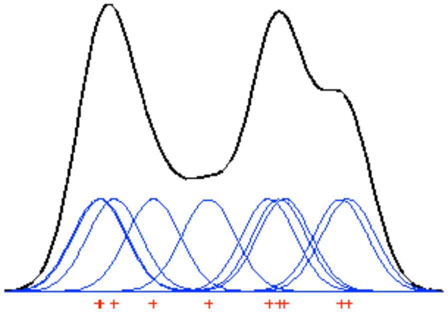


- Histograms [Pearson, 1895]

- Kernel methods [M. Rosenblatt, 1956]

- Metric Entropy [A.N. Kolmogorov, 1960]

- Wavelets
  [Donoho, Johnstone, Kerkyacharian, Picard, ’90’s]

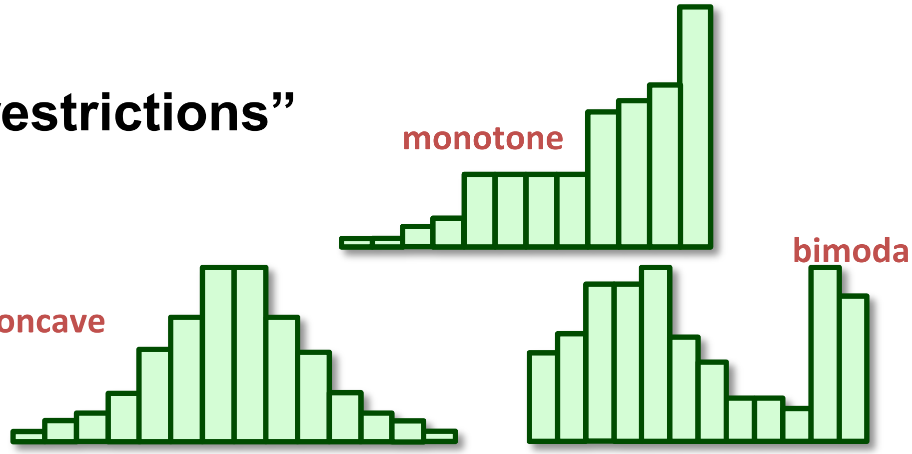


Many others: Nearest Neighbor, Orthogonal Series, …

Focus traditionally on sample size.

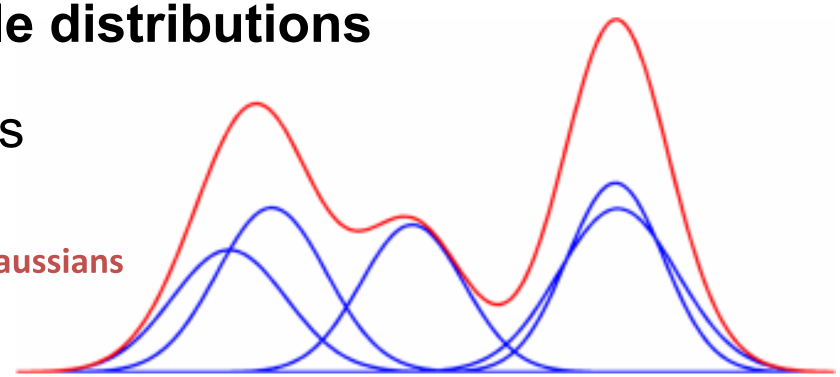
Types of Structured Distributions

- **Distributions with "shape restrictions"**

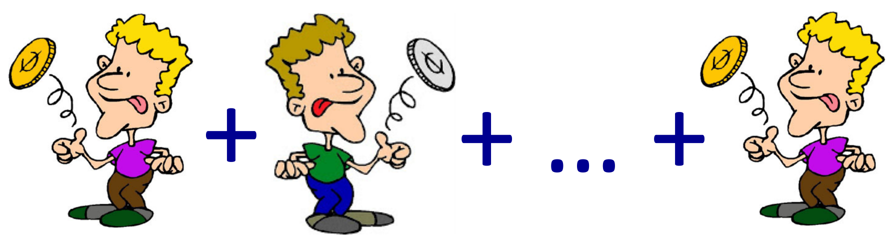


- **Simple combinations of simple distributions**

Mixtures of simple distributions



Sums of simple distributions



History

Nonparametric Estimation under “shape restrictions”

- Long line of work in statistics since the 1950’s
  [Gre’56, Rao69, Weg70, Gro85, Bir87,…]

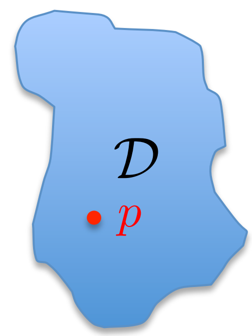
- Shape restrictions studied in early work: monotonicity, unimodality, concavity, convexity, Lipschitz continuity, …

- Very active research area: log-concavity, *k*-monotonicity, …
  [Balabdaoui-Wellner’07, Balabdaoui-Rufibach-Wellner’09, Walther’09, Dumbgen-Rufibach’09, Cule-Samworth’10, Koenker-Mizera’10, Guntuboyina-Sen’13, Doss-Wellner’13, Kim-Samworth’14]

- Standard tool in these settings: MLE

Distribution Learning: Definition

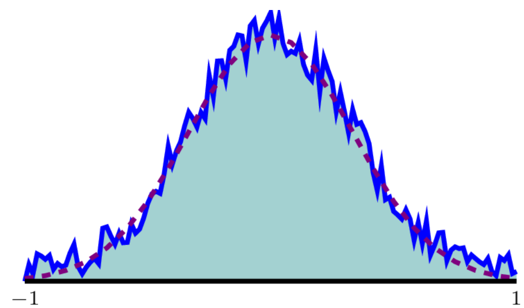
- Learning problem defined by family $\mathcal{D}$ of distributions



- Target distribution $p \in \mathcal{D}$ unknown to learner.
- Learner given sample of IID draws from p.

