Spatial Coupling vs. Block Coding: A Comparison

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Coding: From Theory to Practice
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Research Collaborators: David Mitchell, Michael Lentmaier, and Ali Pusane

Outline

- **LDPC Block Codes**
  - Parity-check matrix and Tanner graph representations, minimum distance bounds, iterative decoding thresholds, regular and irregular code designs, protograph-based constructions
- **Spatially Coupled LDPC Codes**
  - Protograph representation, edge-spreading construction, termination
  - Iterative decoding thresholds, threshold saturation, minimum distance
- **Practical Considerations**
  - Finite-length scaling, window decoding, performance, latency, and complexity comparisons to LDPC block codes, implementation aspects

LDPC Block Codes

Definition by parity-check matrix: [Gallager, '62]

Bipartite graph representation: [Tanner, '81]

$$
H = \left[\begin{array}{cccccccccccccccccccc}
0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&1\\
0&0&0&0&1&1&0&1&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&1&0&1&1&1&0&0&0&0&0&0&0&0&0\\
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1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&1&0\\
\hline
0&0&1&1&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
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1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&1&0\\
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0&0&0&0&0&0&0&0&1&0&1&0&0&0&1&0&0&0&1&0\\
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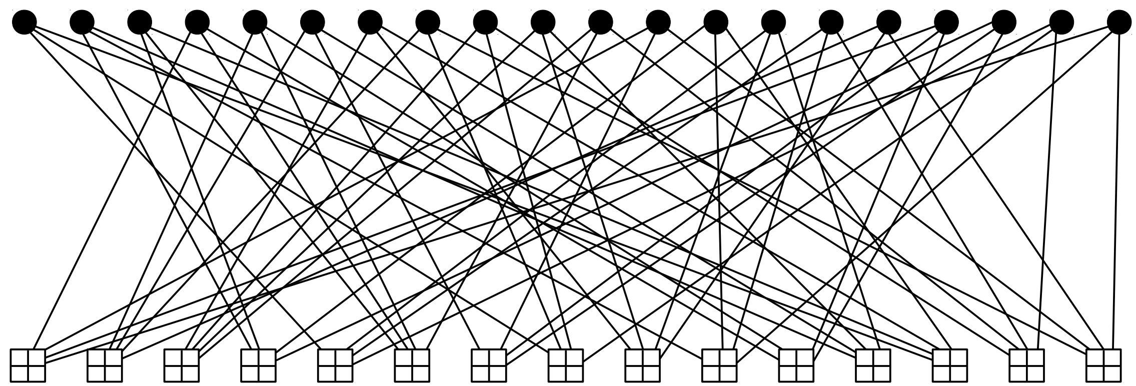
\end{array}\right]_{15 \times 20}
$$

$n = 20$ **variable nodes** of degree $J = 3$



$l = 15$ **check nodes** of degree $K = 4$

Code: $\{\mathbf{v} \mid \mathbf{v}\mathbf{H}^{\mathrm{T}} = \mathbf{0}\}$

(J,K)-regular LDPC code: $R \geq 1 - \dfrac{J}{K}$

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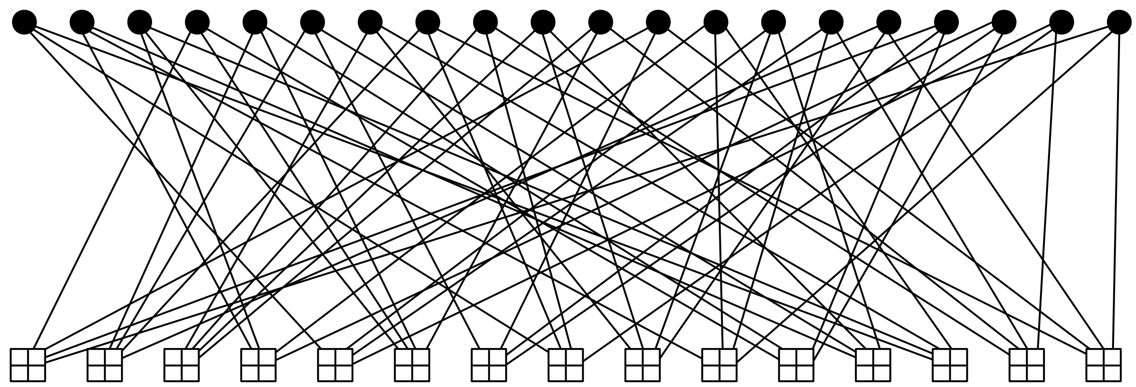

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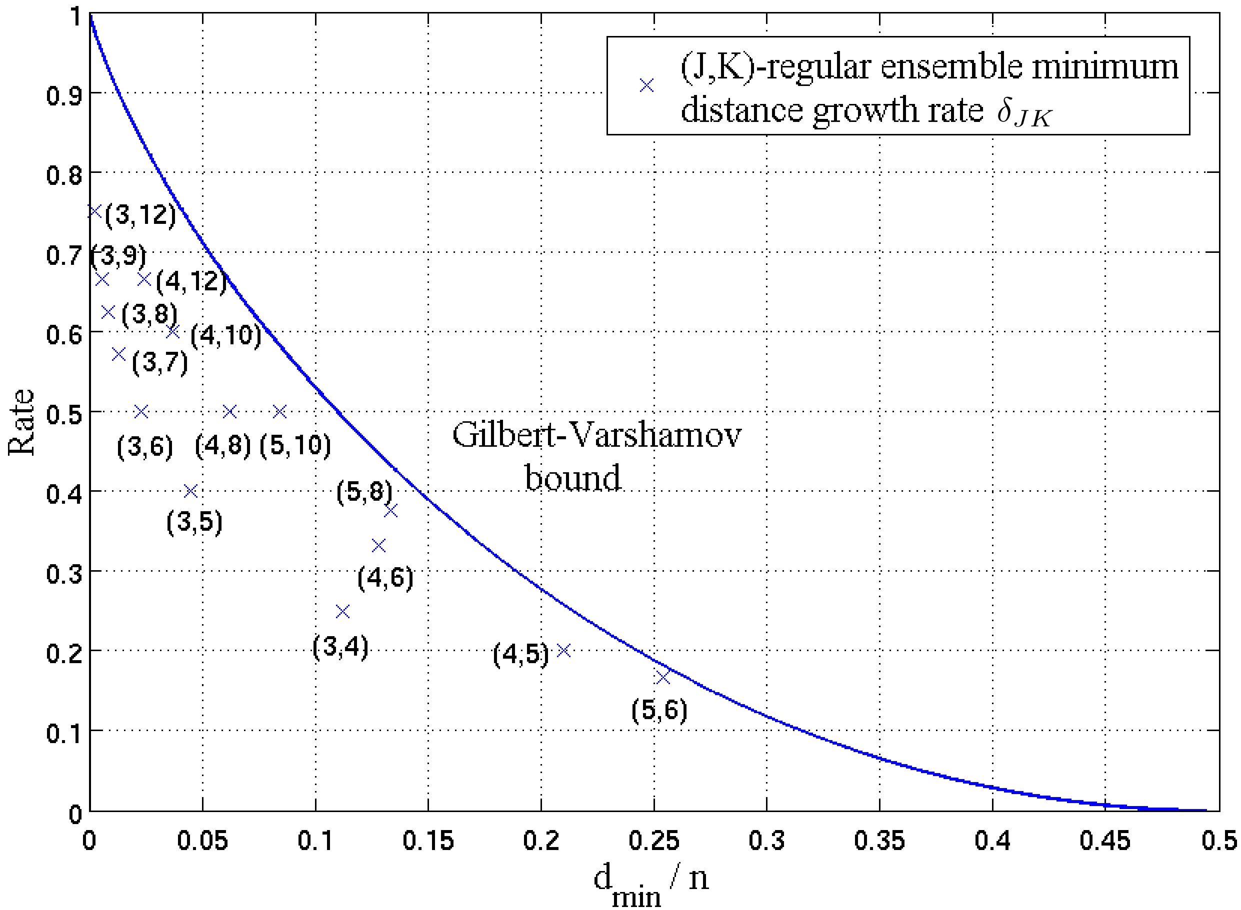
Code: $\{\mathbf{v} \mid \mathbf{v}\mathbf{H}^{\mathrm{T}} = \mathbf{0}\}$

(J,K)-regular LDPC code: $R \geq 1 - \dfrac{J}{K}$

- Graph-based codes can be decoded **iteratively** with **low-complexity** by exchanging messages in the graph using **Belief Propagation (BP).**

Code Ensembles – Minimum Distance Growth Rates

- For an **asymptotically good** code ensemble, the minimum distance grows linearly with the block length n



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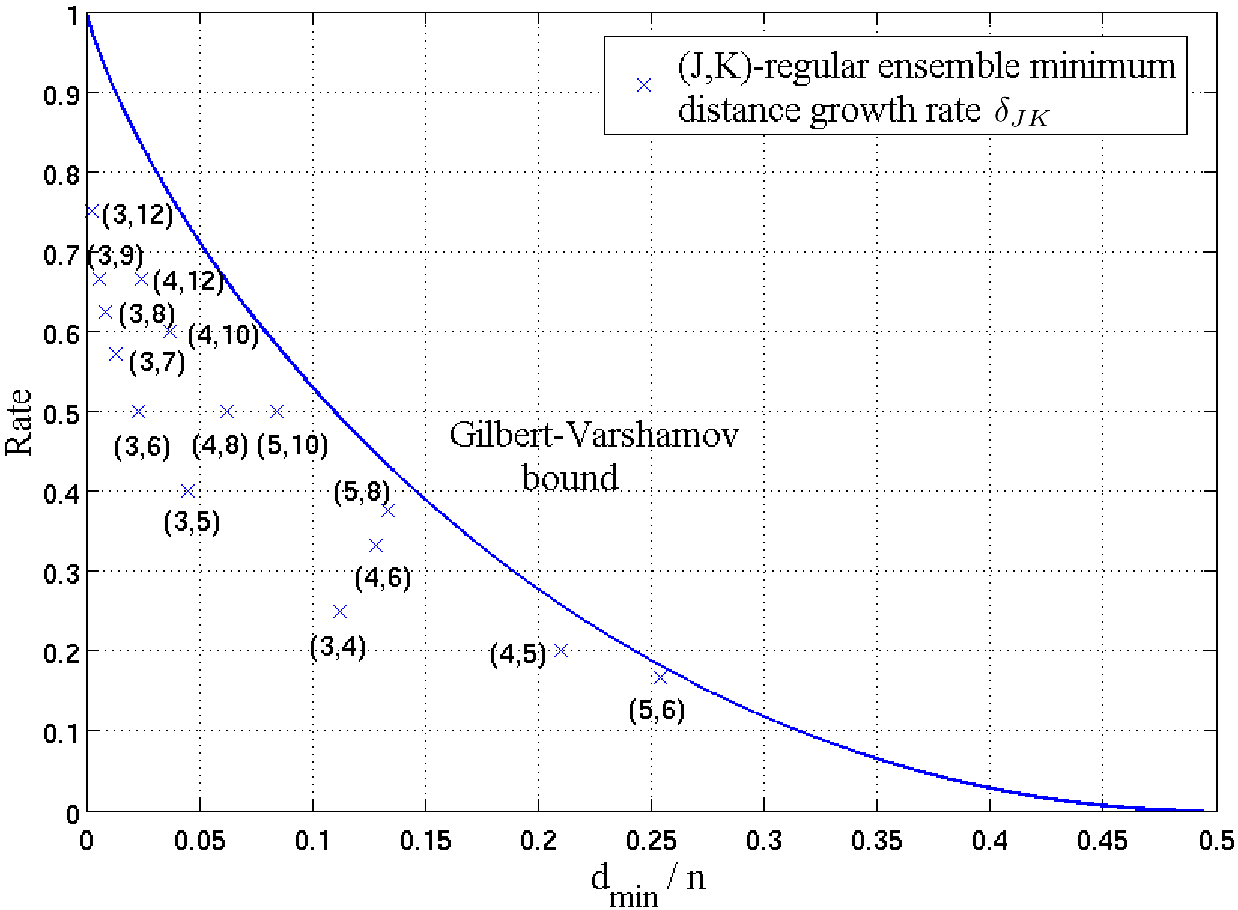

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$$\boxed{d_{\min} \geq n\delta_{JK}}$$

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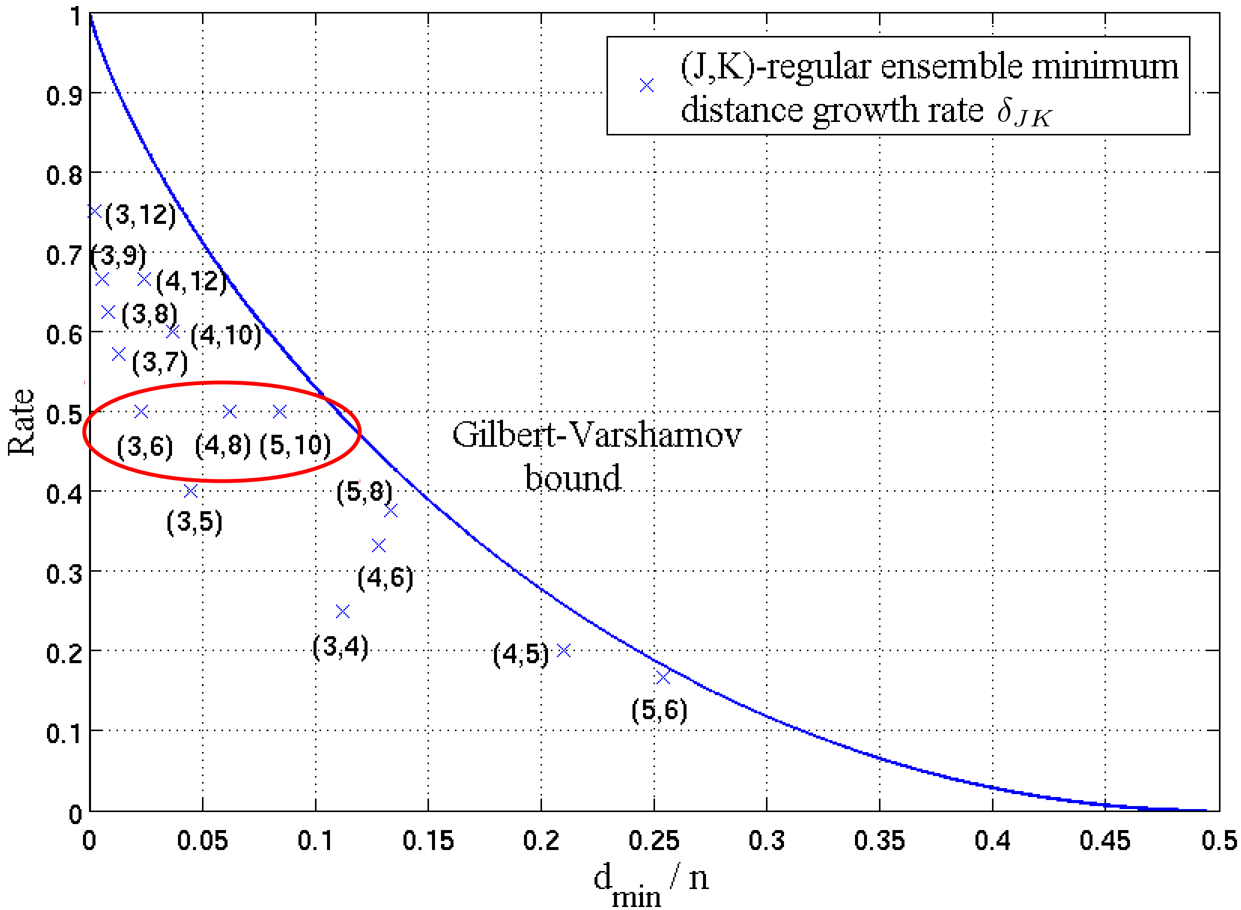

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- As the density of (J,K)-regular ensembles increases, δ_{JK} **approaches the Gilbert-Varshamov bound**.

Thresholds of (J,K)-regular LDPC Block Code Ensembles

- Iterative decoding thresholds can be calculated for (J,K)-regular LDPC block code ensembles using density evolution (DE).

BEC thresholds

J	K	Rate	ε^*	$\varepsilon_{\mathrm{Sh}}$
3	6	0.5	0.429	0.5
4	8	0.5	0.383	0.5
5	10	0.5	0.341	0.5
3	5	0.4	0.517	0.6
4	6	0.333	0.506	0.667
3	4	0.25	0.647	0.75

AWGNC thresholds

J	K	Rate	$(E_b/N_0)^*$	$(E_b/N_0)_{\mathrm{Sh}}$
3	6	0.5	1.11	0.184
4	8	0.5	1.61	0.184
5	10	0.5	2.04	0.184
3	5	0.4	0.96	−0.229
4	6	0.333	1.67	−0.480
3	4	0.25	1.00	−0.790

[RU01] T. J. Richardson, and R. Urbanke, “The capacity of low-density parity-check codes under message passing decoding”, *IEEE Transactions on Information Theory*, vol. 47 no. 2, Feb. 2001.

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- There exists a relatively **large gap to capacity**.
- Iterative decoding thresholds get **further from capacity** as the graph **density increases**.

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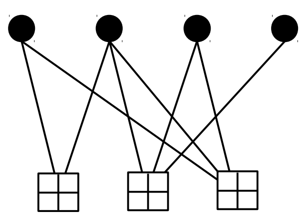
Protograph Representation

- **Compact representation** of a structured LDPC block code ensemble with code length $n = Mb_v$ and code design rate $R \geq (b_v - b_c)/b_v$

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}_{b_c \times b_v}$$

base matrix

b_v variable nodes



b_c check nodes

protograph

[Tho05] J. Thorpe, "Low-Density Parity-Check (LDPC) codes constructed from protographs", *Jet Propulsion Laboratory INP Progress Report*, Vol. 42-154 Aug. 2003.

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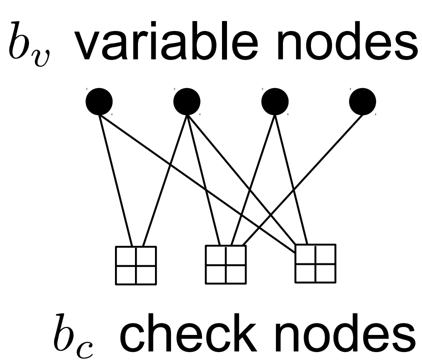
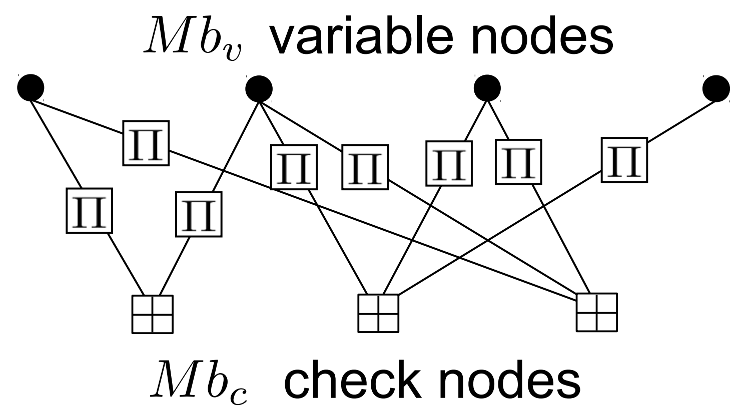
base matrix **parity-check matrix**



protograph

lifting factor M



Tanner graph

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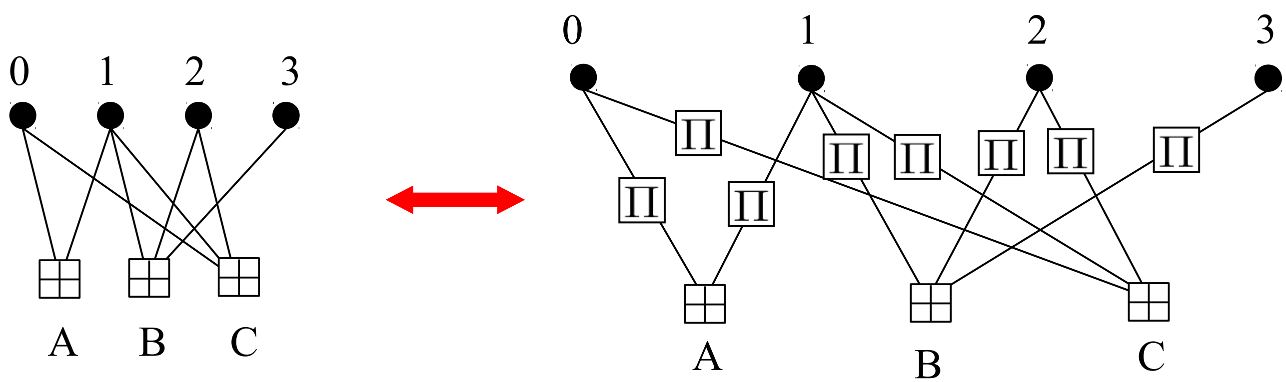
Protograph Analysis

- By design, every member of a protograph-based ensemble **preserves the structure** of the base protograph.