Output: with probability $\geq 9/10$ output h satisfying

$$\|h - p\|_1 \leq \epsilon.$$



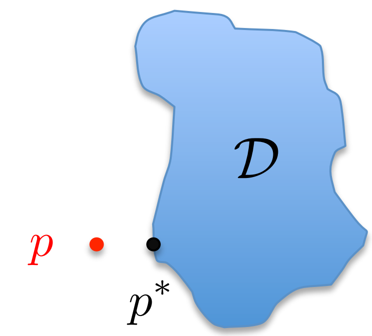
Goal: Sample optimal & computationally efficient algorithms

Agnostic Learning: Definition

- Learning problem defined by class $\mathcal{D}$ of distributions

- Target distribution p unknown to learner and let

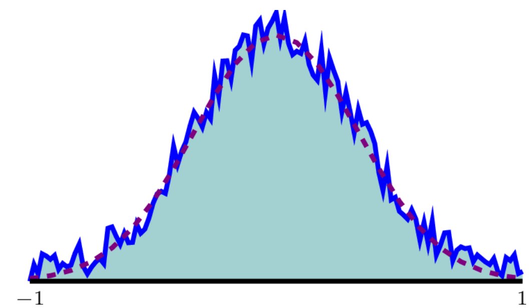
$$\mathrm{OPT} = \inf_{q \in \mathcal{D}} \|p - q\|_1$$



- Learner given sample of IID draws from p

Output: with probability $\geq 9/10$ output h satisfying

$$\|h - p\|_1 \leq \mathrm{OPT} + \epsilon.$$

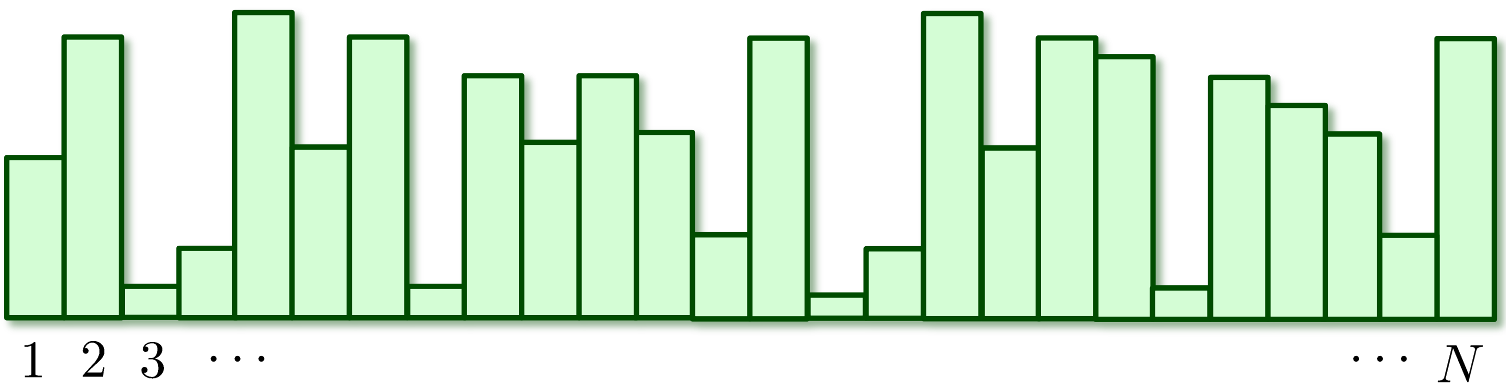


Goal: Sample optimal & computationally efficient algorithms

Learning Arbitrary Discrete Distributions

Let $\mathcal{D}$ be the set of all distributions over $[N]$.

What is the best learning algorithm?

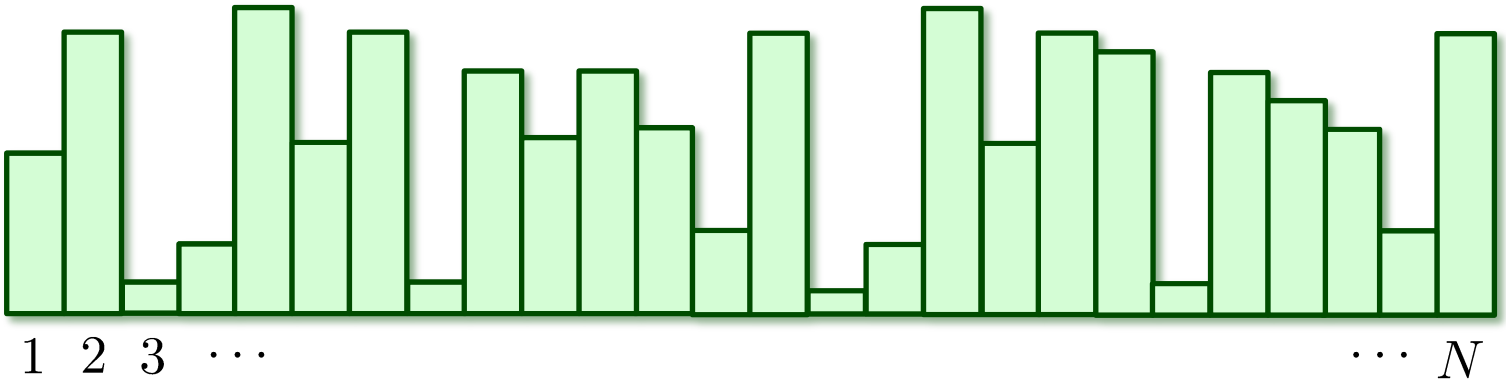


Simple answer (folklore):

- Algorithm with sample (and time) complexity $O(N/\epsilon^2)$.
- Information theoretic lower bound of $\Omega(N/\epsilon^2)$.

Learning Arbitrary Discrete Distributions

Learning an *arbitrary* distribution over $[N]$:
Sample size $\Theta(N/\epsilon^2)$
necessary and sufficient



When can we do better?

Which distributions are easy to learn, which are hard (and why)?

Structure and Statistical Estimation

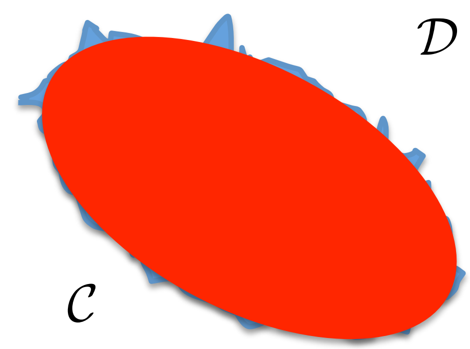
General Recipe for Statistical Estimation:

Given a "complex" distribution family $\mathcal{D}$.

1. Find a "canonical" class of distributions $\mathcal{C}$ that approximates $\mathcal{D}$ well.

(For every $p \in \mathcal{D}$ there is $q \in \mathcal{C}$ such that $p \approx q$.)

2. Use samples from p to estimate it as if it was q.



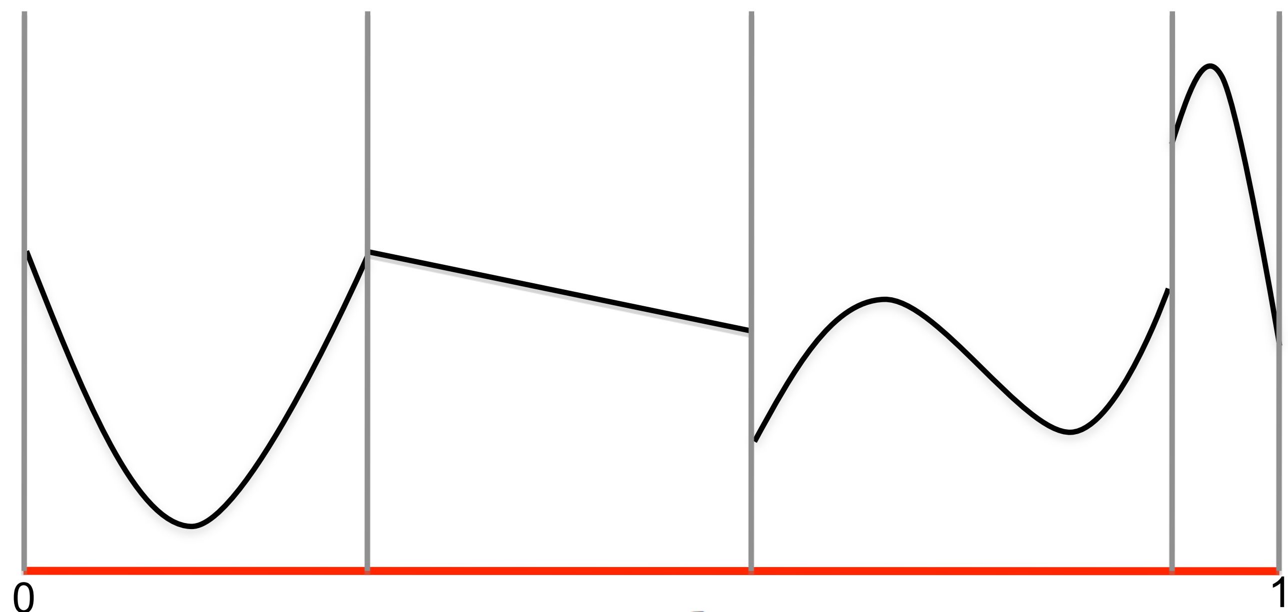
Reduction-based approach.

Main difficulty: Algorithm for $\mathcal{C}$ should be **robust to error** in the data.

Question: Which "canonical" class should we use?