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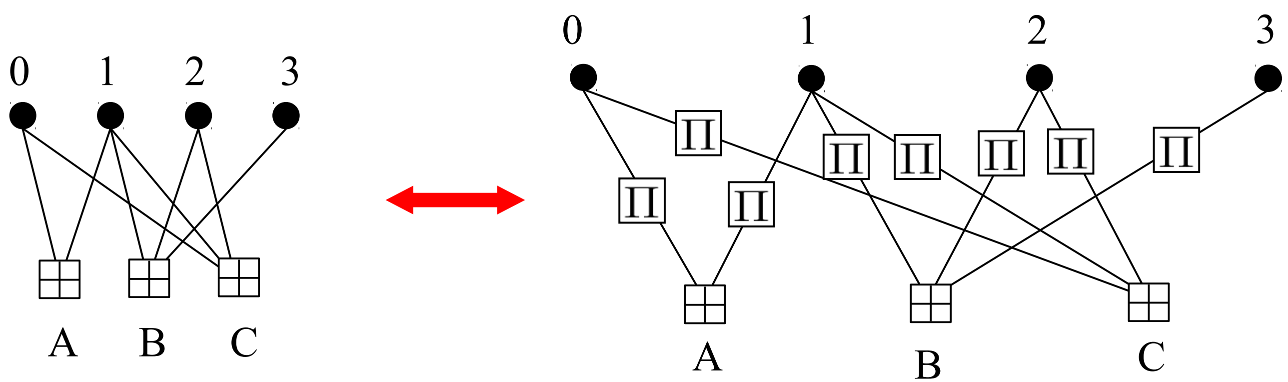

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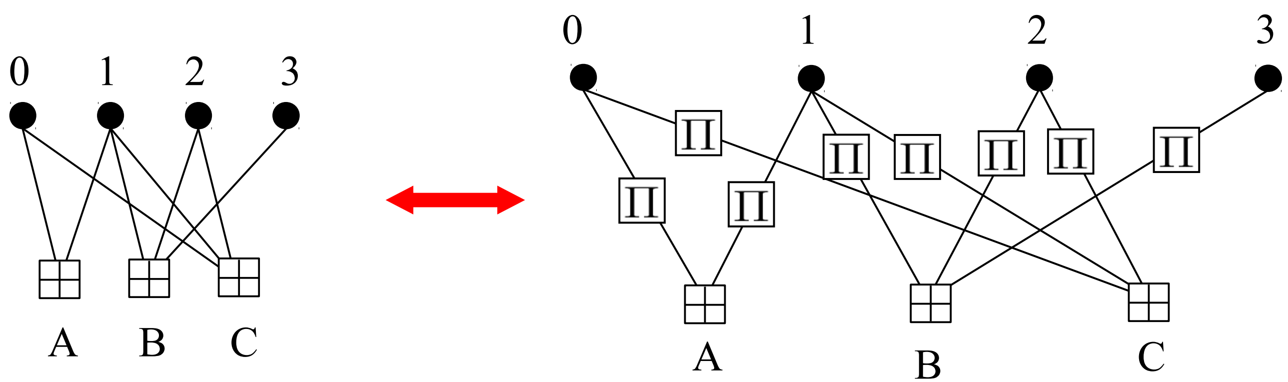

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➡ From the protograph, an expression [Divsalar '06] can be obtained for the **ensemble average weight enumerator**,

$$\overline{A}(z) = \sum_{d=0}^{n} \overline{A_d} z^d, \quad \left(\begin{array}{l} \overline{A_d} = \text{avg. number of} \\ \text{codewords of weight } d \end{array} \right)$$

which can be used to test if the ensemble is **asymptotically good**.

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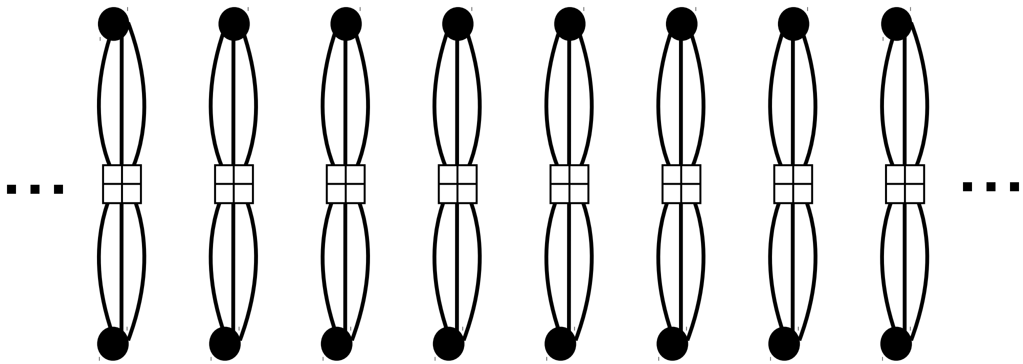
Spatially Coupled LDPC Code Ensembles

- Consider transmission of consecutive blocks (protograph representation):



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$(3{,}6)$-regular
LDPC-BC
base matrix

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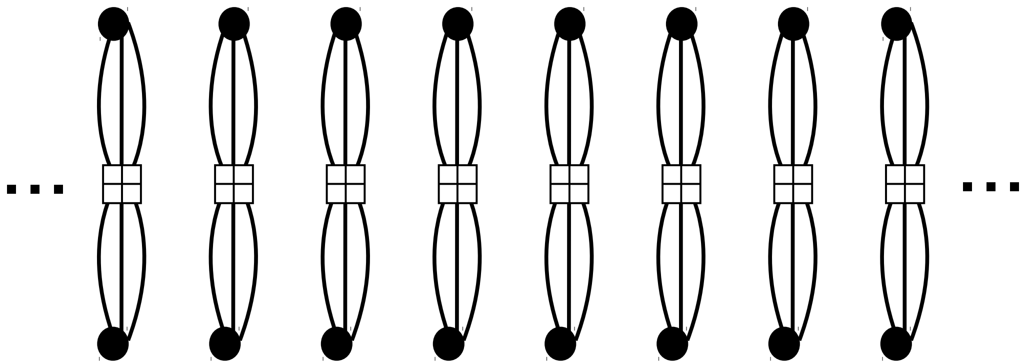

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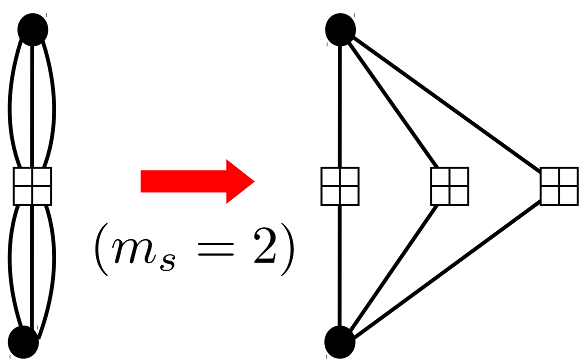

$$\mathbf{B} = \begin{bmatrix} 3 & 3 \end{bmatrix} \xrightarrow{(m_s = 2)} \begin{bmatrix} \mathbf{B}_0 \\ \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

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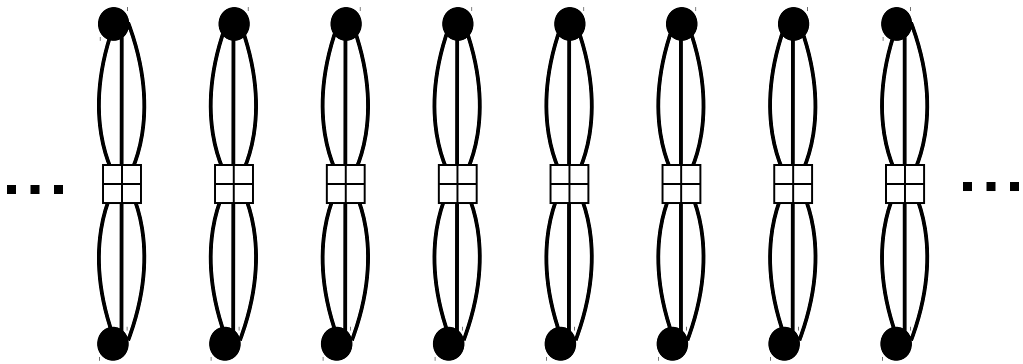

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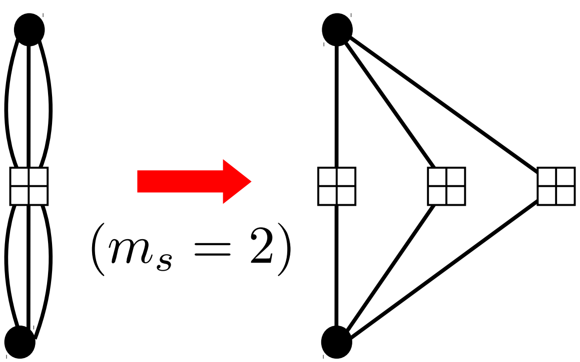

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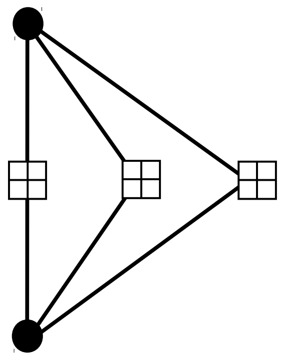
- **Spreading constraint:** $\displaystyle\sum_{i=0}^{m_s} \mathbf{B}_i = \mathbf{B}$

Spatially Coupled LDPC Code Ensembles

- Transmission of consecutive spatially coupled (SC) blocks results in a **convolutional protograph**:



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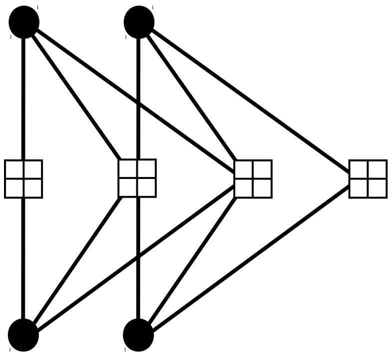

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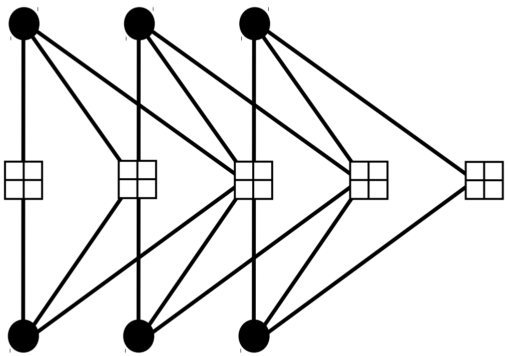

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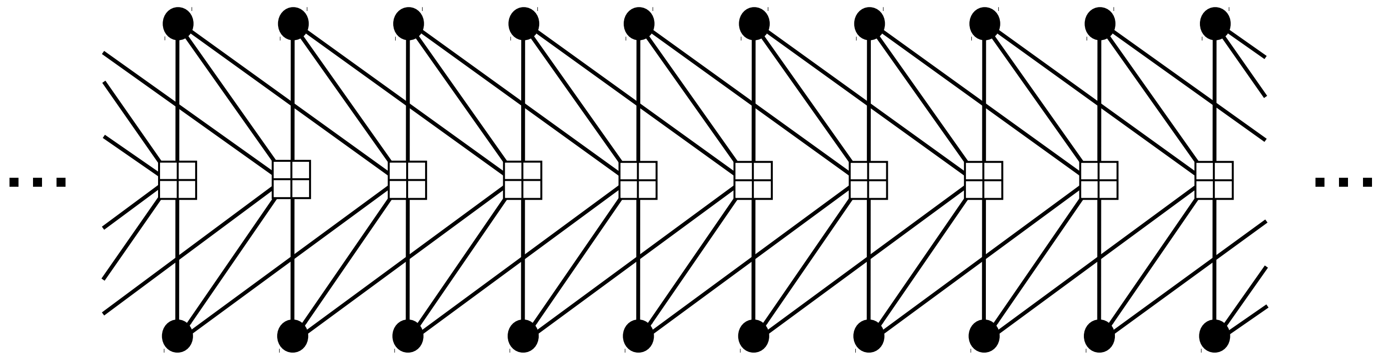

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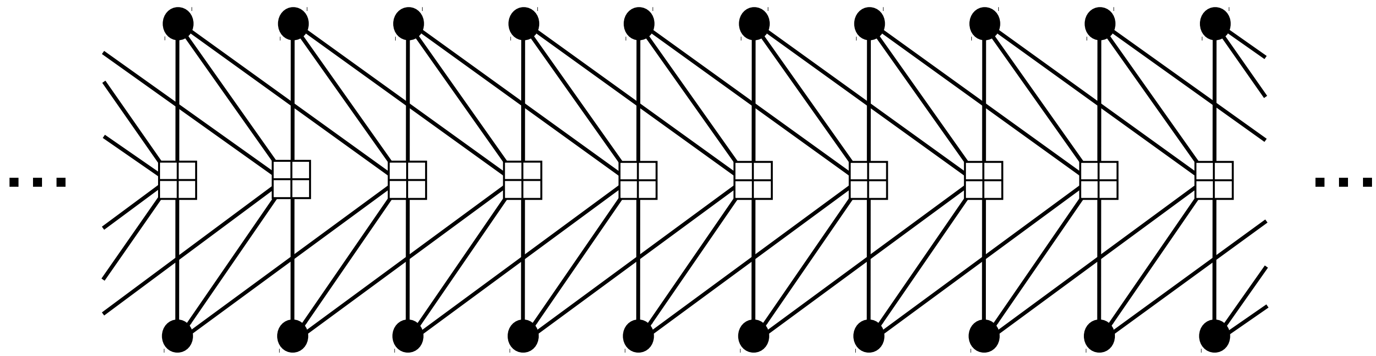

- The bi-infinite convolutional protograph corresponds to a bi-infinite **convolutional base matrix**:

$$\mathbf{B}_{[-\infty,\infty]} = \begin{bmatrix} \ddots & & \ddots & \ddots & & & \\ & \mathbf{B}_{m_s} & \cdots & \mathbf{B}_1 & \mathbf{B}_0 & & \\ & & \ddots & & \ddots & \ddots & \\ & & & \mathbf{B}_{m_s} & \cdots & \mathbf{B}_1 & \mathbf{B}_0 \\ & & & & \ddots & & \ddots & \ddots \end{bmatrix}$$

$\mathbf{B}_i$ has size $b_c \times b_v$

Rate:

$$R = \frac{b_v - b_c}{b_v}$$

Constraint length:

$$\nu_s = b_v(m_s + 1)$$

Terminated Spatially Coupled Codes

- Consider **terminating** $\mathbf{B}_{[-\infty,\infty]}$ to a (block code) **base matrix** of length Lb_v:

$$\mathbf{B}_{[0,L-1]} = \begin{bmatrix} \mathbf{B}_0 & & \\ \vdots & \ddots & \\ \mathbf{B}_{m_s} & & \mathbf{B}_0 \\ & \ddots & \vdots \\ & & \mathbf{B}_{m_s} \end{bmatrix}_{(L+m_s)b_c \times Lb_v}$$

($\mathbf{B}_i$ is a $b_c \times b_v$ matrix)

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$$R_L = \frac{Lb_v - (L+m_s)b_c}{Lb_v}.$$

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- For large L, R_L approaches the **unterminated** code rate $R = (b_v - b_c)/b_v$.

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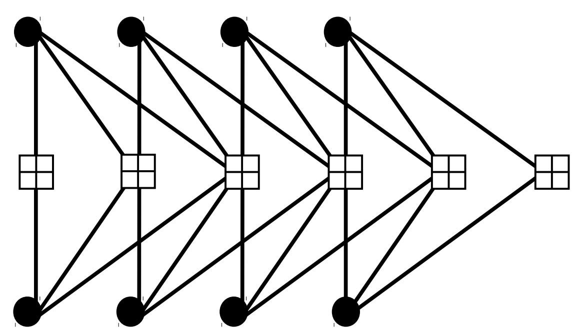
- For large L, R_L approaches the **unterminated** code rate $R = (b_v - b_c)/b_v$.
- **Example:** $(3{,}6)$-regular base matrix $\mathbf{B} = [3 \;\; 3]$, $m_s = 2$, $L = 4$, $R_4 = 1/4$

$$\mathbf{B}_{[0,3]} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$



(**check node degrees lower at the ends**)

Terminated Spatially Coupled Codes

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- Consider **terminating** $\mathbf{B}_{[-\infty,\infty]}$ to a (block code) **base matrix** of length Lb_v:

$$\mathbf{B}_{[0,L-1]} = \begin{bmatrix} \mathbf{B}_0 & & \\ \vdots & \ddots & \\ \mathbf{B}_{m_s} & & \mathbf{B}_0 \\ & \ddots & \vdots \\ & & \mathbf{B}_{m_s} \end{bmatrix}_{(L+m_s)b_c \times Lb_v}$$

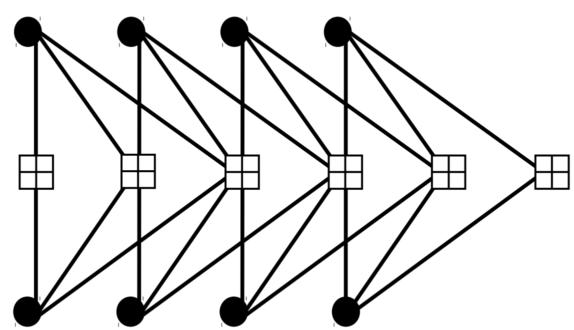
($\mathbf{B}_i$ is a $b_c \times b_v$ matrix)

Code rate:

$$R_L = \frac{Lb_v - (L+m_s)b_c}{Lb_v}.$$

- For large L, R_L approaches the **unterminated** code rate $R = (b_v - b_c)/b_v$.
- **Example:** $(3,6)$-regular base matrix $\mathbf{B} = [3 \;\; 3]$, $m_s = 2$, $L = 4$, $R_4 = 1/4$

$$\mathbf{B}_{[0,3]} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

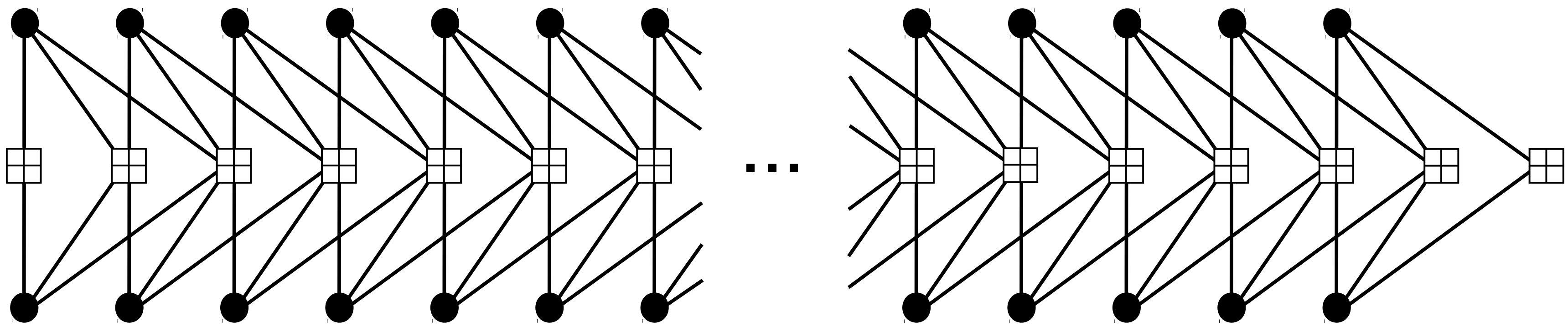


(**check node degrees lower at the ends**)

- Codes can be **lifted** to **different lengths and rates** by varying M and L.

Wave-like Decoding of Terminated Spatially Coupled Codes

- Variable nodes all have the **same degree** as the underlying block code.
- Check nodes with **lower degrees** (at the ends) improve the BP decoder.