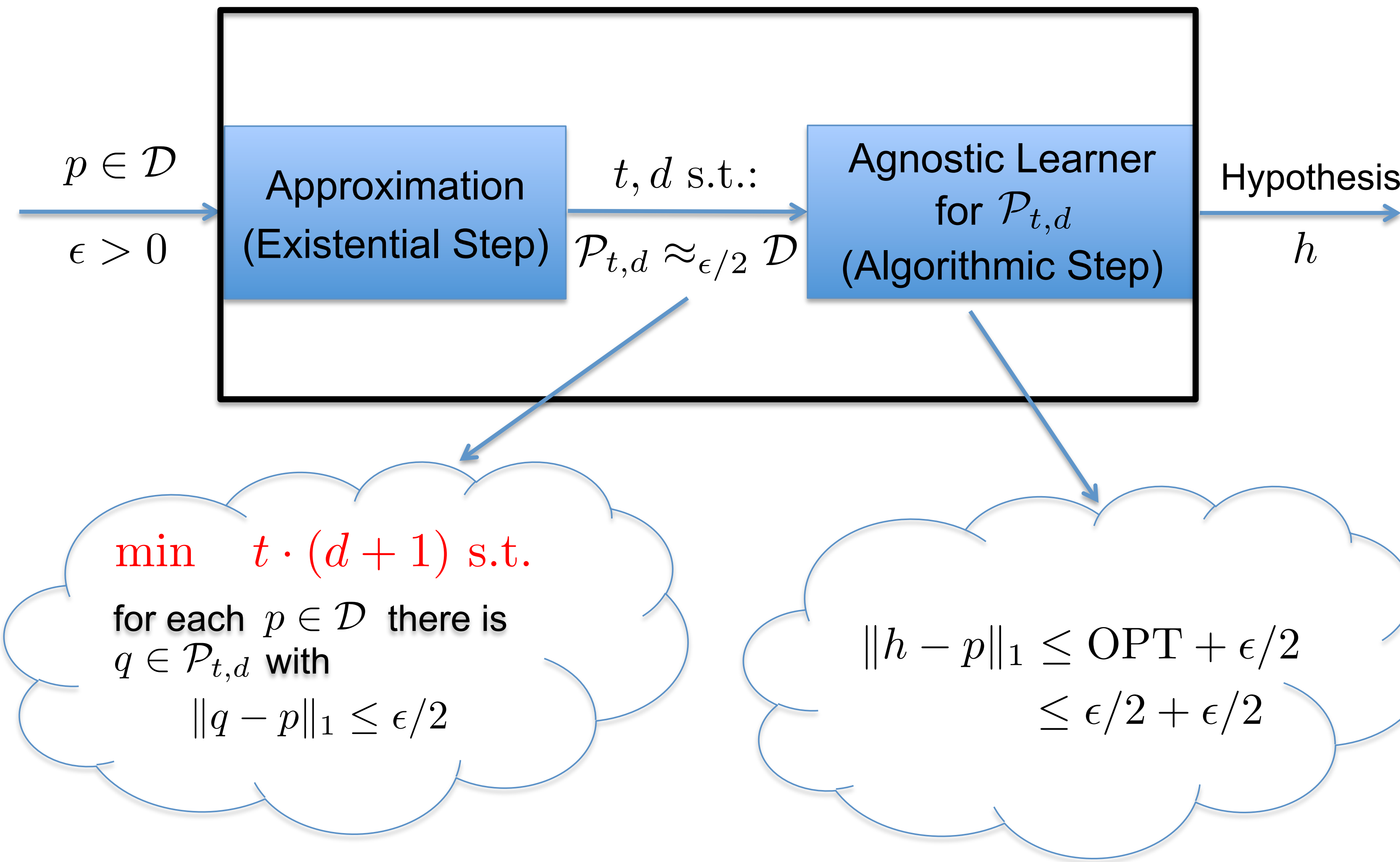
Piecewise polynomial distributions

- Distribution p is **t-piecewise degree-d** if there exists a partition of the domain into t intervals such that within each interval, the density of p is a degree-d polynomial.
- Let $\mathcal{P}_{t,d}$ be the family of all such distributions.



$$t = 4, d = 3$$

Overview of Framework



Why Piecewise Polynomials?

- Analogy with PAC learning of Boolean functions
  [Linial-Mansour-Nisan'93]
- Common method in statistics: fitting splines to data
  [Wegman-Wright'83, Stone et al.'90's, Willet-Nowak'07]
- Gives sample optimal and computationally efficient estimators for wide range of distribution families

Results: Learning Structured Families

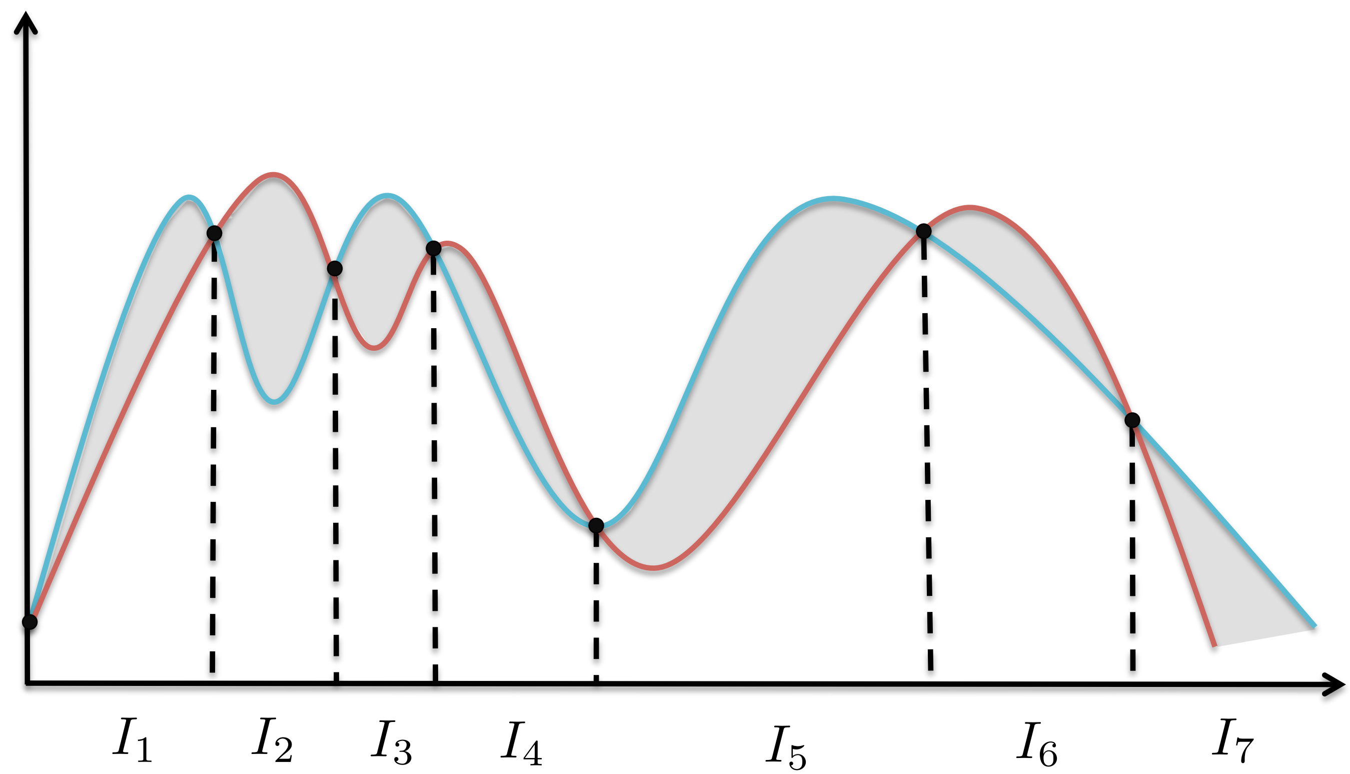
Previous work
(parameter estimation):
Moitra-Valiant'10
$(1/\epsilon)^{\Omega(k)}$

Distribution	Sample Size	Parameters	Reference
	$\epsilon^3)$	$t = \log n/\epsilon, d = 0$	Birgé'87
	$/\epsilon^3)$	$t = k \log n/\epsilon, d = 0$	Daskalakis-D-Servedio'12
hazard rate	$O(\log(n/\epsilon)/\epsilon^3)$	$t = \log(n/\epsilon), d = 0$	Chan-D-Servedio-Sun' 13
log-concave k-mixture	$O(k/\epsilon^{5/2})$	$t = k/\sqrt{\epsilon}, d = 1$	Chan-D-Servedio-Sun' 14, D-Kane'15
Gaussian k-mixture	$\widetilde{O}(k/\epsilon^2)$	$t = k, d = \log(k/\epsilon)$	Chan-D-Servedio-Sun' 14
Poisson/Binomial k-mixture	$\widetilde{O}(k/\epsilon^2)$	$t = k, d = \log(k/\epsilon)$	Daskalakis-D-Stewart'15
Besov spaces	$O(1/\epsilon^{2+1/\alpha})$	$t = \epsilon^{-1/\alpha}, d = \lceil \alpha \rceil$	Devore' 98
k-monotone	$O(k/\epsilon^{2+1/k})$	$t = k, d = \epsilon^{-1/k}$	Konovalov-Leviatan'07

Statistical Performance: Intuition (I)

Question: Let *p*, *q* be probability density functions. How many samples are required to distinguish between them?

Partial Answer: If *p*, *q* have a few "crossings", distinguishing is easy.

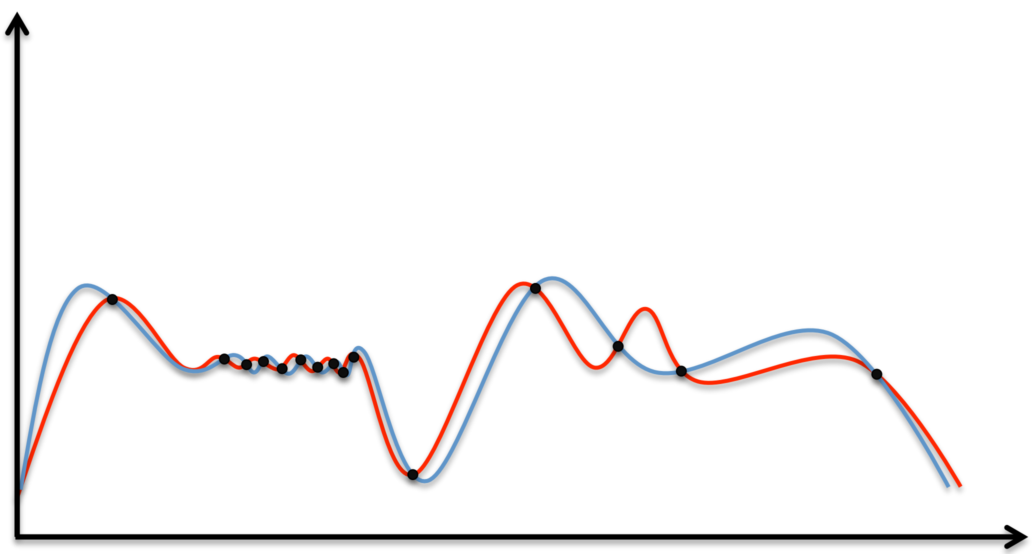


Statistical Performance: Intuition (II)

Question: Let p, q be probability density functions. How many samples are required to distinguish between them?

Partial Answer: If p, q have a few "crossings", distinguishing is easy.

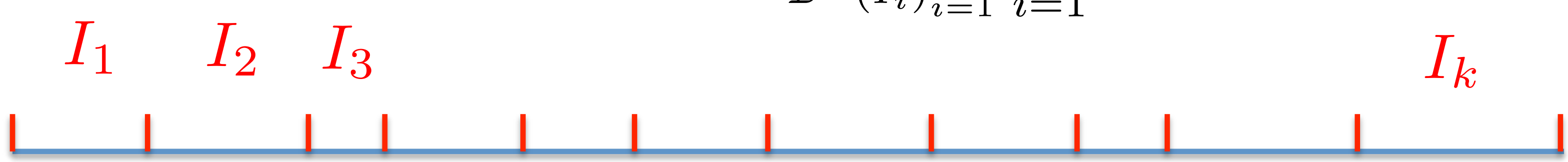
Typically, unbounded many crossings, but only a few are important.



"Complexity measure" for learning a distribution family

Definition. For $p, q : \mathbb{R} \to \mathbb{R}_+$ and $k \geq 1$, we define the $\mathcal{A}_k$ - distance between p, q as follows:

$$\|p - q\|_{\mathcal{A}_k} = \sup_{\mathcal{I}=(I_i)_{i=1}^k} \sum_{i=1}^{k} |p(I_i) - q(I_i)|$$



Upper Bound on Sample Complexity: For a family of one-dimensional distributions $\mathcal{D}$ and $\epsilon > 0$, let $k = k(\mathcal{D}, \epsilon)$ be the smallest integer such that for any $p, q \in \mathcal{D}$ it holds

$$\|p - q\|_1 \approx_\epsilon \|p - q\|_{\mathcal{A}_k}.$$

Then, the parameter k is an upper bound on the sample complexity of agnostic learning for $\mathcal{D}$.