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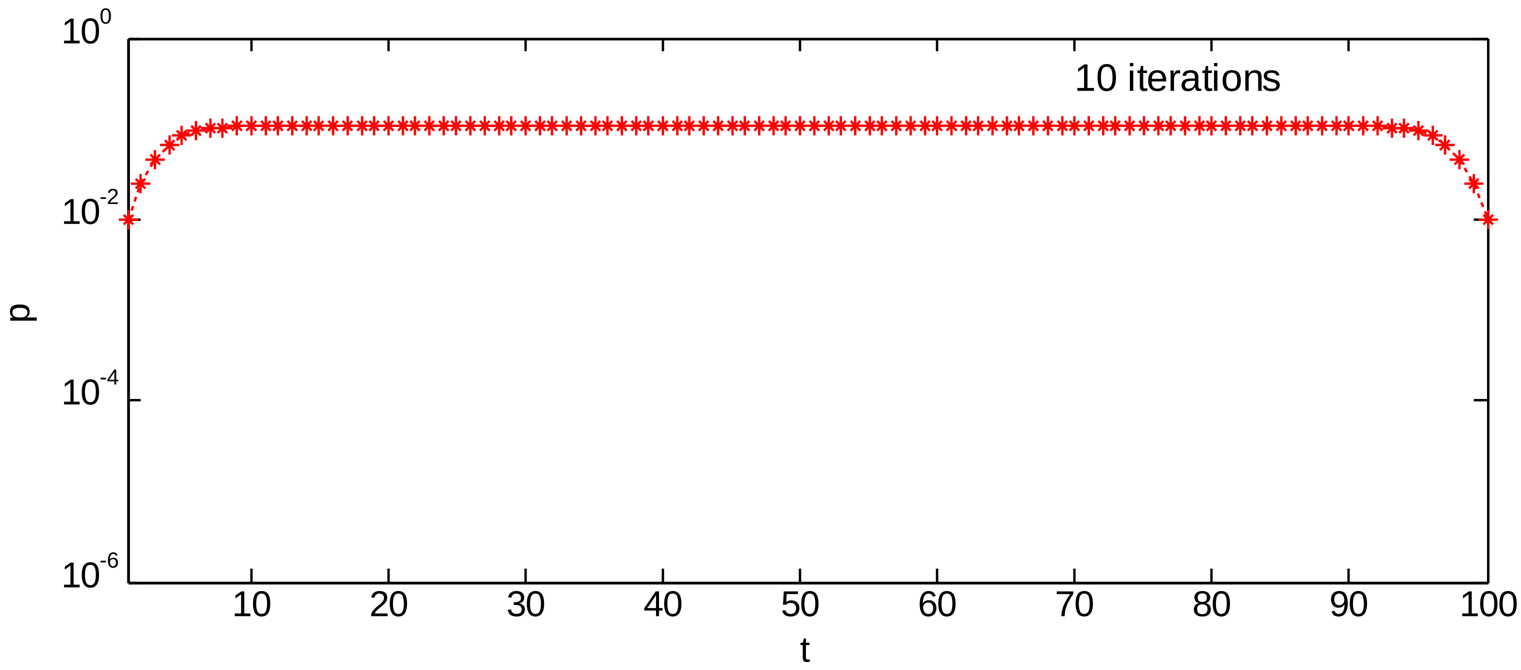
Evolution of message probabilities: (3,6)-regular SC-LDPC code $(L = 100)$



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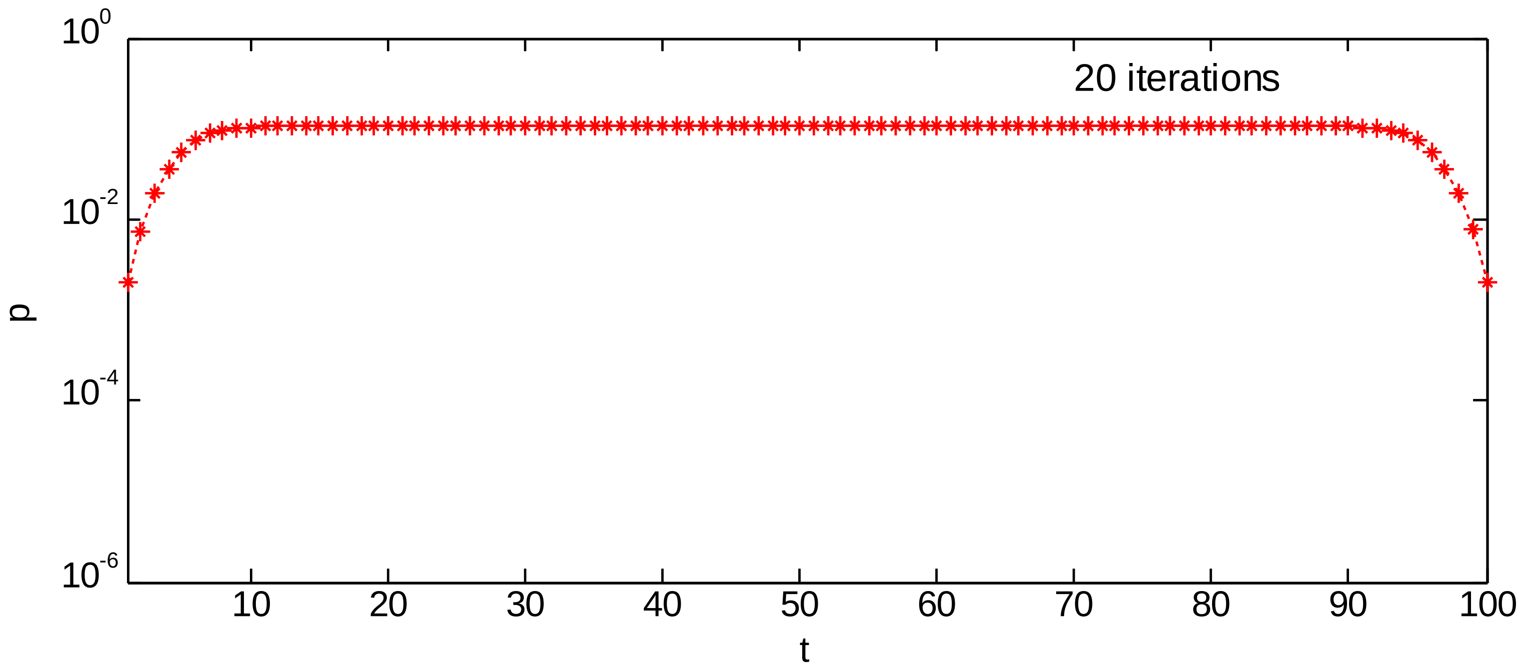

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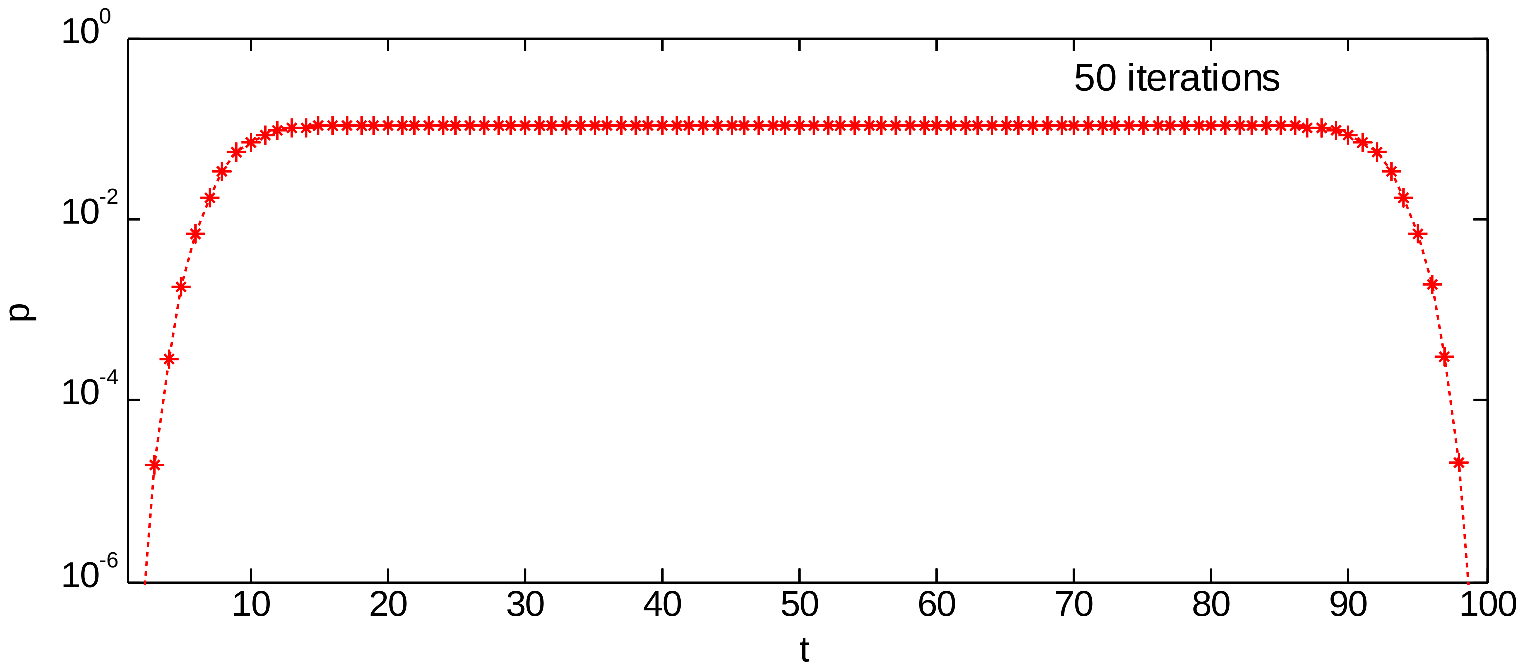

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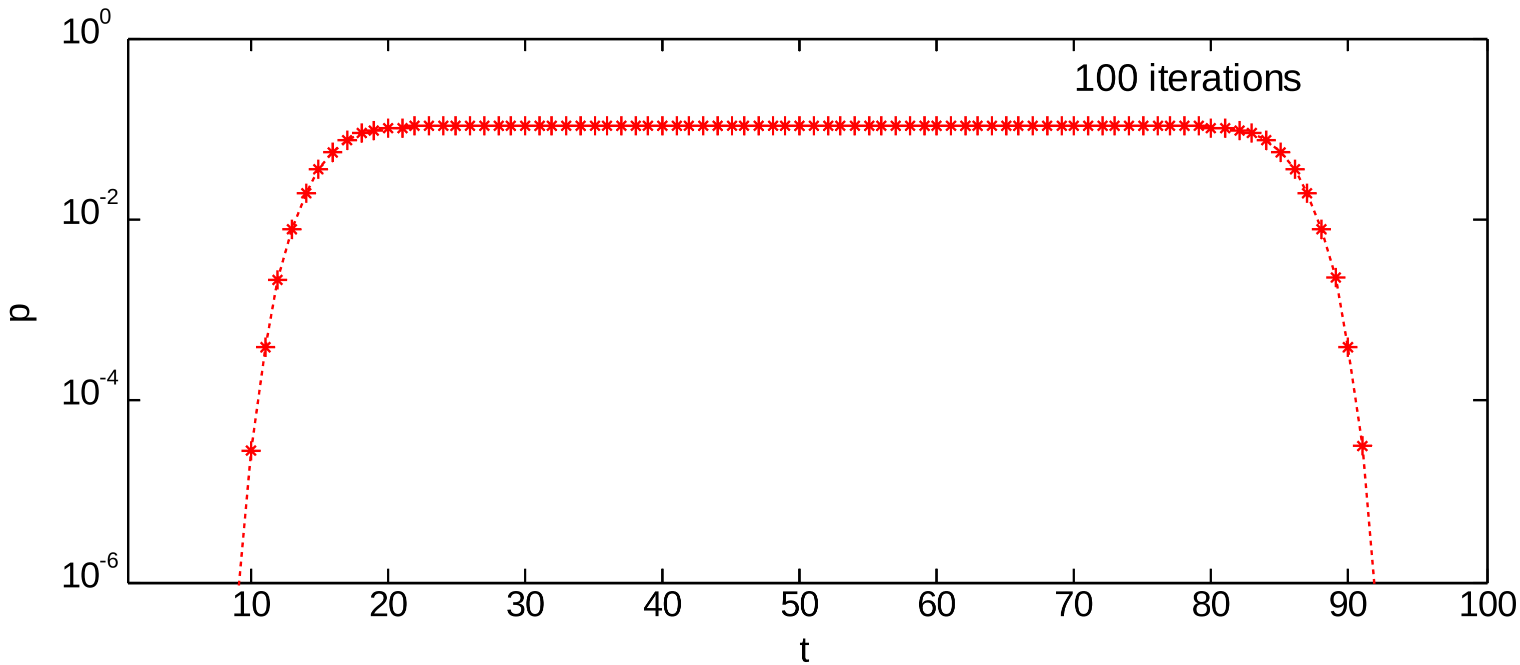

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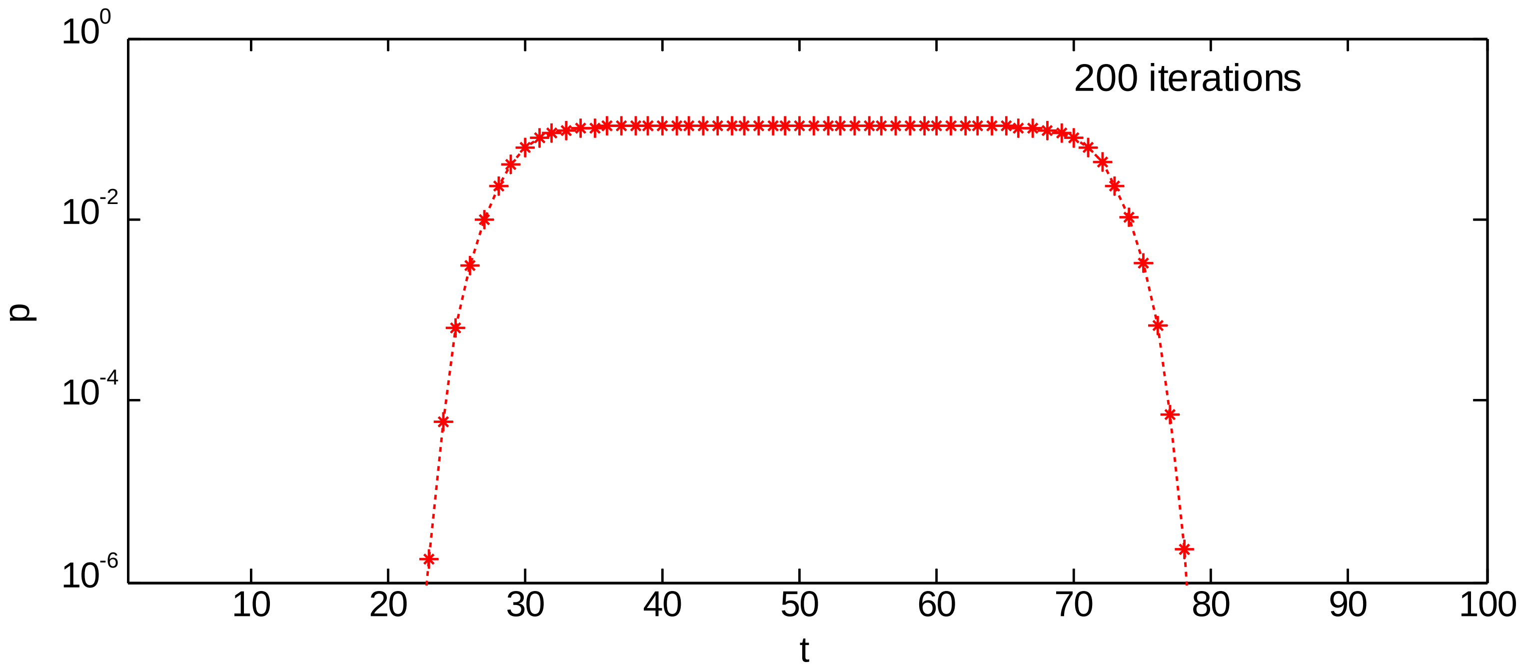

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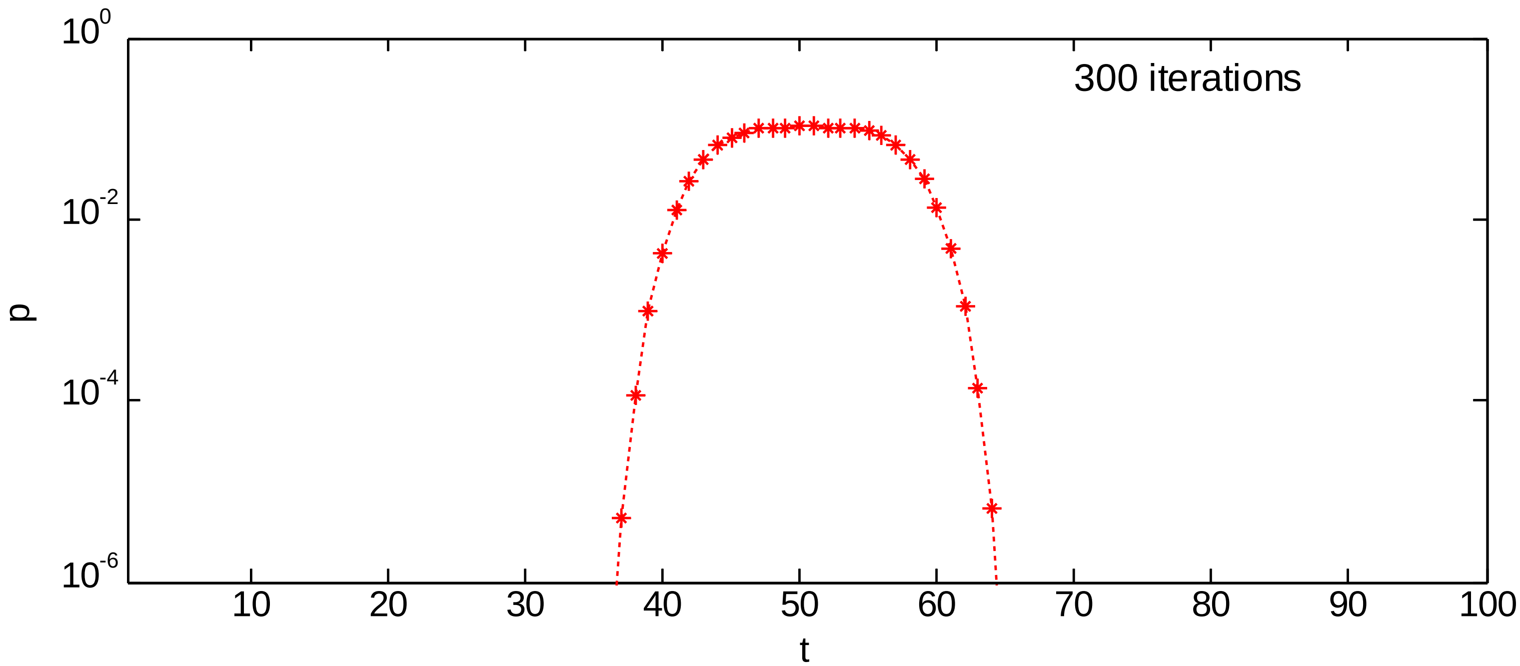

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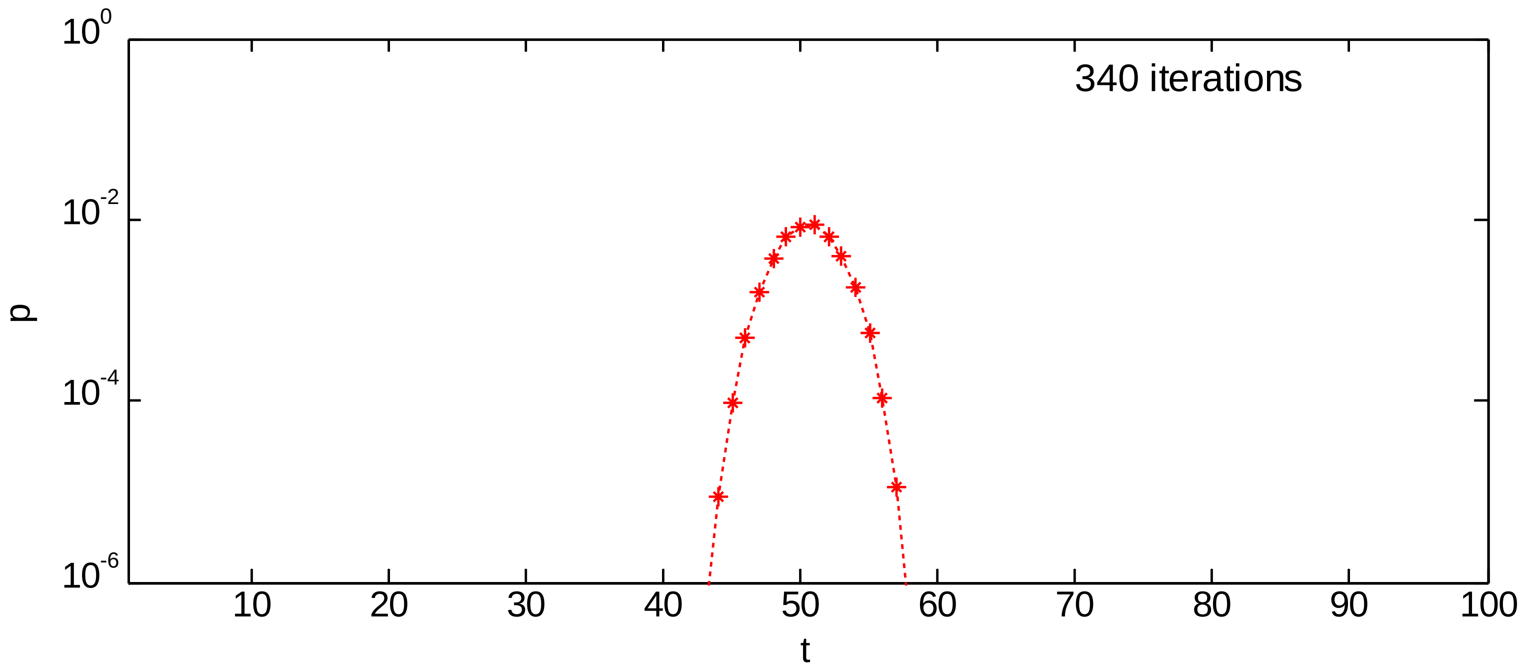