Statistical Estimator

Lemma. For any $\mathcal{D}$ and $\epsilon > 0$, let $k = k(\mathcal{D}, \epsilon)$ be such that for any $p, q \in \mathcal{D}$ it holds $\|p - q\|_1 \leq \|p - q\|_{\mathcal{A}_k} + \epsilon$. Then there exists an agnostic learning algorithm for $\mathcal{D}$ using $O(k/\epsilon^2)$ samples.

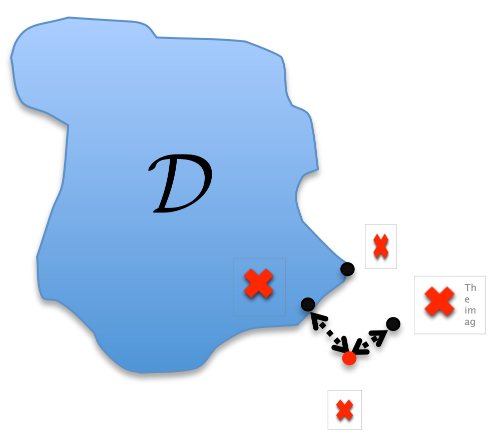
Proof Sketch.

Consider the following procedure:

1. Draw $m = \Omega(k/\epsilon^2)$ samples from p and let $\widehat{p}_m$ be the empirical distr.
2. Compute $h \in \mathcal{D}$ that minimizes $\|h - \widehat{p}_m\|_{\mathcal{A}_k}$.

Analysis:

Empirical Process Theory (Vapnik, Chervonenkis, Dudley ~70's)



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Difficulties in Implementing Estimator

For any $\mathcal{D}$ and $\epsilon > 0$, let $k = k(\mathcal{D}, \epsilon)$ be such that for any $p, q \in \mathcal{D}$ it holds $\|p - q\|_1 \leq \|p - q\|_{\mathcal{A}_k} + \epsilon$.

Algorithm:

1. Draw $m = \Omega(k/\epsilon^2)$ samples from p and let $\widehat{p}_m$ be the empirical distr.
2. Compute $h \in \mathcal{D}$ that minimizes $\|h - \widehat{p}_m\|_{\mathcal{A}_k}$.

Main Issues:

1. How do we bound the value of $k = k(\mathcal{D}, \epsilon)$?

2. How do we efficiently perform the "projection" step?
   (**Non-convex optimization problem**)

Solution: Replace $\mathcal{D}$ by $\mathcal{P}_{t,d}$ such that $\mathcal{D} \approx_{\epsilon/2} \mathcal{P}_{t,d}$

Agnostically Learning Piecewise Polynomials

Application of general framework for $\mathcal{C} = \mathcal{P}_{t,d}$ and $k = O(t(d+1))$.

1. Draw $m = \Omega(t(d+1)/\epsilon^2)$ samples from p.
2. Compute $h \in \mathcal{P}_{t,d}$ that minimizes $\|h - \widehat{p}_m\|_{\mathcal{A}_k}$.

Still non-convex optimization problem…

Main Algorithmic Contribution:

Polynomial time algorithm for Step 2.

Agnostically Learning Piecewise Polynomials

Theorem [Chan-D-Servedio-Sun, STOC'14]