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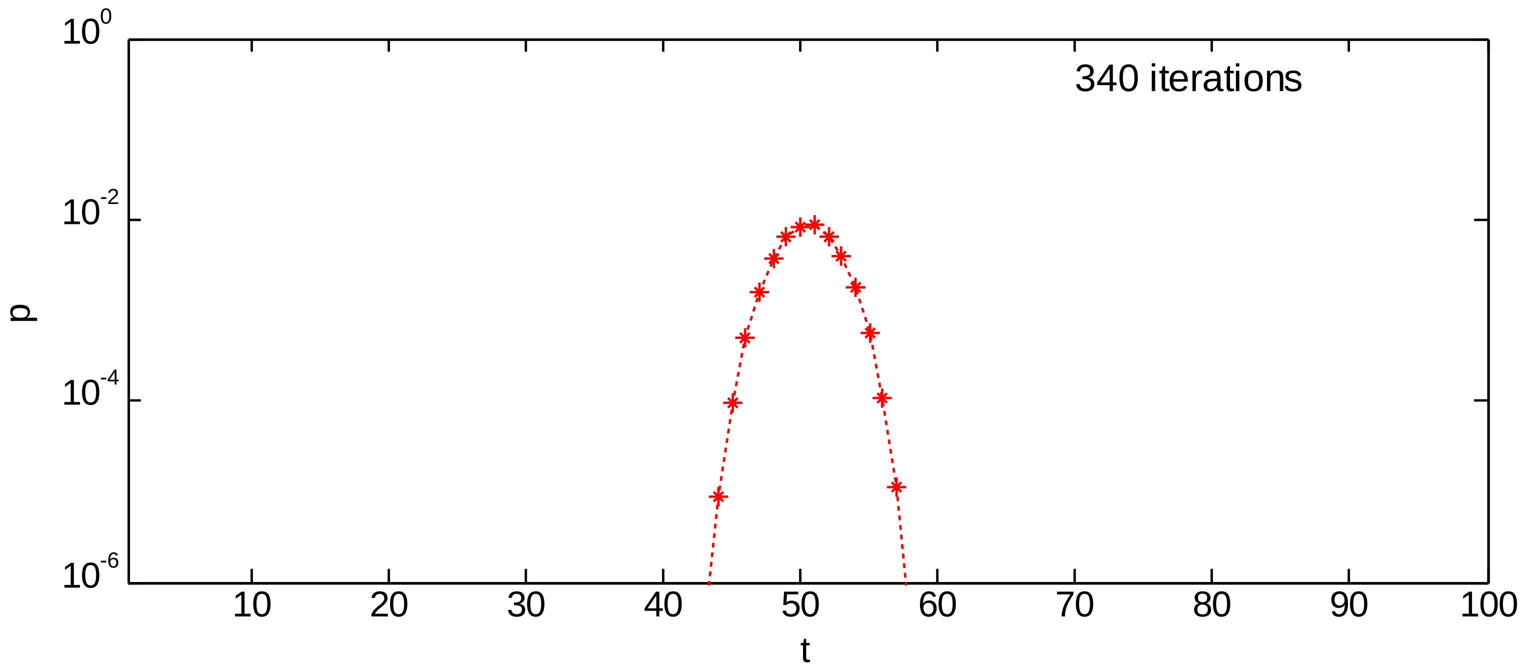

- Note: the **fraction** of **lower degree** nodes tends to zero as $L \to \infty$, i.e., the codes are **asymptotically regular**.

Thresholds of Terminated Spatially Coupled Codes

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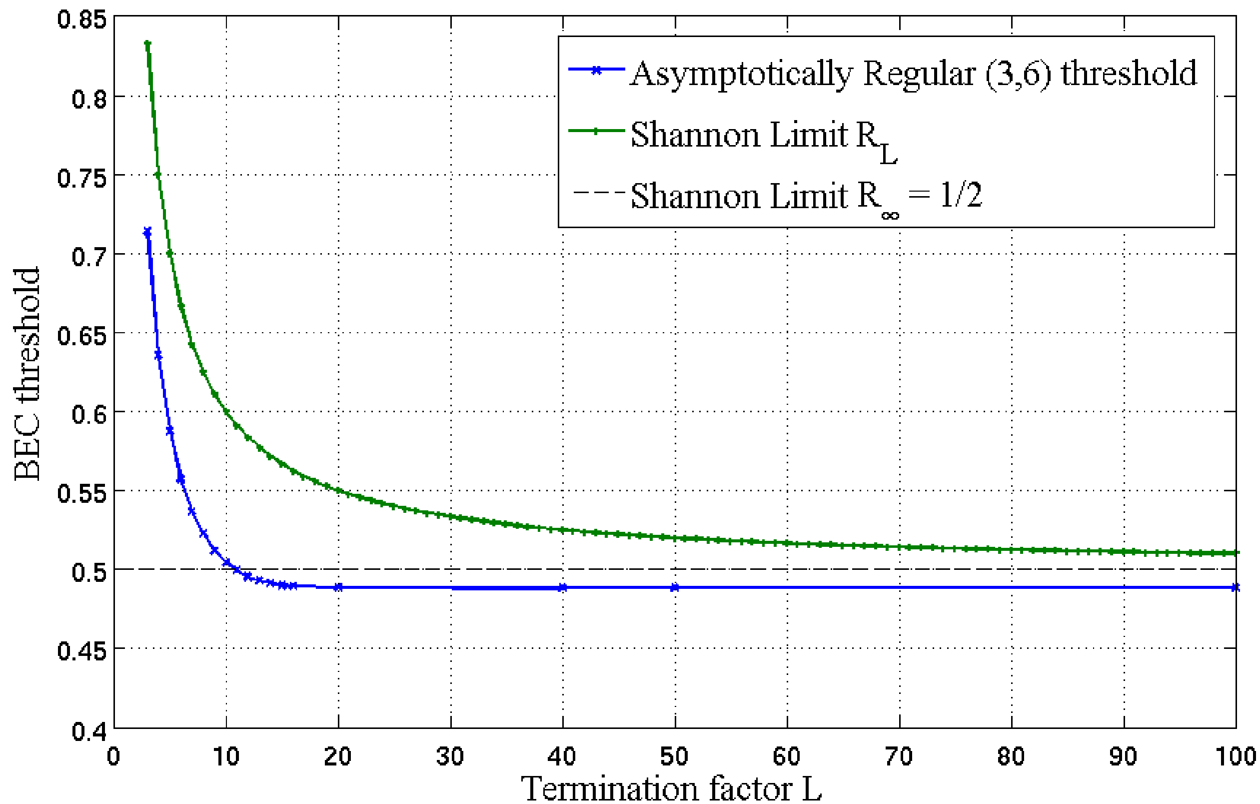
- **Density evolution** can be applied to the protograph-based ensembles with $M \rightarrow \infty$ [Sridharan et al. '04]:

Example: BEC



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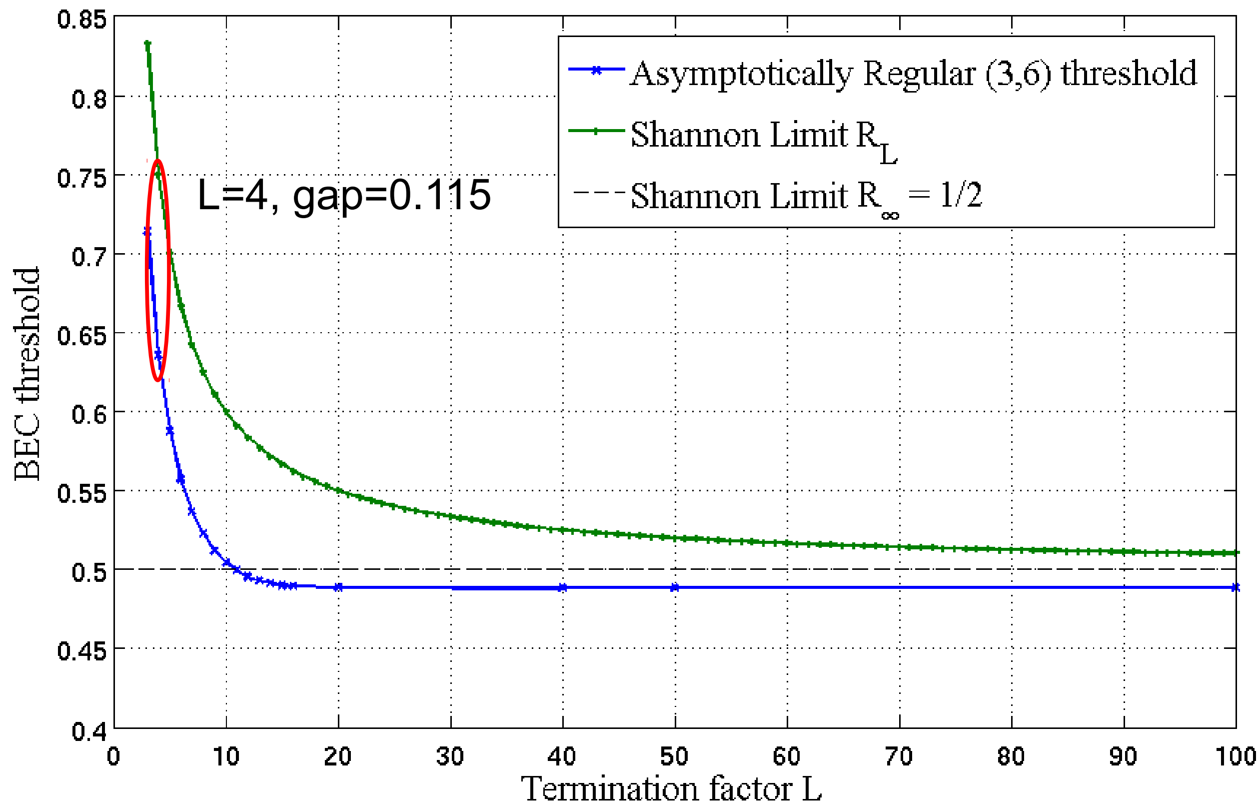

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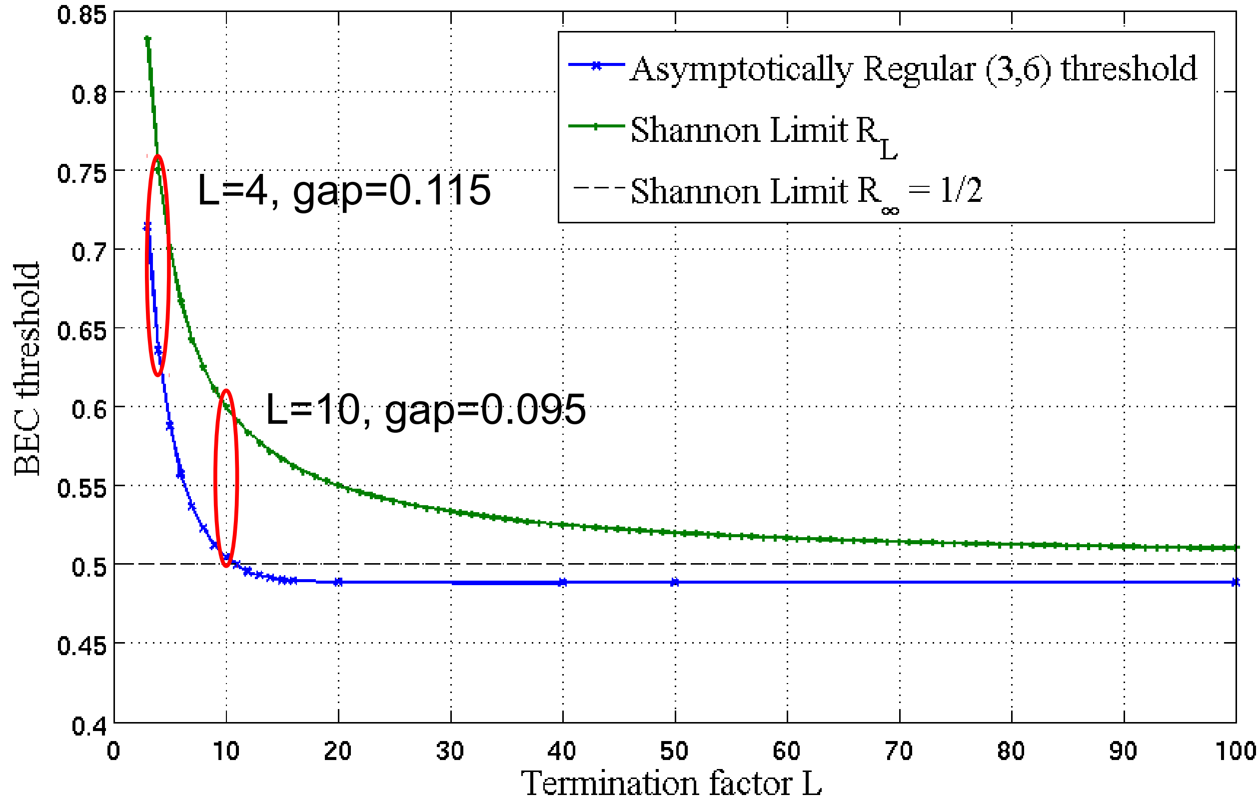

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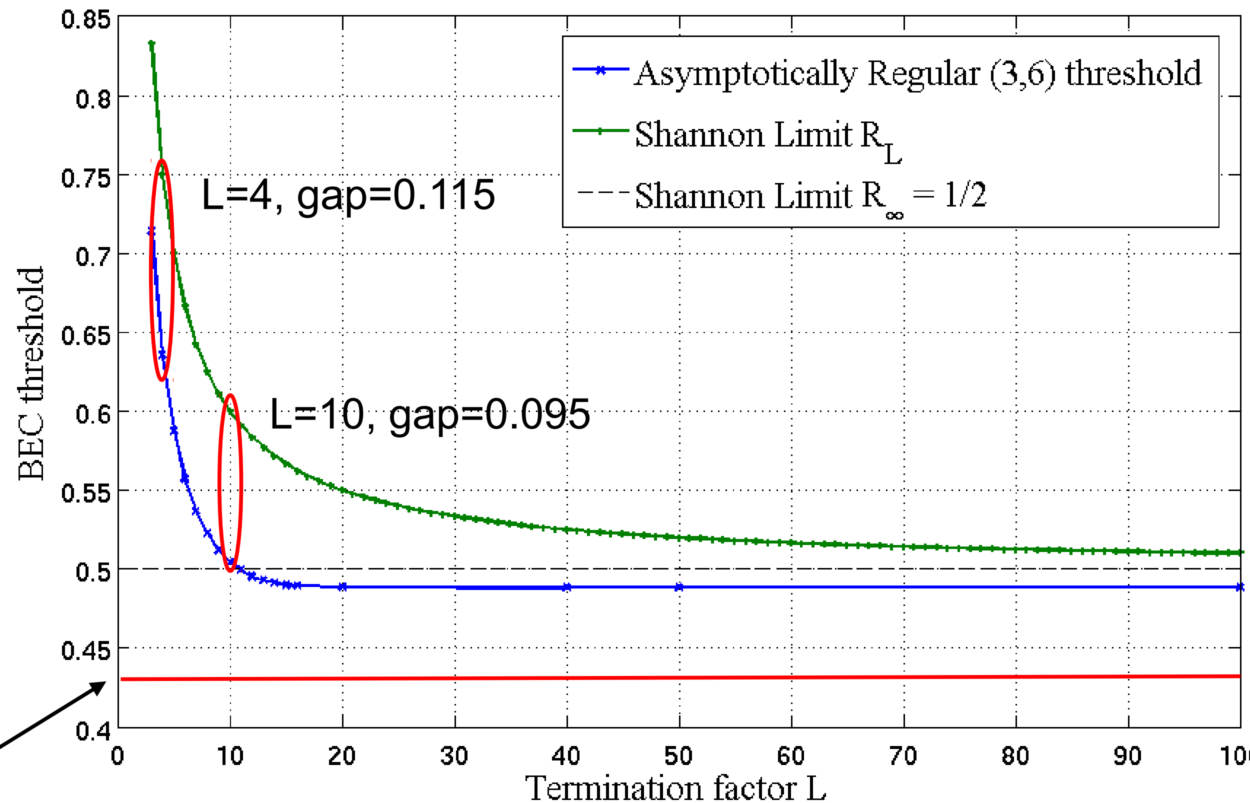
⋮

$L \to \infty$, $R \to 1/2$
$\varepsilon^* = 0.488$, $\varepsilon_{\text{Sh}} = 0.5$



(3,6)-regular block code:

$\varepsilon^* = 0.429$

Thresholds of Terminated Spatially Coupled Codes

Iterative decoding thresholds (structured protograph-based ensembles)

BEC

(J, K)	ϵ^*_{SC}	$\epsilon^*_{\mathrm{blk}}$
(3,6)	0.488	0.429
(4,8)	0.497	0.383
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AWGN

(J, K)	$E_b/N_{o\,\mathrm{SC}}$	$E_b/N_{o\,\mathrm{blk}}$
(3,6)	0.46 dB	1.11 dB
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- We observe a **significant improvement** in the thresholds of SC-LDPC codes compared to the associated LDPC block codes (LDPC-BCs) due to the lower degree check nodes at the ends of the graph and wave-like decoding.

[LSCZ10] M. Lentmaier, A. Sridharan, D. J. Costello, Jr., and K.Sh. Zigangirov, "Iterative decoding threshold analysis for LDPC convolutional codes," *IEEE Trans. Inf. Theory*, 56:10, Oct. 2010.

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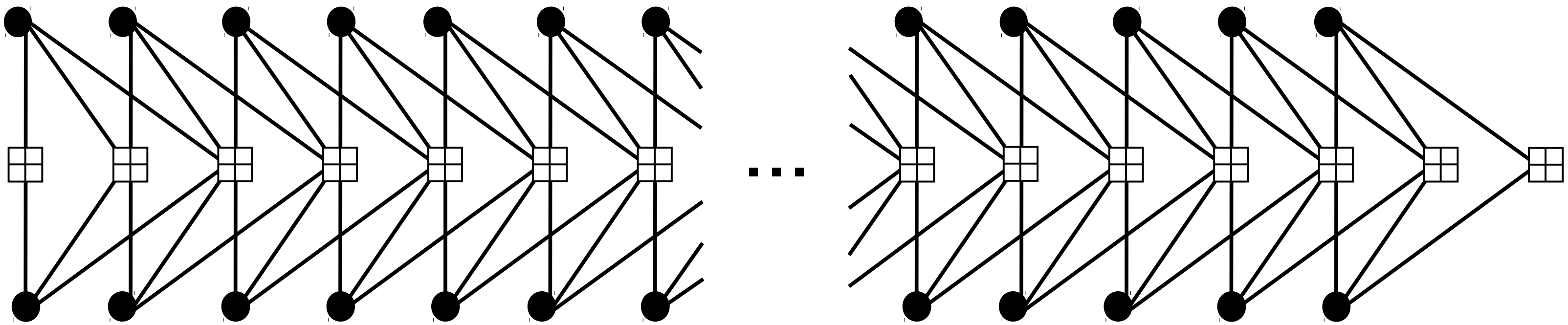
Why are Terminated Spatially Coupled Thresholds Better?

- When symbols are perfectly known (BEC), all adjacent edges can be removed from the Tanner graph.