There exists an agnostic learning algorithm for $\mathcal{P}_{t,d}$ that uses

$$\widetilde{O}(t(d+1)/\epsilon^2)$$

samples and runs in time

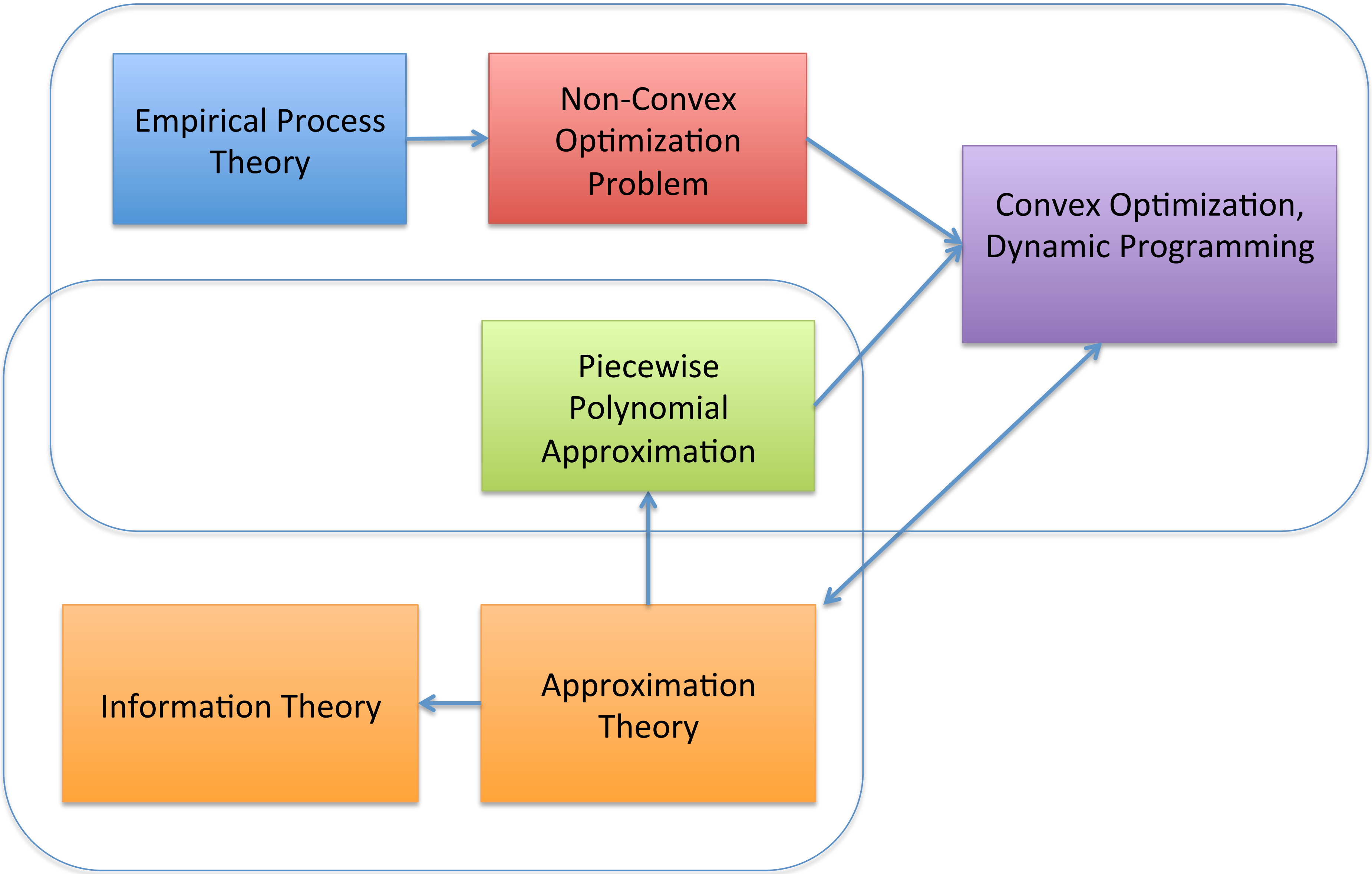
$$\mathrm{poly}(t, d+1, 1/\epsilon).$$

Moreover, $\Omega(t(d+1)/\epsilon^2)$ samples are information-theoretically necessary.

Recent Progress:

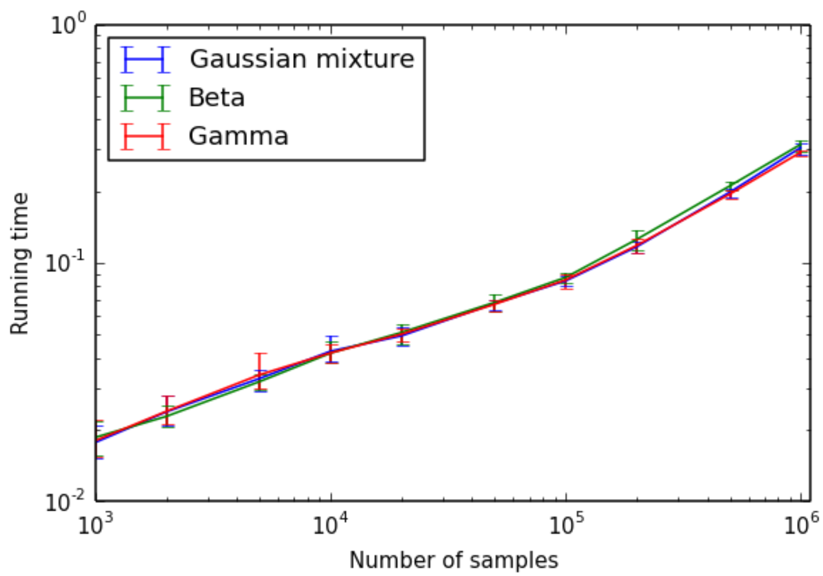
- Piecewise constant: near-linear time [Chan-D-Servedio-Sun, NIPS'14]
- **General Case:** $O(t(d+1)/\epsilon^2)$ samples and $(t/\epsilon^2) \cdot \mathrm{poly}(d)$ time. **[Acharya-D-Li-Schmidt'15]**

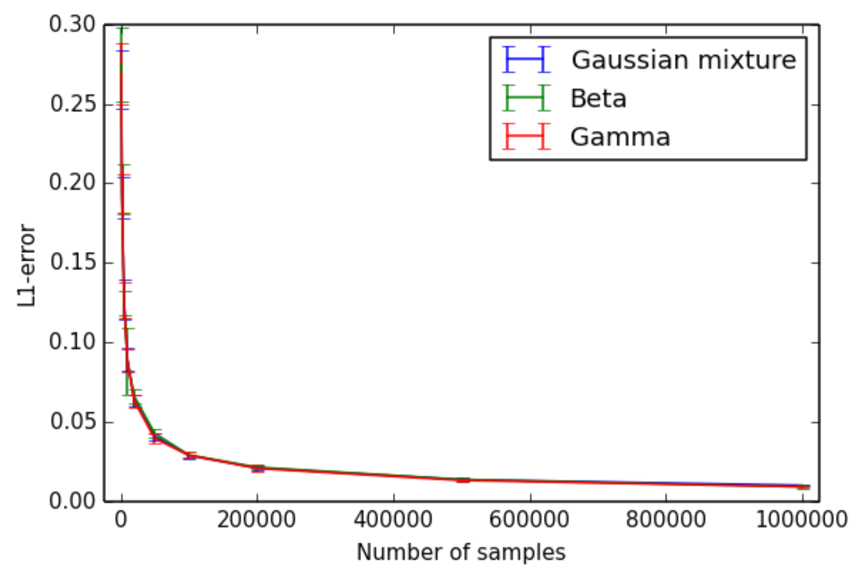
Overview of Techniques



Illustrative Empirical Results
[Acharya-D-Li-Schmidt '15]

Predictive performance of straightforward implementation:
speed-up over recent implementations of the MLE.





Application in Databases: Succinct Representation of Data

[Acharya-D-Hegde-Li-Schmidt, PODS'15]: **Approximating Data Distributions by Histograms**

Classical problem in databases

[Gibbons-Matias-Poosala' 97, Jagadish et al. '98, Chaudhuri-Motwani-Narasayya '98, Thaper-Guha-Indyk-Koudas '02, Gilbert et al. '02, Guha-Koudas-Shim '06, Indyk-Levi-Rubinfeld'12.]

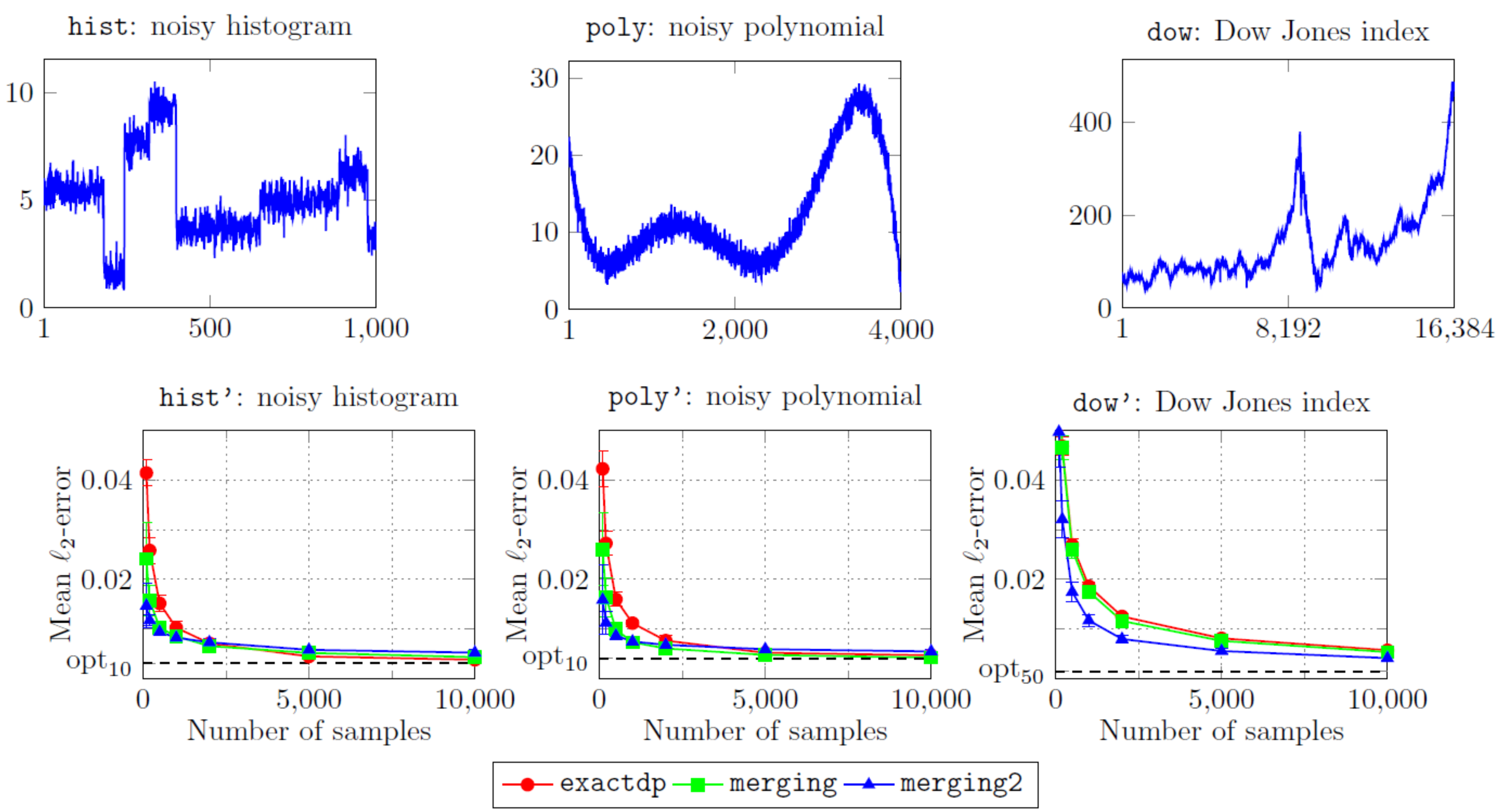
Goal: Given data distribution, construct a succinct approximation (histogram). Minimize computation time, approximation error.

Our Result: Sample optimal, sample-linear time algorithm with optimal error (up to small constant factor).

Experimental Evaluation: Outperforms all previous algorithms for the problem by one to two orders of magnitude.

Empirical Results (I) [Acharya-D-Hegde-Li-Schmidt, PODS'15]

- Two synthetic and one real-word data set (same as [Guha-Koudas-Shim'06])



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Additional Applications of Framework

Hypothesis Testing (Property Testing)

- Testing Identity of Structured Distributions [D-Kane-Nikishkin'15a, '15b]

"Given samples from a structured distribution, is it uniform?"

"Given samples from two structured distributions, are they the identical?"

- Testing Shape Restrictions [Canonne-D-Gouleakis-Rubinfeld'15]

"Given samples from a (potentially arbitrary) distribution, is it structured?"

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Future Directions

Broad Context:

Complexity theory for statistical estimation

Specific Challenges:

- Agnostic proper learning
- “Instance optimal” (adaptive) algorithms
- Tradeoffs between sample size and computational efficiency

Thank you for your attention!