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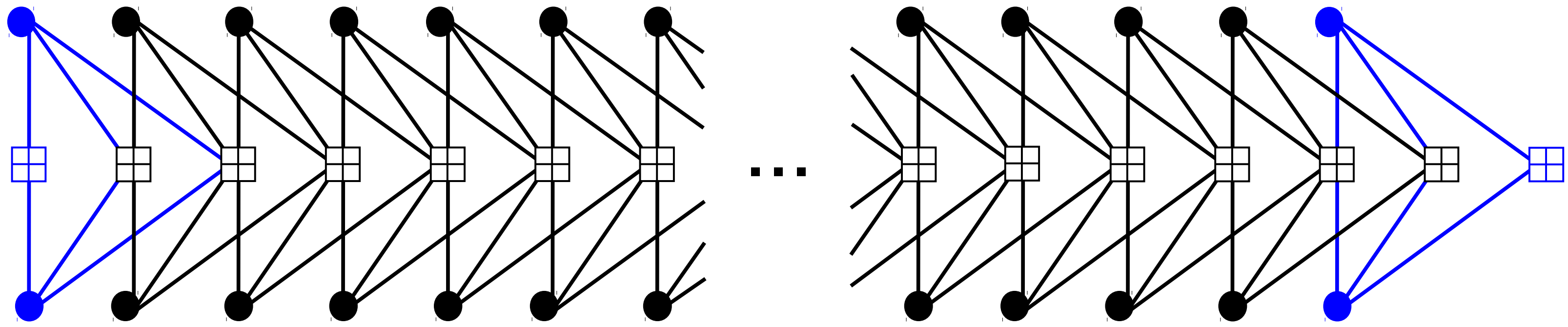

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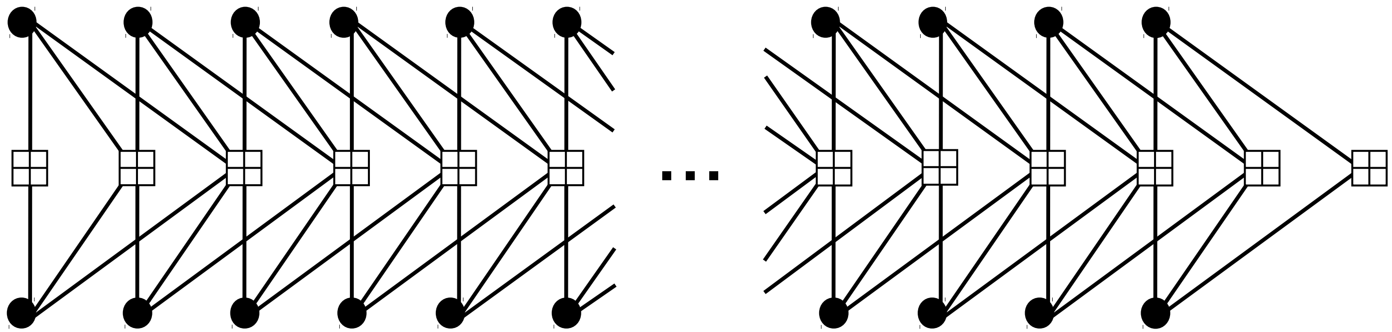

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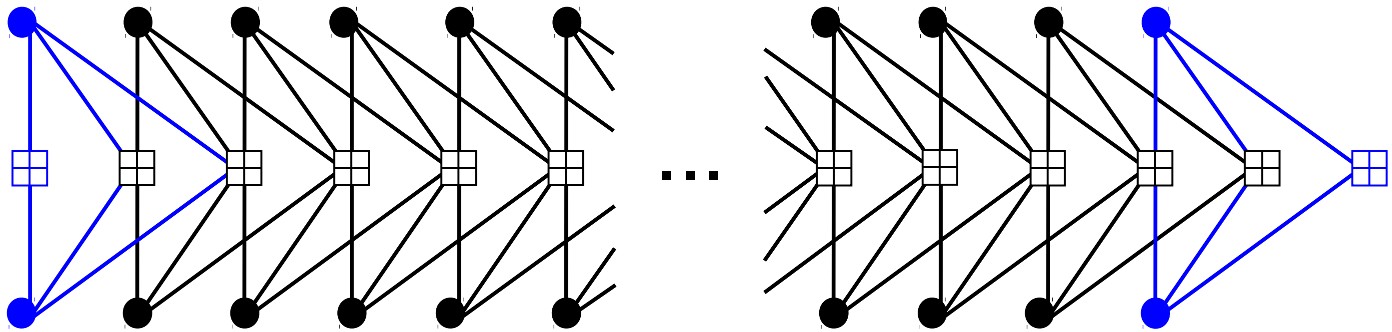

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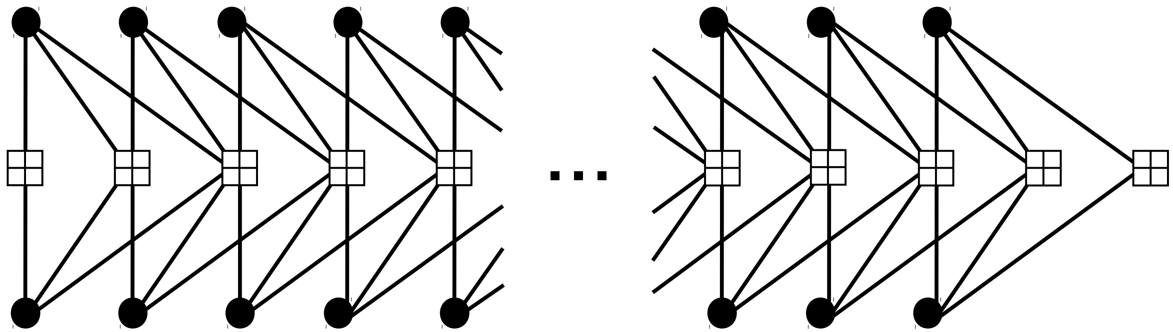

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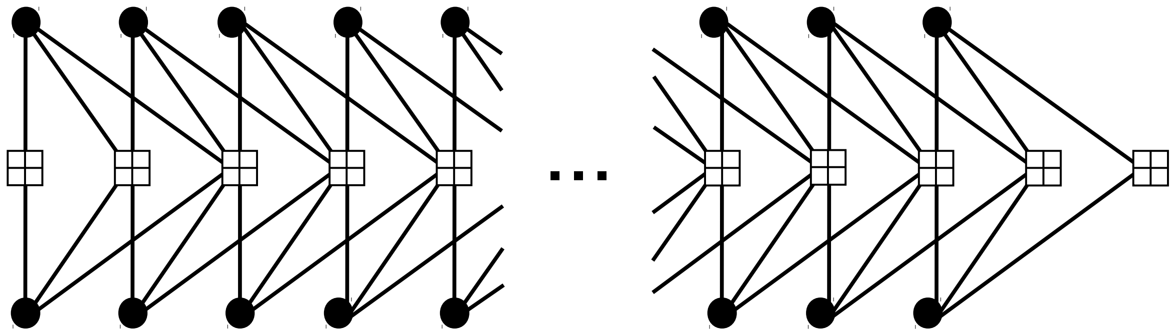

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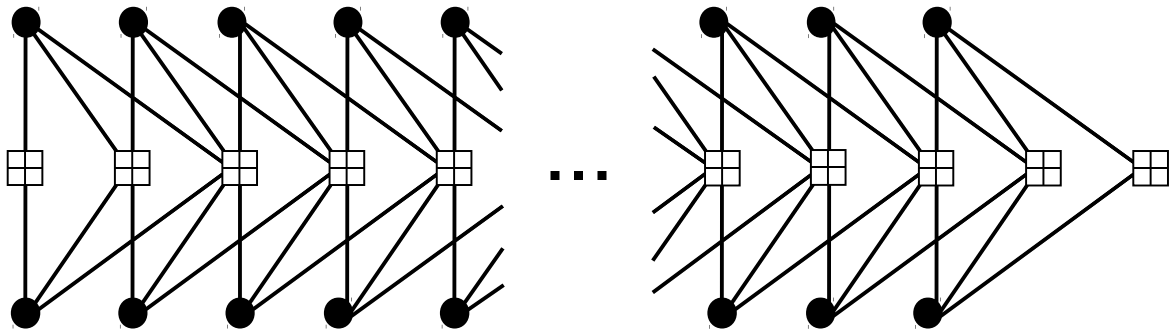

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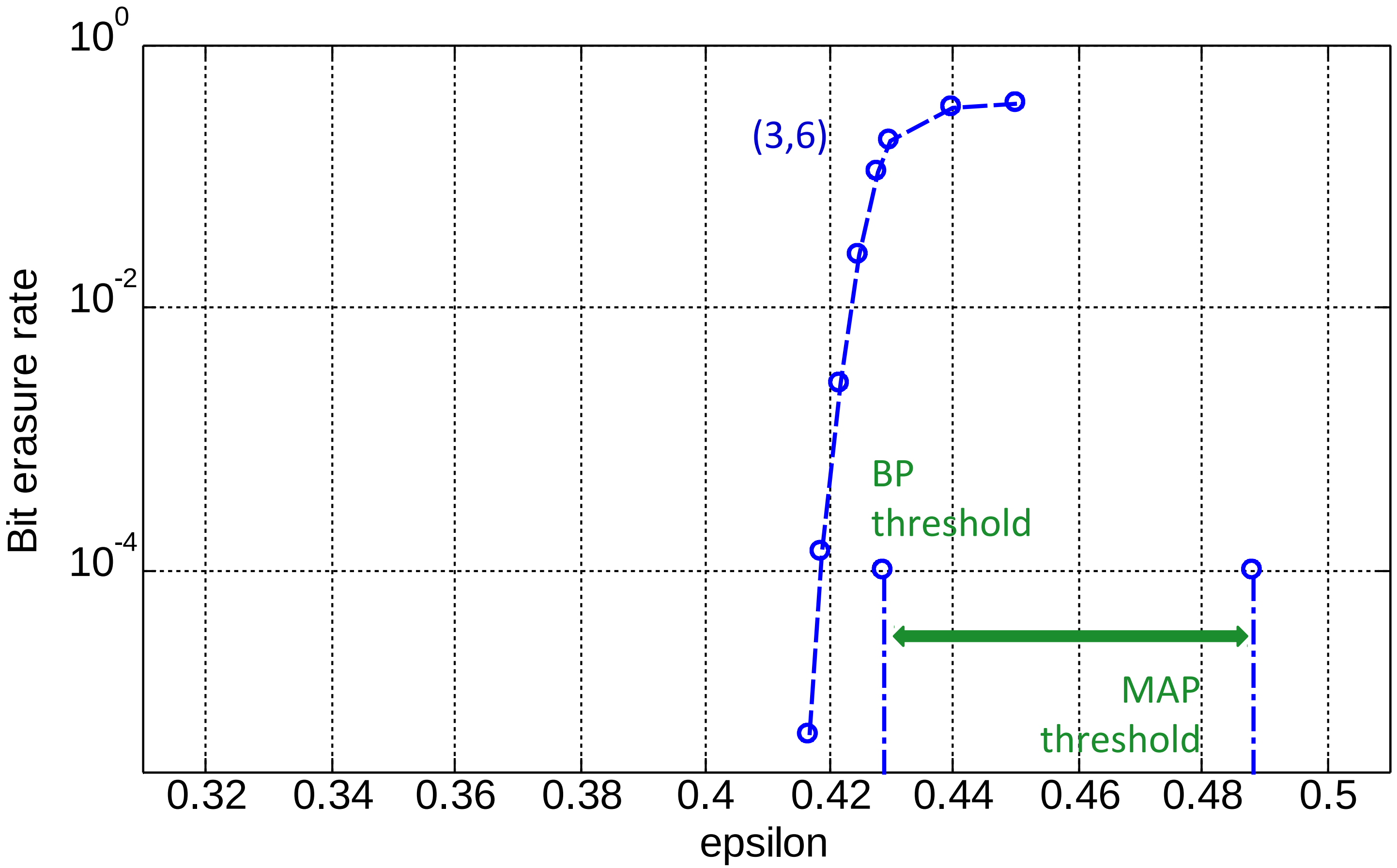
- For a more random-like ensemble, this has been proven analytically, first for the BEC [KRU11], then for all BMS channels [KRU13].

[LSCZ10] M. Lentmaier, A. Sridharan, D. J. Costello, Jr., and K.Sh. Zigangirov, "Iterative decoding threshold analysis for LDPC convolutional codes," *IEEE Trans. Inf. Theory*, 56:10, Oct. 2010.

[KRU11] S. Kudekar, T. J. Richardson and R. Urbanke, "Threshold saturation via spatial coupling: why convolutional LDPC ensembles perform so well over the BEC", *IEEE Trans. on Inf. Theory*, 57:2, 2011

[KRU13] S. Kudekar, T. J. Richardson and R. Urbanke, "Spatially coupled ensembles universally achieve capacity under belief propagation", *IEEE Trans. on Inf. Theory*, 59:12, 2013.

Threshold Saturation (BEC)



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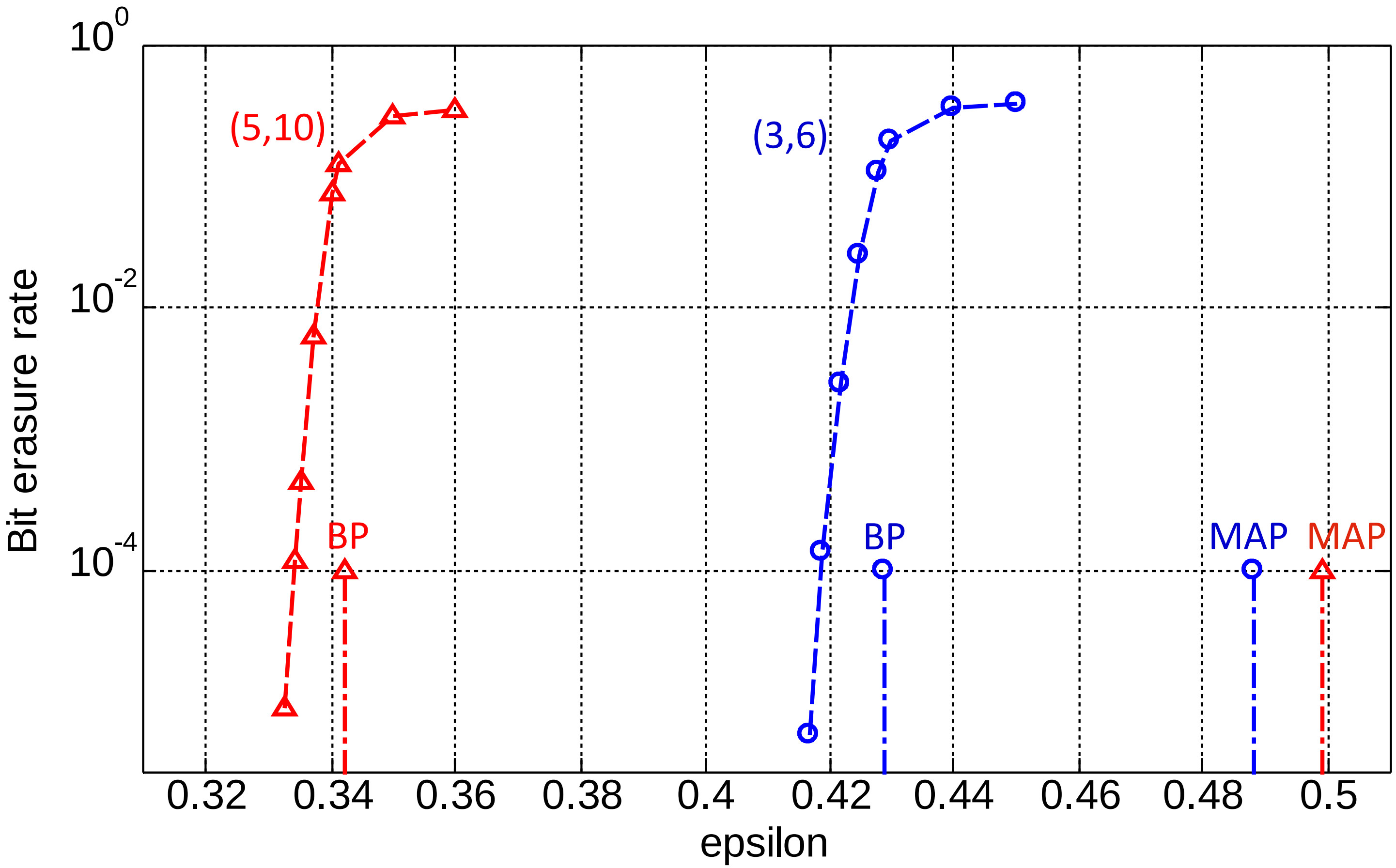

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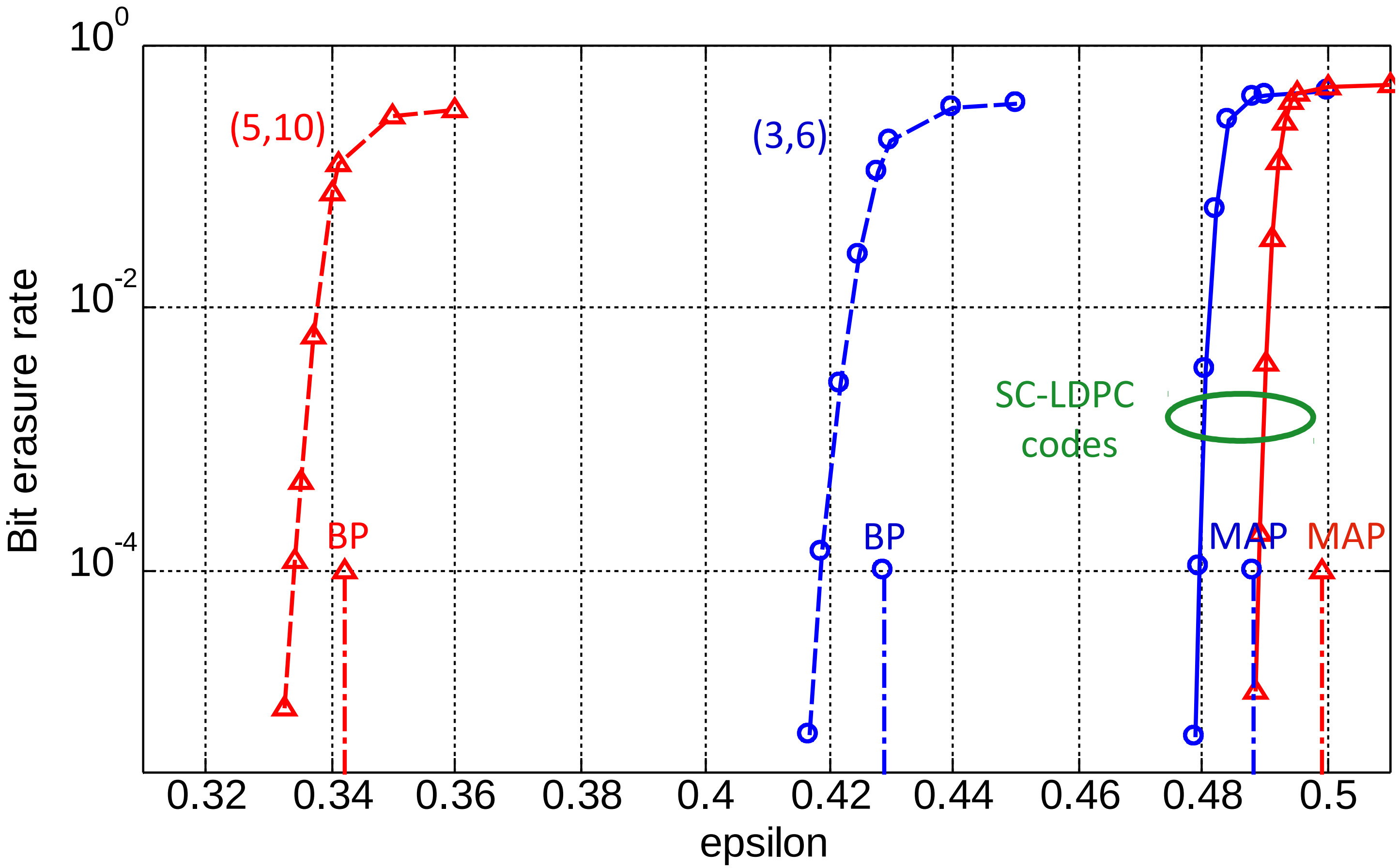
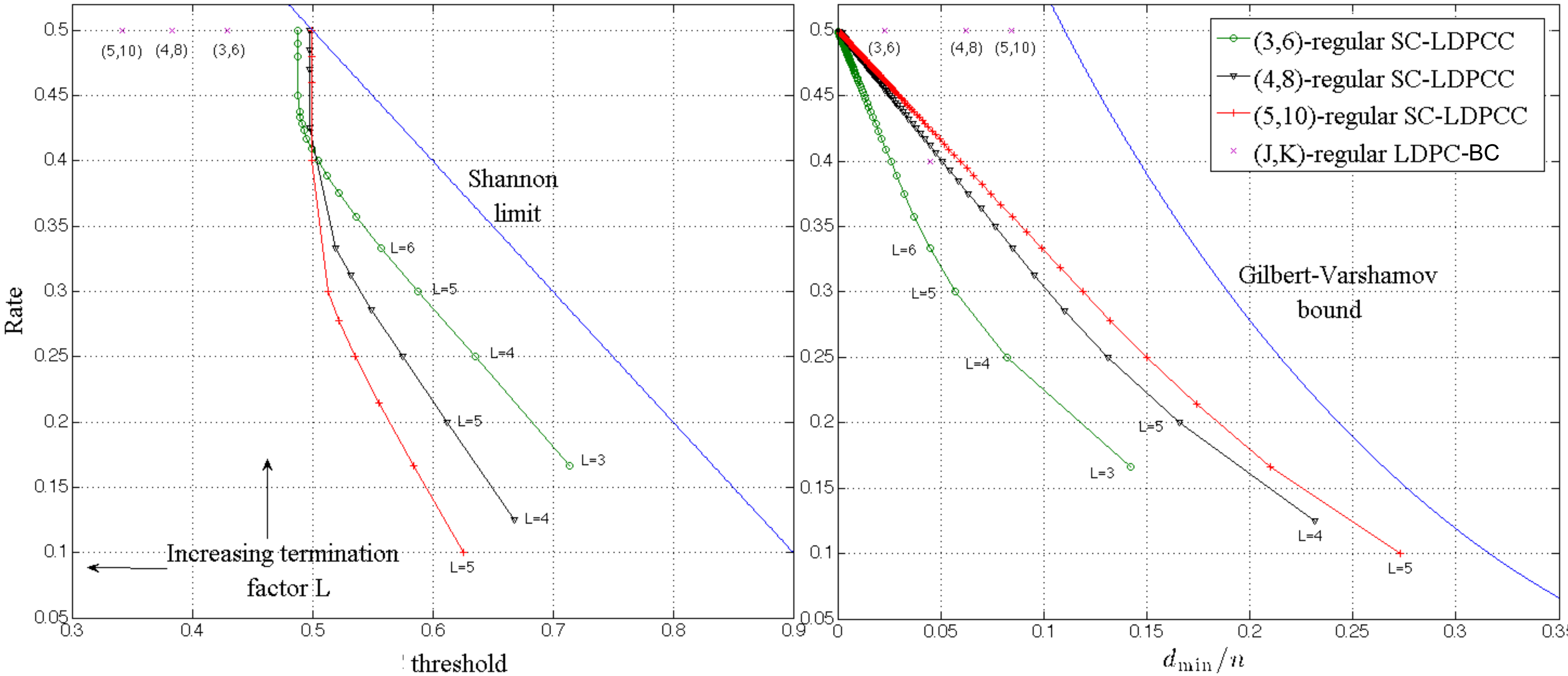


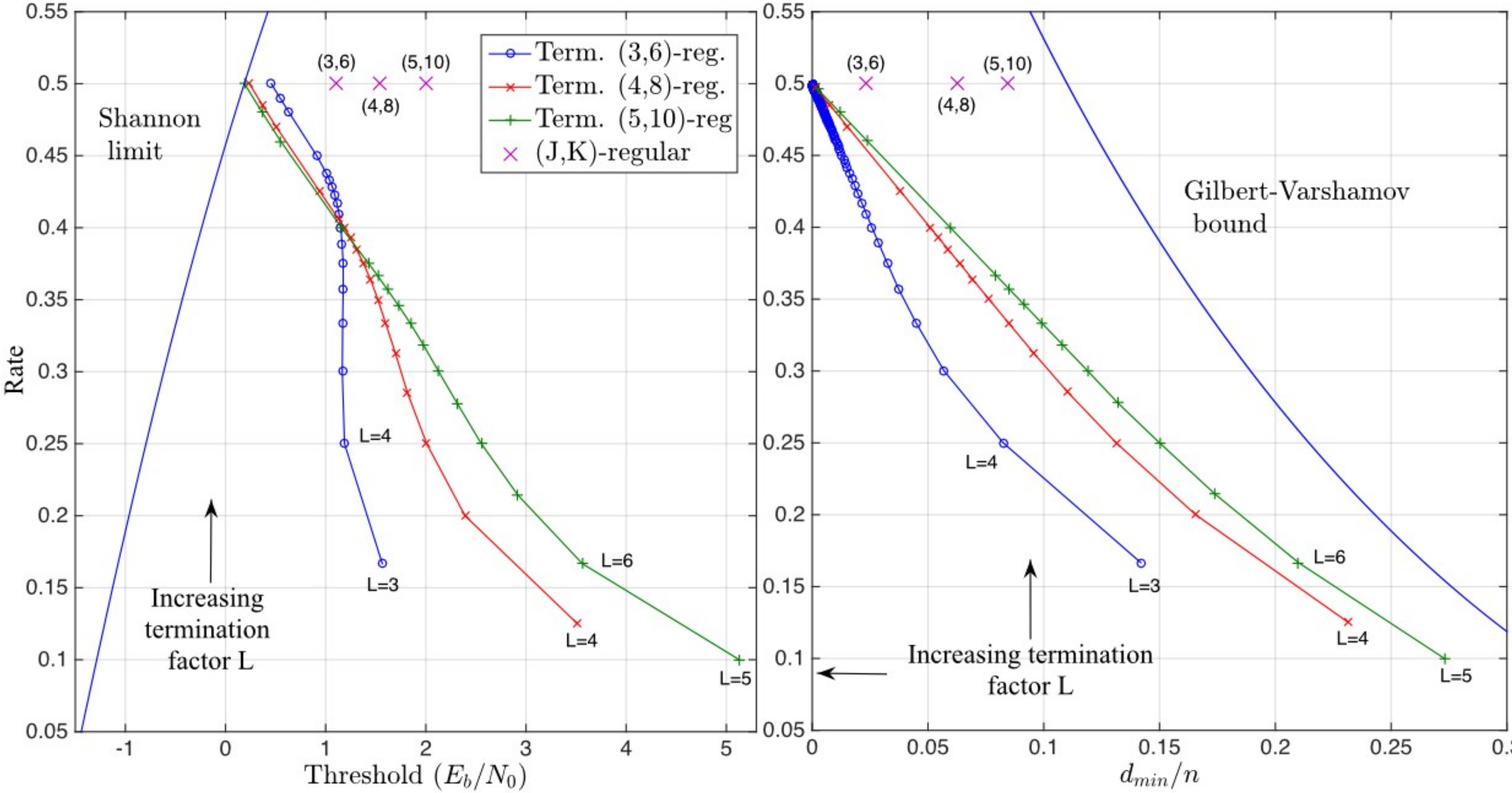
BEC Thresholds vs Distance Growth

- By increasing J and K, we obtain **capacity achieving** (J,K)-regular SC-LDPC code ensembles with linear minimum distance growth.



AWGNC Thresholds

- Similar results are obtained for the AWGNC



[MLC10] D. G. M. Mitchell, M. Lentmaier and D. J. Costello, Jr., "AWGN Channel Analysis of Terminated LDPC Convolutional Codes", *Proc. Information Theory and Applications Workshop*, San Diego, Feb. 2011.

Distance Measures for SC-LDPC Codes

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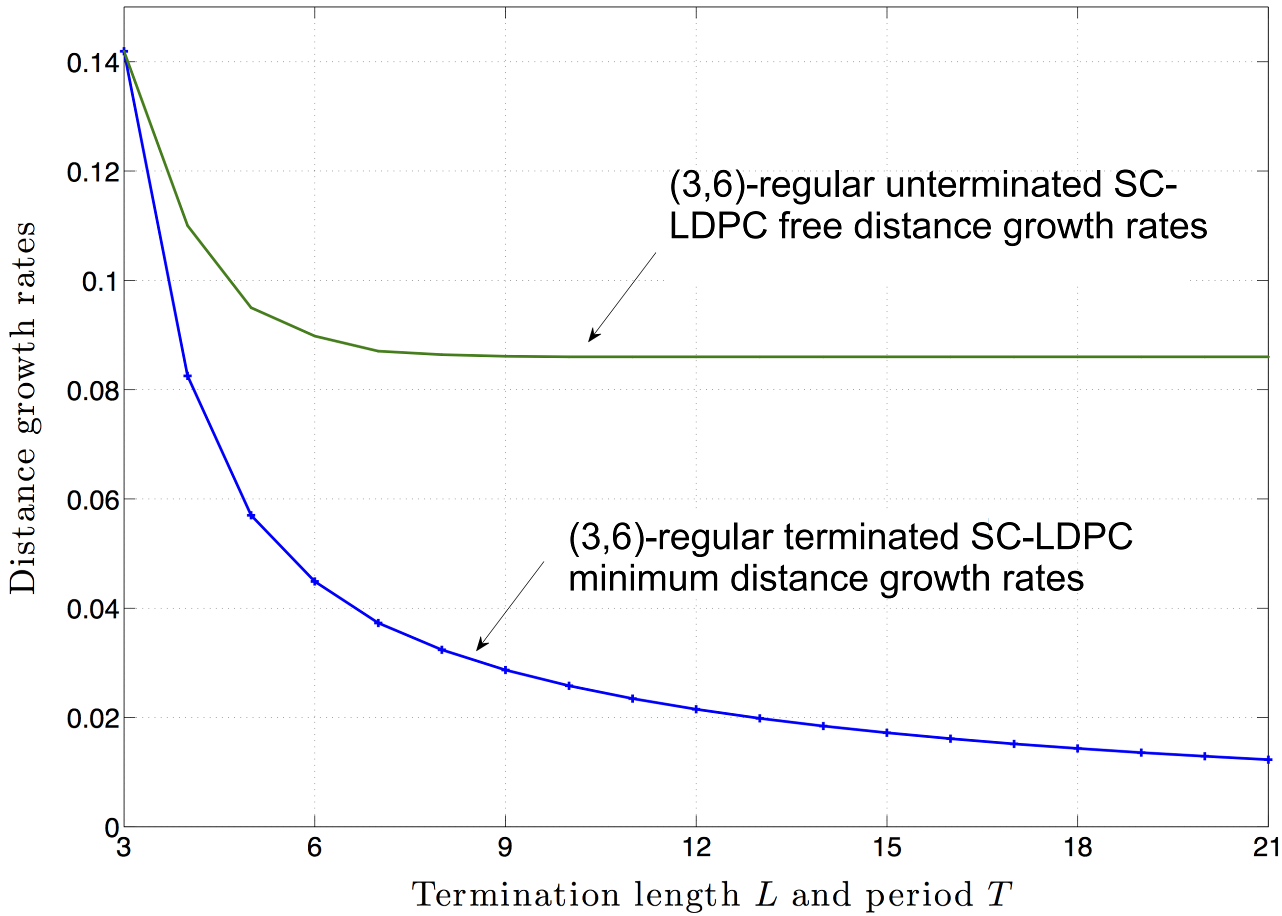
- As $L \to \infty$ the minimum distance growth rates of **terminated** SC-LDPC code ensembles tend to zero. However, the free distance growth rates of the **unterminated** ensembles remain constant.



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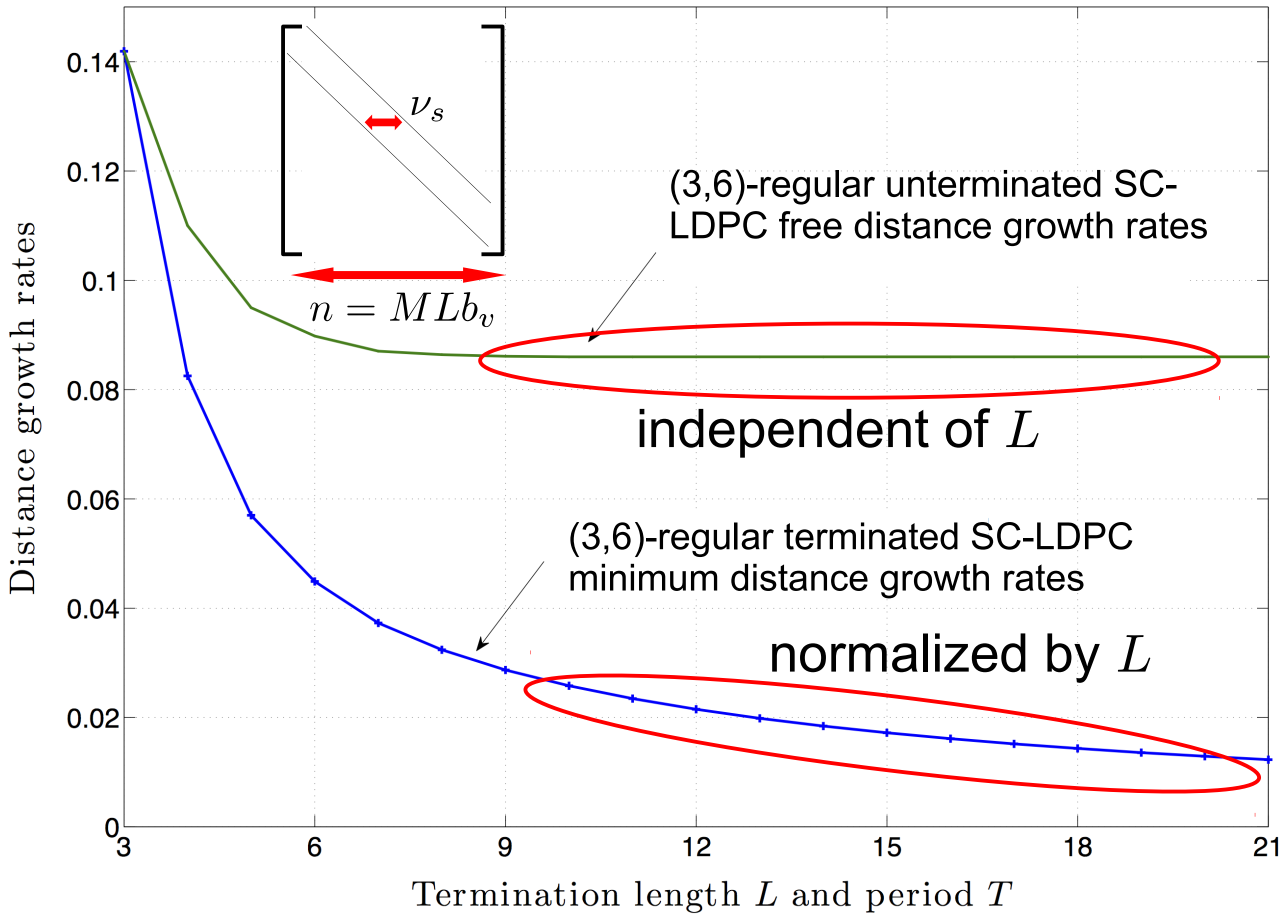

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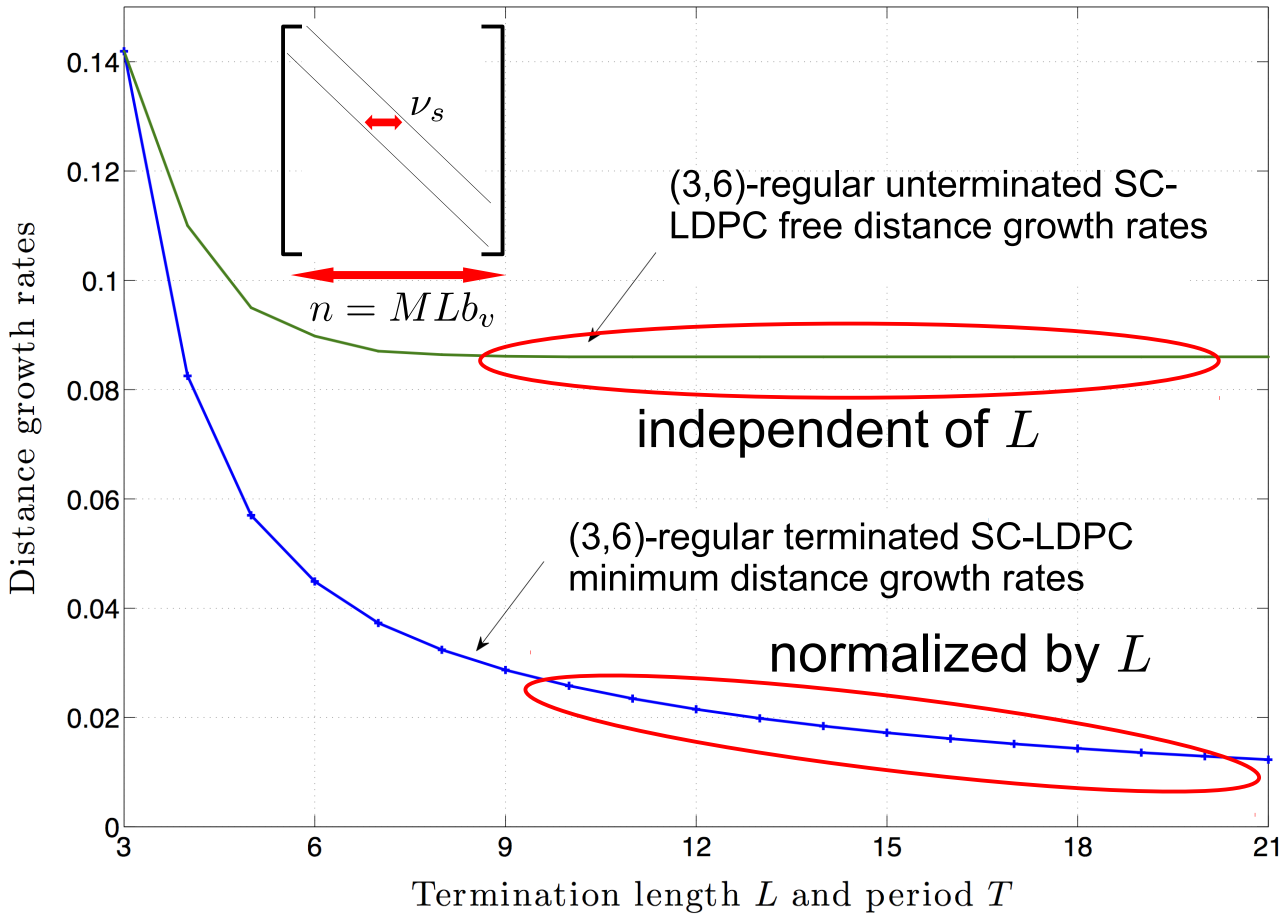

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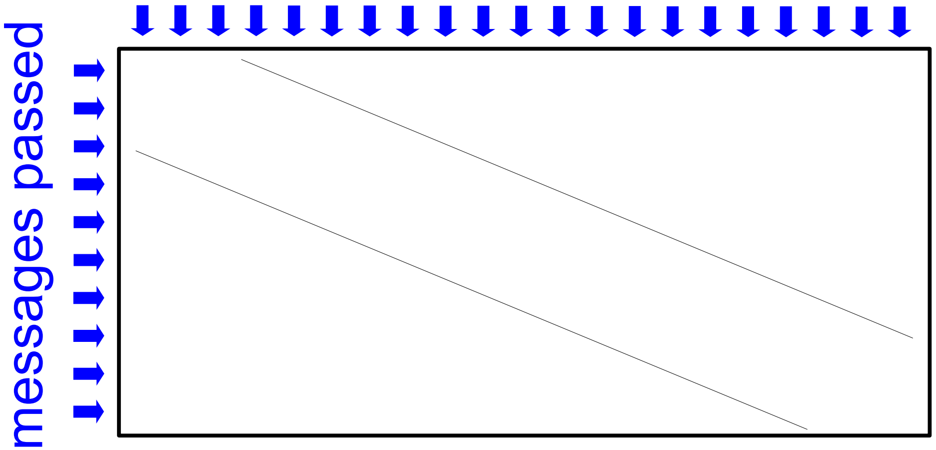
- An appropriate distance measure for 'convolutional-like' terminated ensembles should be independent of L.

Outline

- **LDPC Block Codes**
  - ➡ Parity-check matrix and Tanner graph representations, minimum distance bounds, iterative decoding thresholds, regular and irregular code designs, protograph-based constructions
- **Spatially Coupled LDPC Codes**
  - ➡ Protograph representation, edge-spreading construction, termination
  - ➡ Iterative decoding thresholds, threshold saturation, minimum distance
- **Practical Considerations**
  - ➡ Finite-length scaling, window decoding, performance, latency, and complexity comparisons to LDPC block codes, implementation aspects

Decoding SC-LDPC Codes

- SC-LDPC codes can be decoded with standard iterative decoding schedules.



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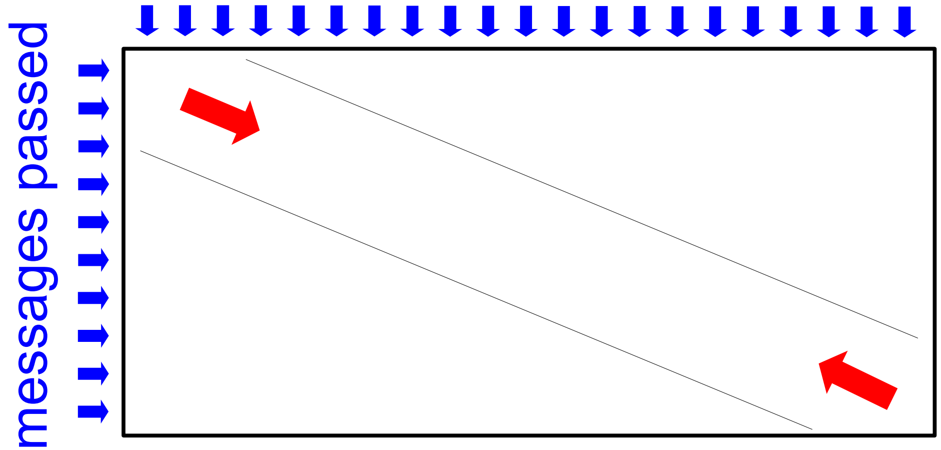

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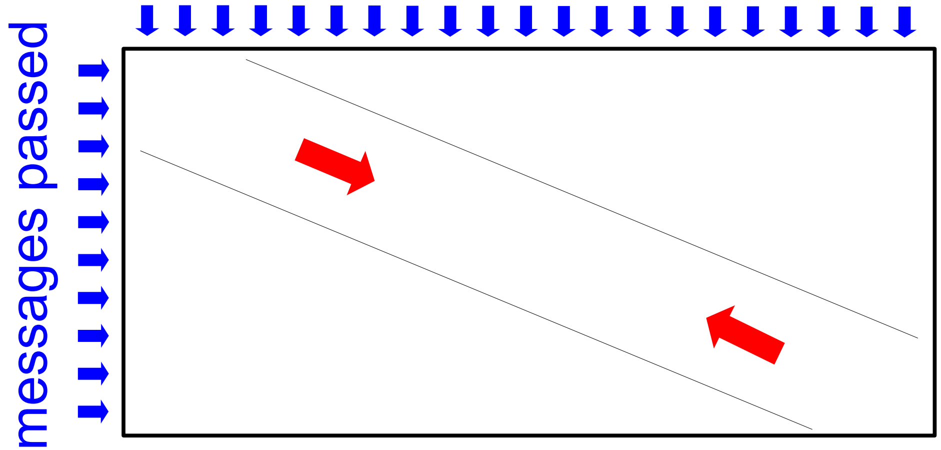

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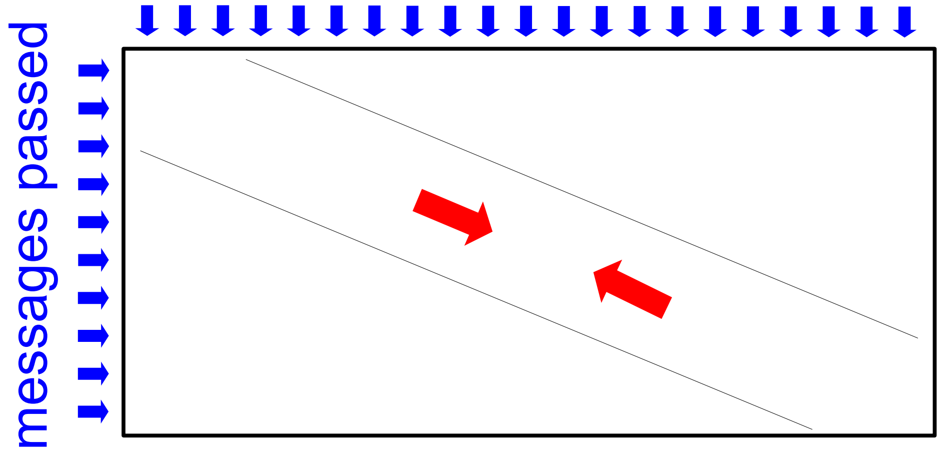

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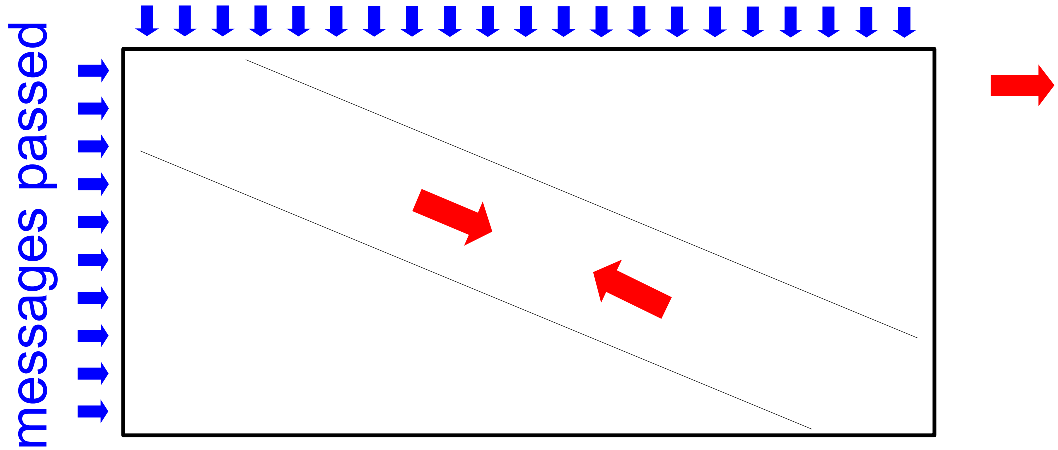

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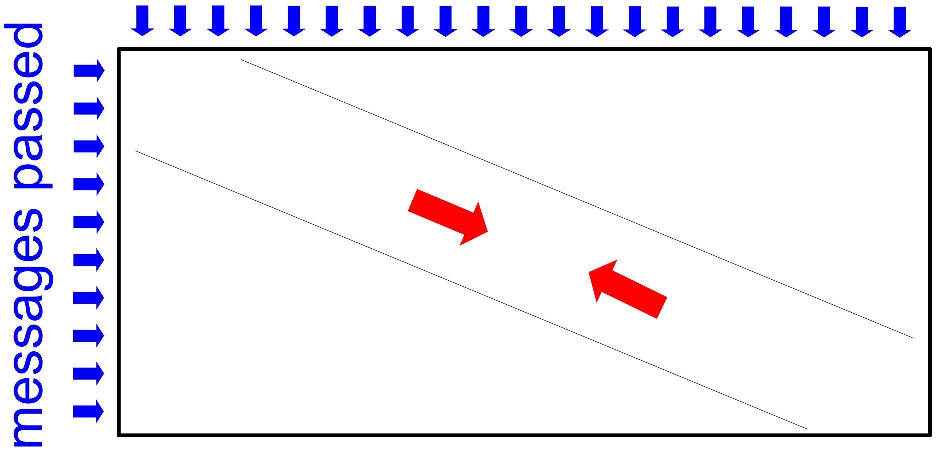

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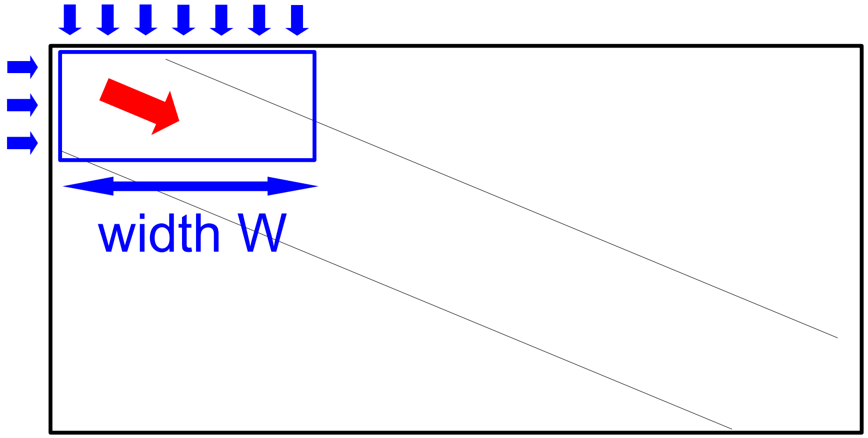

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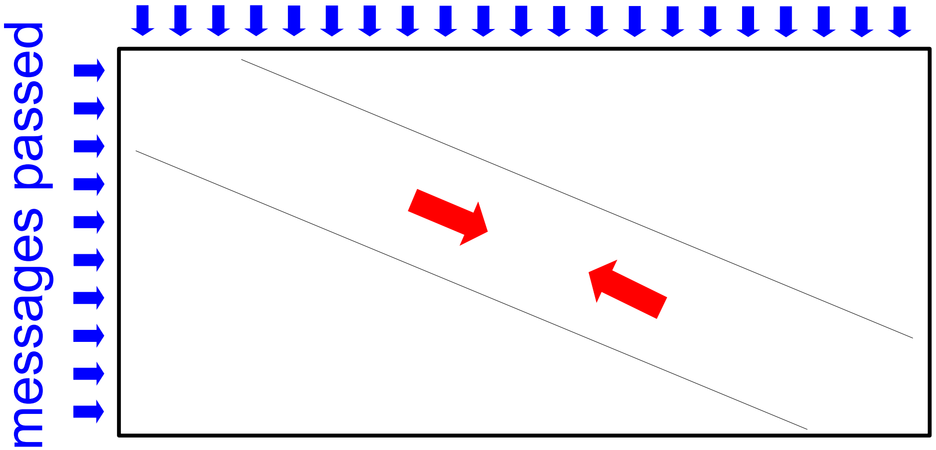

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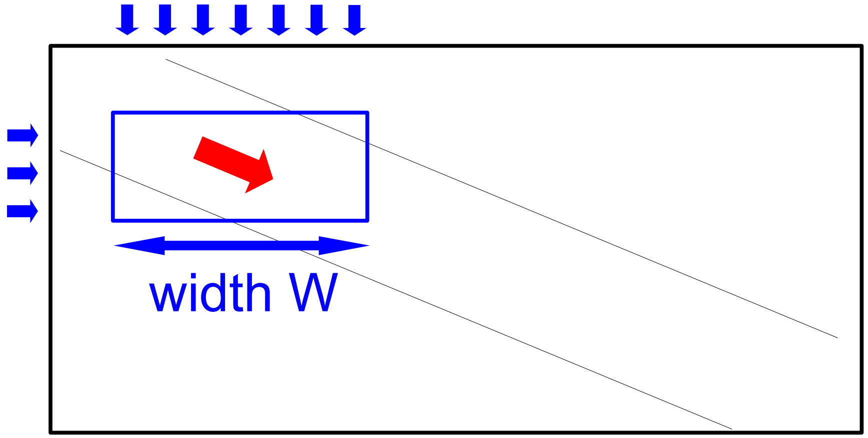

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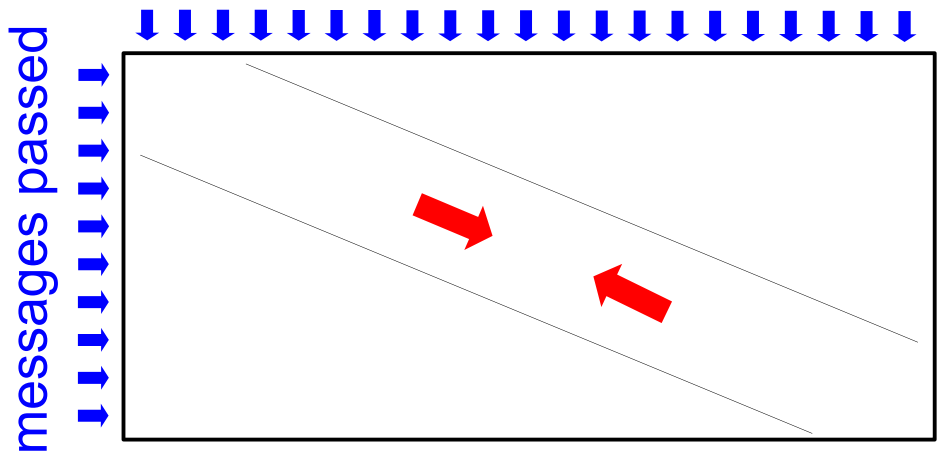
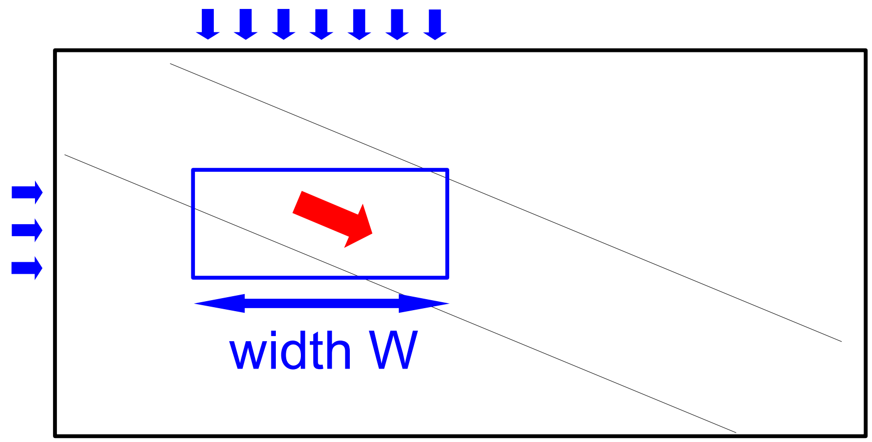
Decoding SC-LDPC Codes

- SC-LDPC codes can be decoded with standard iterative decoding schedules.



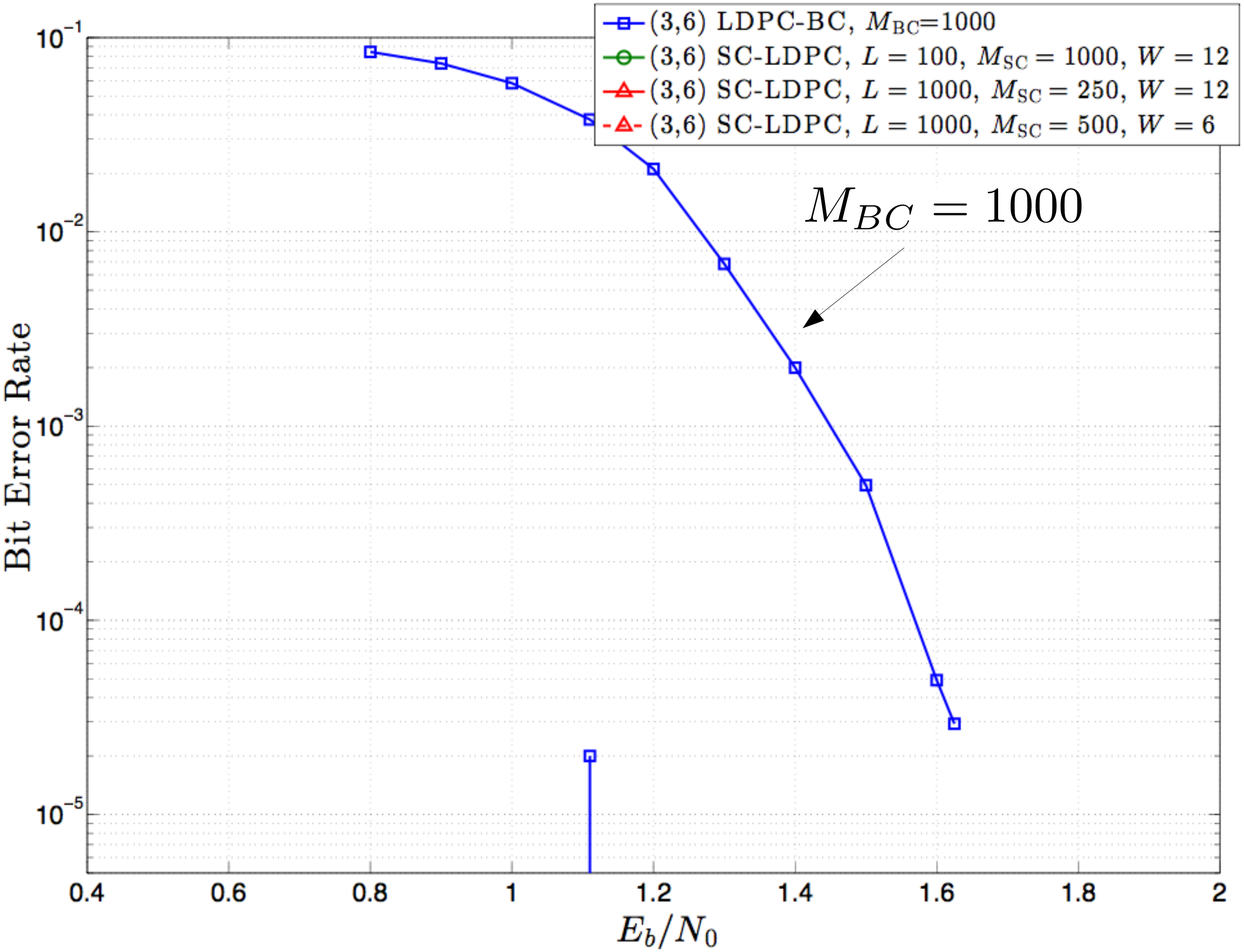
➡ Reliable messages from the ends propagate through the graph toward the center as iterations proceed.

- The **highly localized (convolutional) structure** is well-suited for efficient decoding schedules that reduce memory and latency requirements.



➡ **Sliding window decoding** (WD) updates nodes only within a localized window and then the window shifts across the graph [Lentmaier et al '10, Iyengar et al '12].

Window Decoding Performance



Latencies:
LDPC: $6M_{BC}$
SC-LDPC: $2M_{SC}W$

[LPF11] M. Lentmaier, M. M. Prenda, and G. Fettweis, "Efficient Message Passing Scheduling for Terminated LDPC Convolutional Codes", *Proc. IEEE ISIT*, St. Petersburg, Russia, July 2011.

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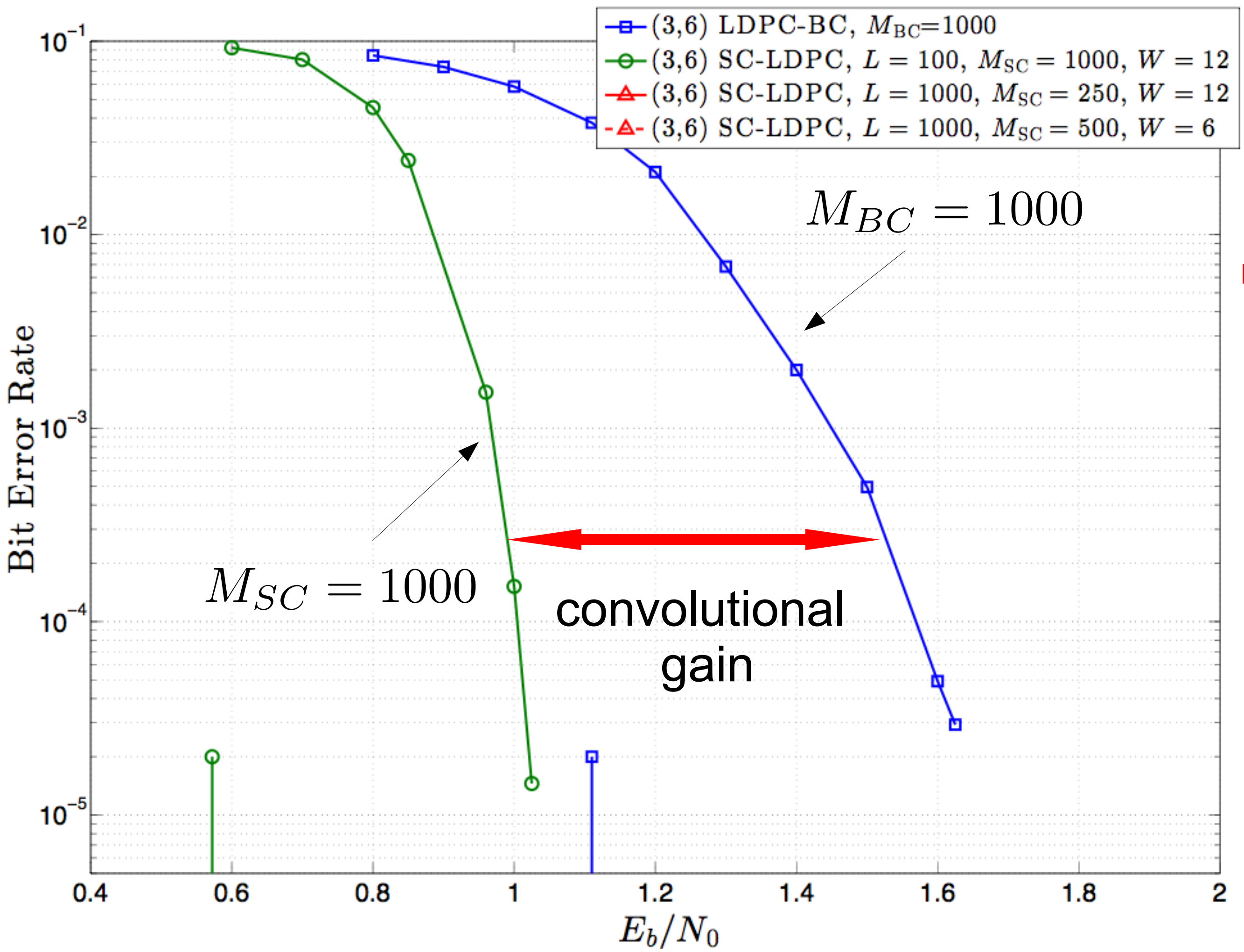

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For **equal lifting factors**, SC-LDPC codes display a large **convolutional gain** at the cost of increased latency.

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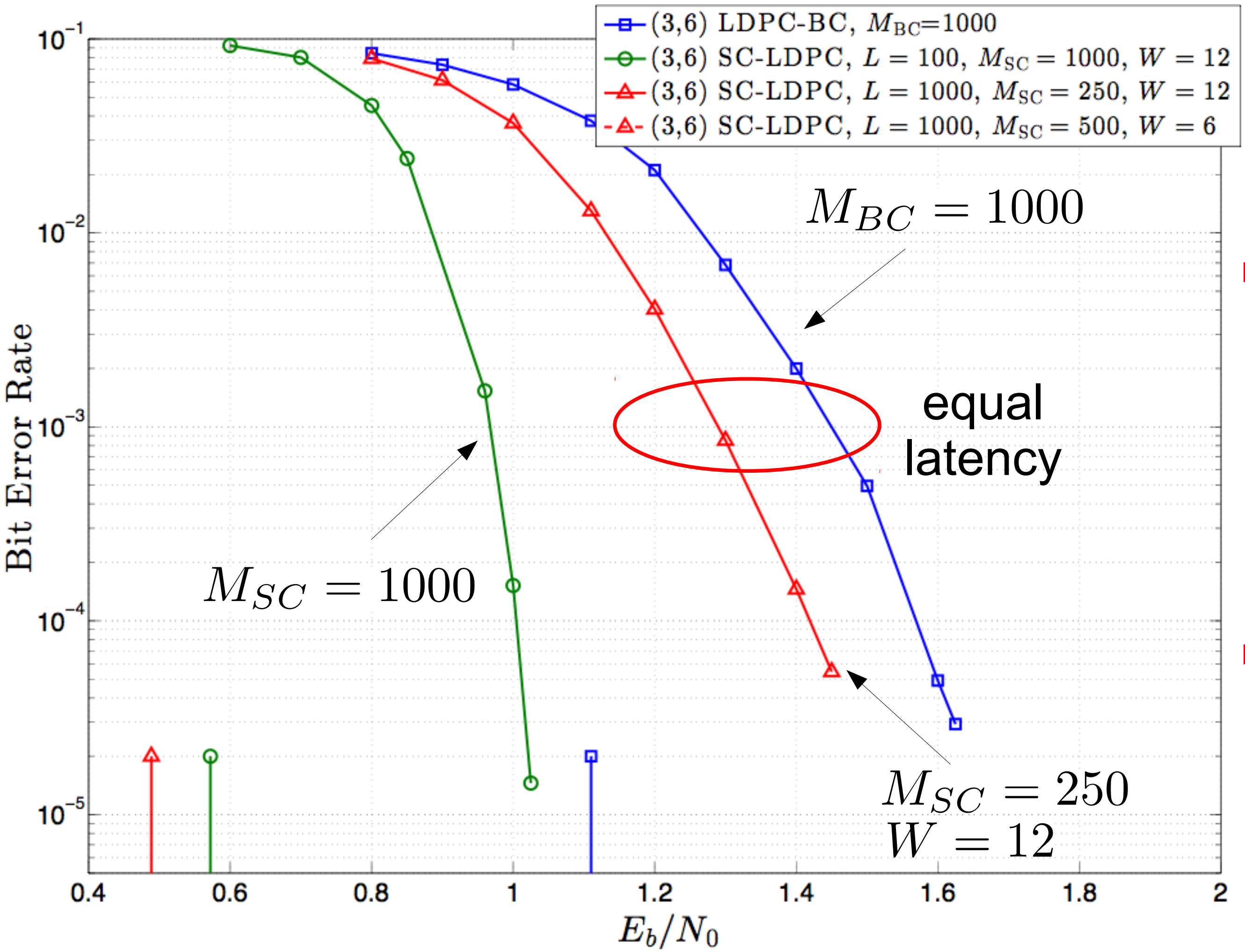

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- For **equal lifting factors**, SC-LDPC codes display a large **convolutional gain** at the cost of increased latency.
- For **equal latency**, SC-LDPC codes still display a significant **performance gain**.

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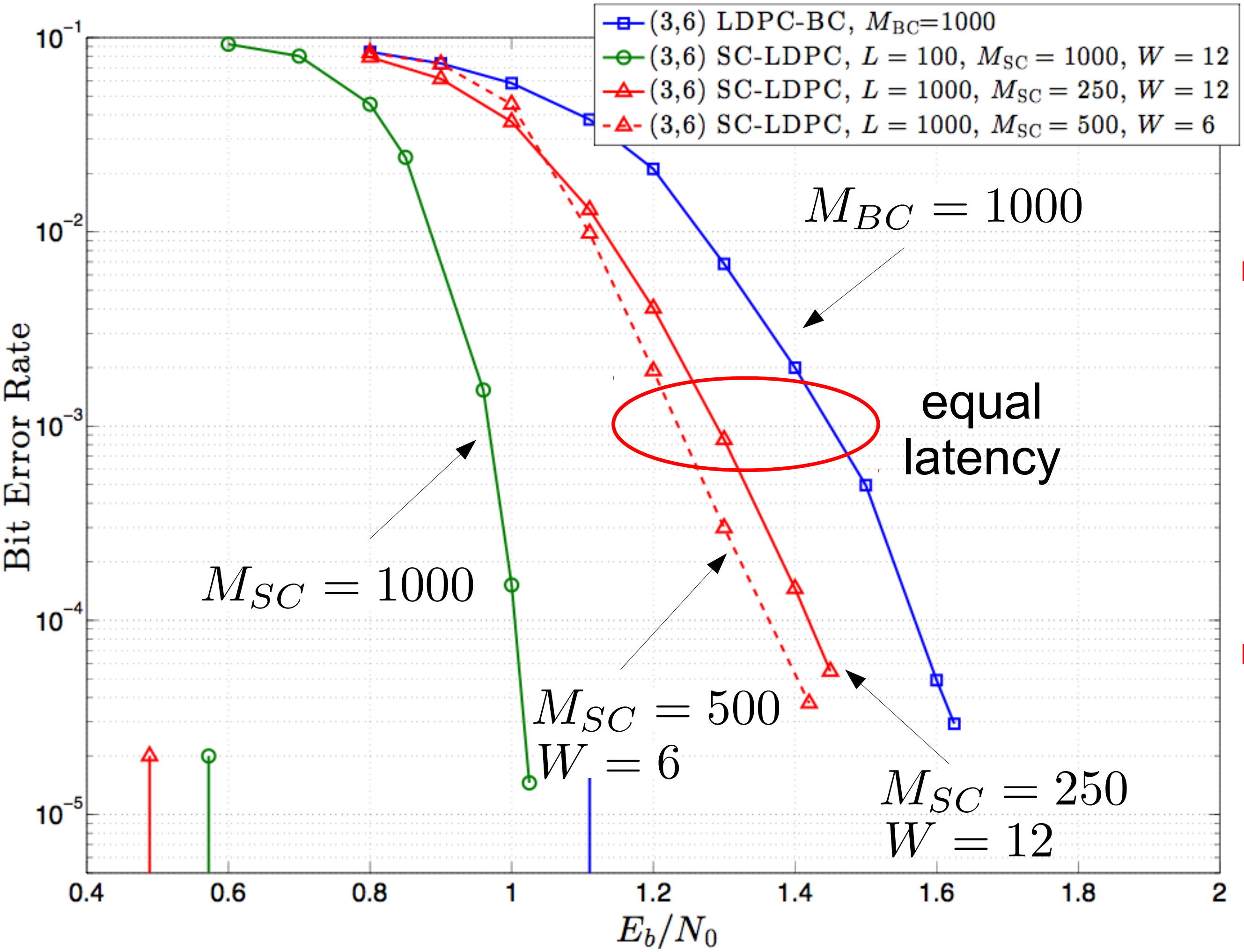

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Finite-Length Scaling for SC-LDPC Codes

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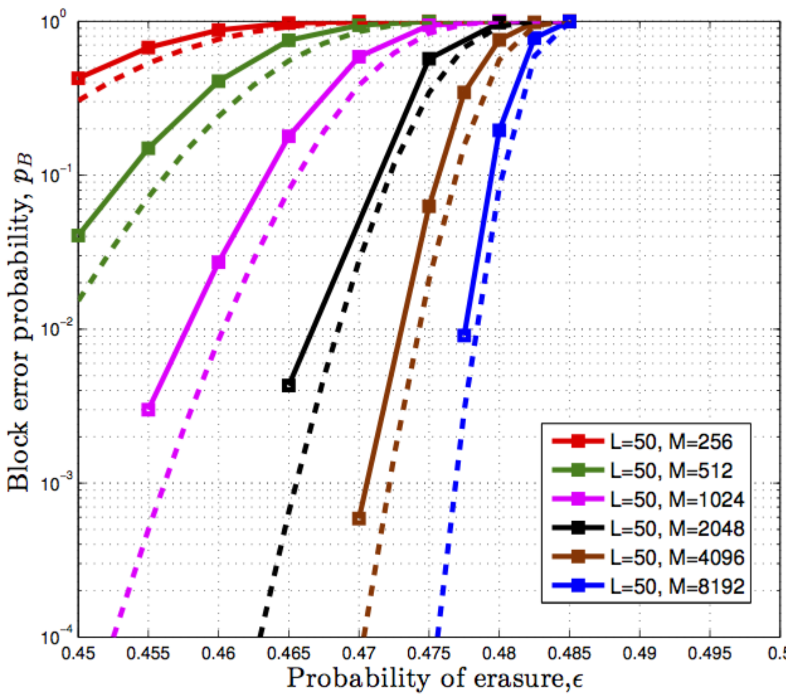
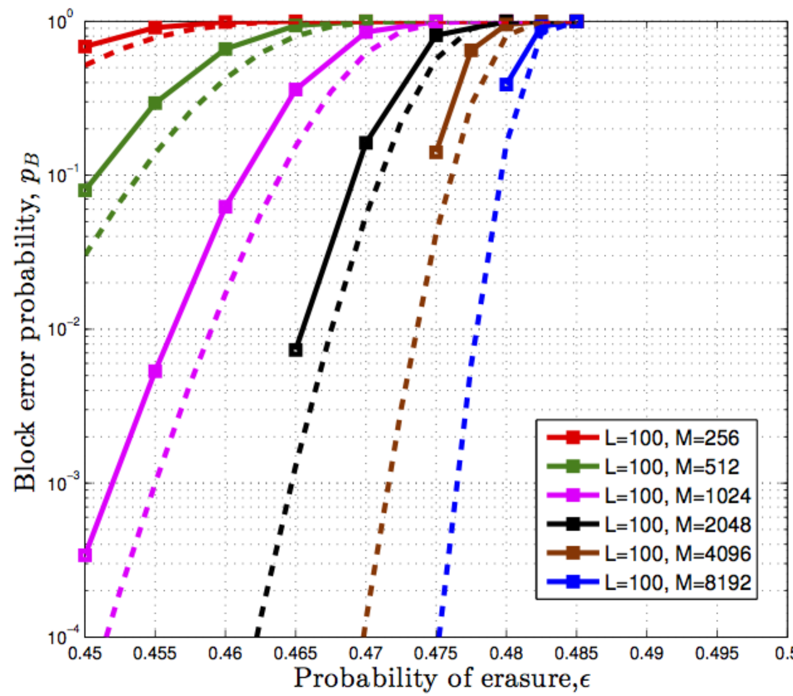
- For the BEC, approximate analytical expressions obtained for the error probability of SC-LDPC codes compare well to simulated results.





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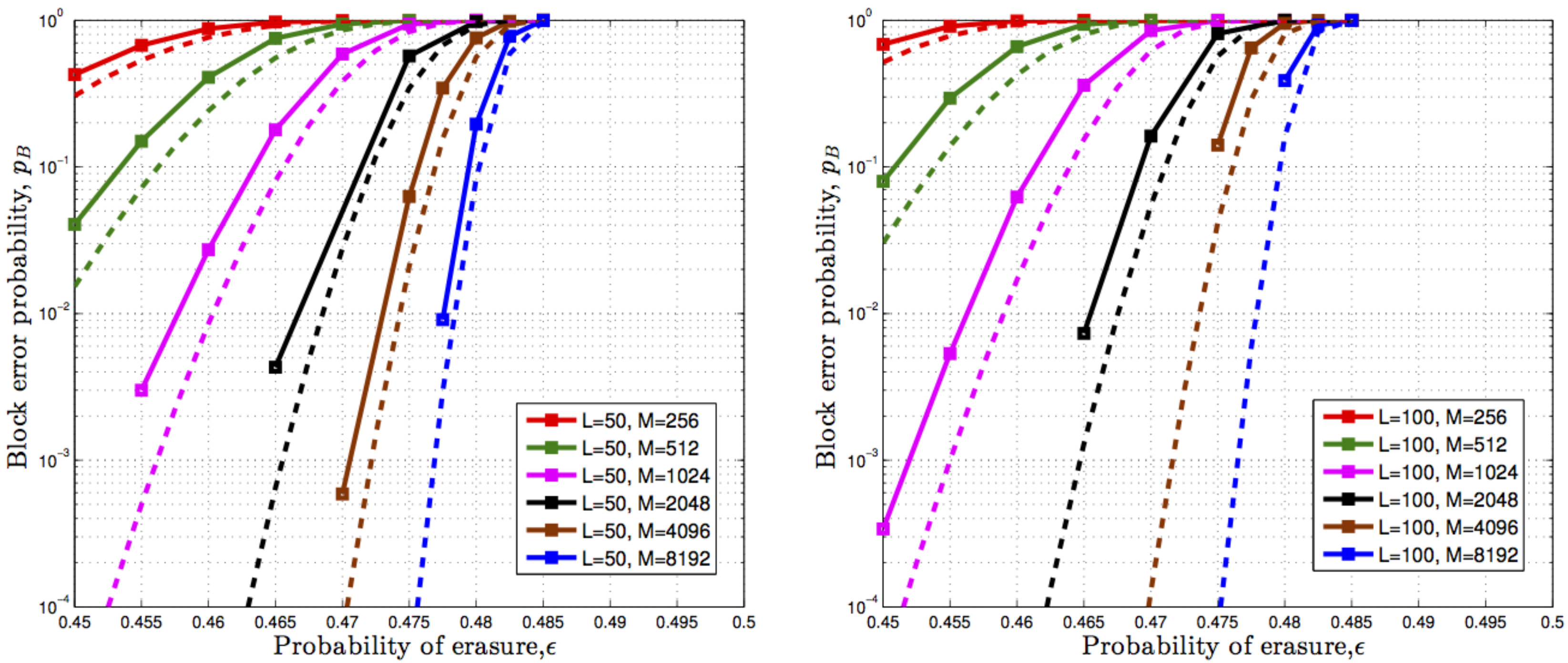

- The scaling law is a useful engineering tool to gain insight into the design of SC-LDPC codes.

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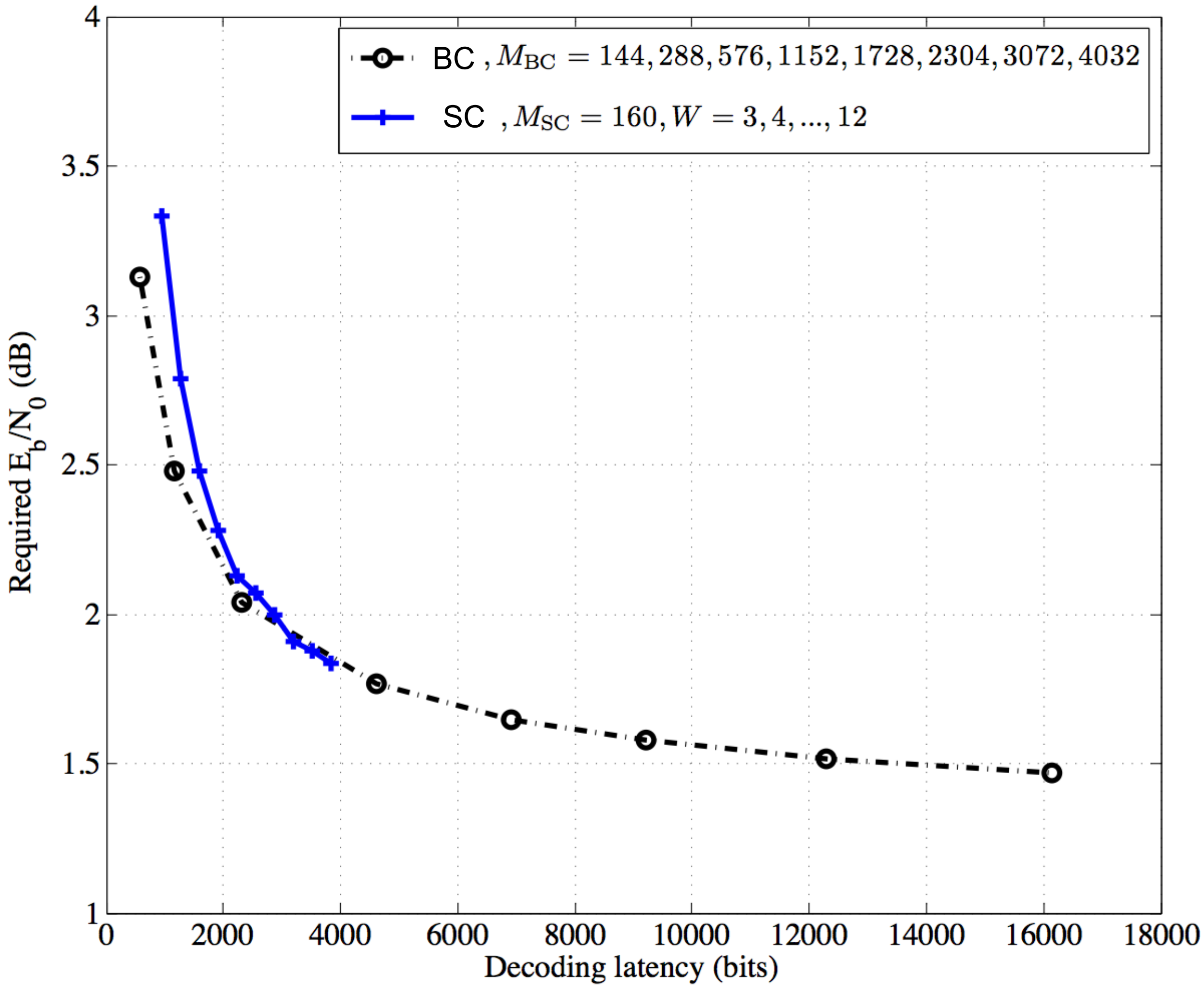
Equal Latency Comparison for (3,6)-Regular LDPC Codes

- Required E_b/N_0 to achieve a BER of 10^{-5} as a function of latency:



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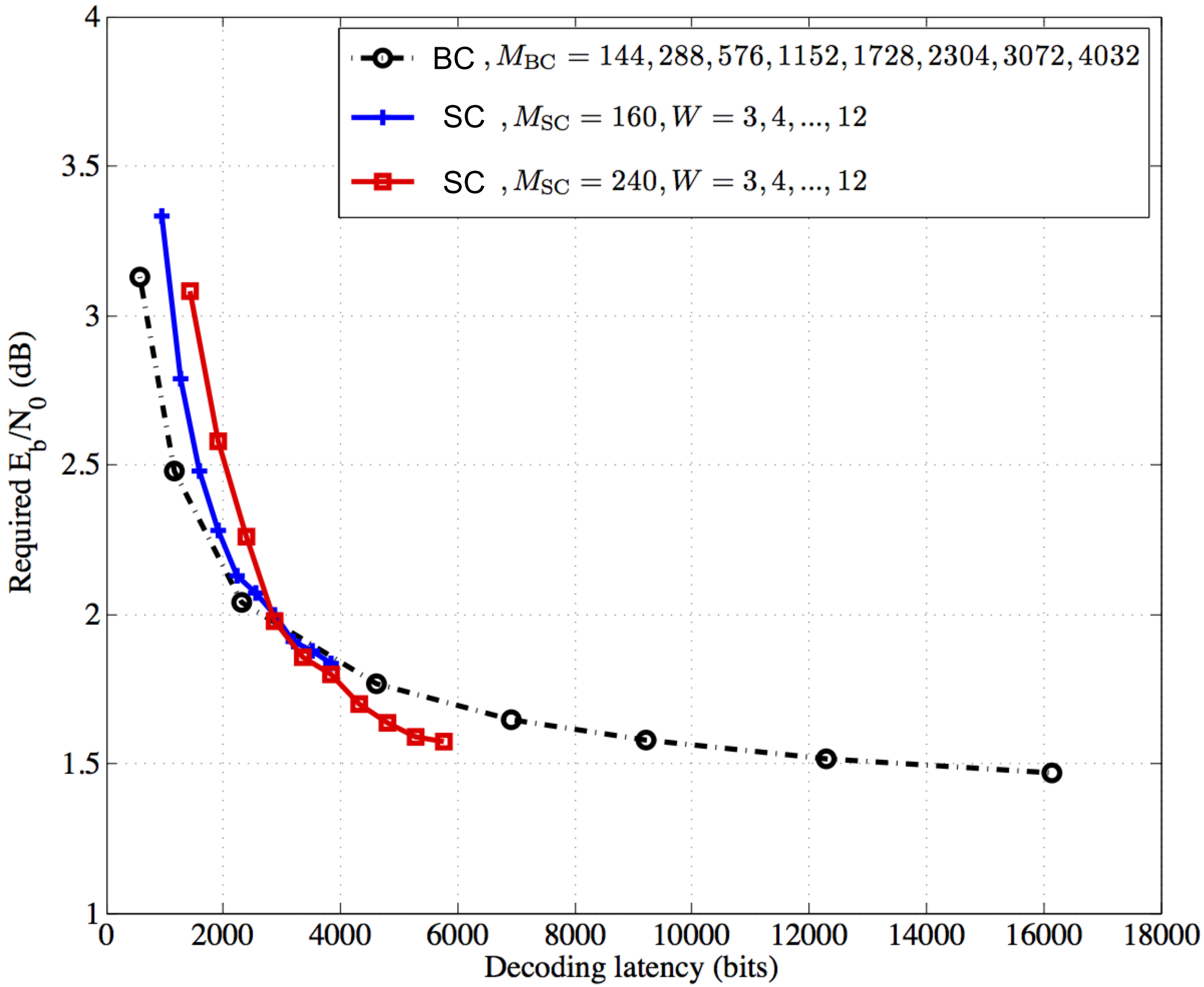

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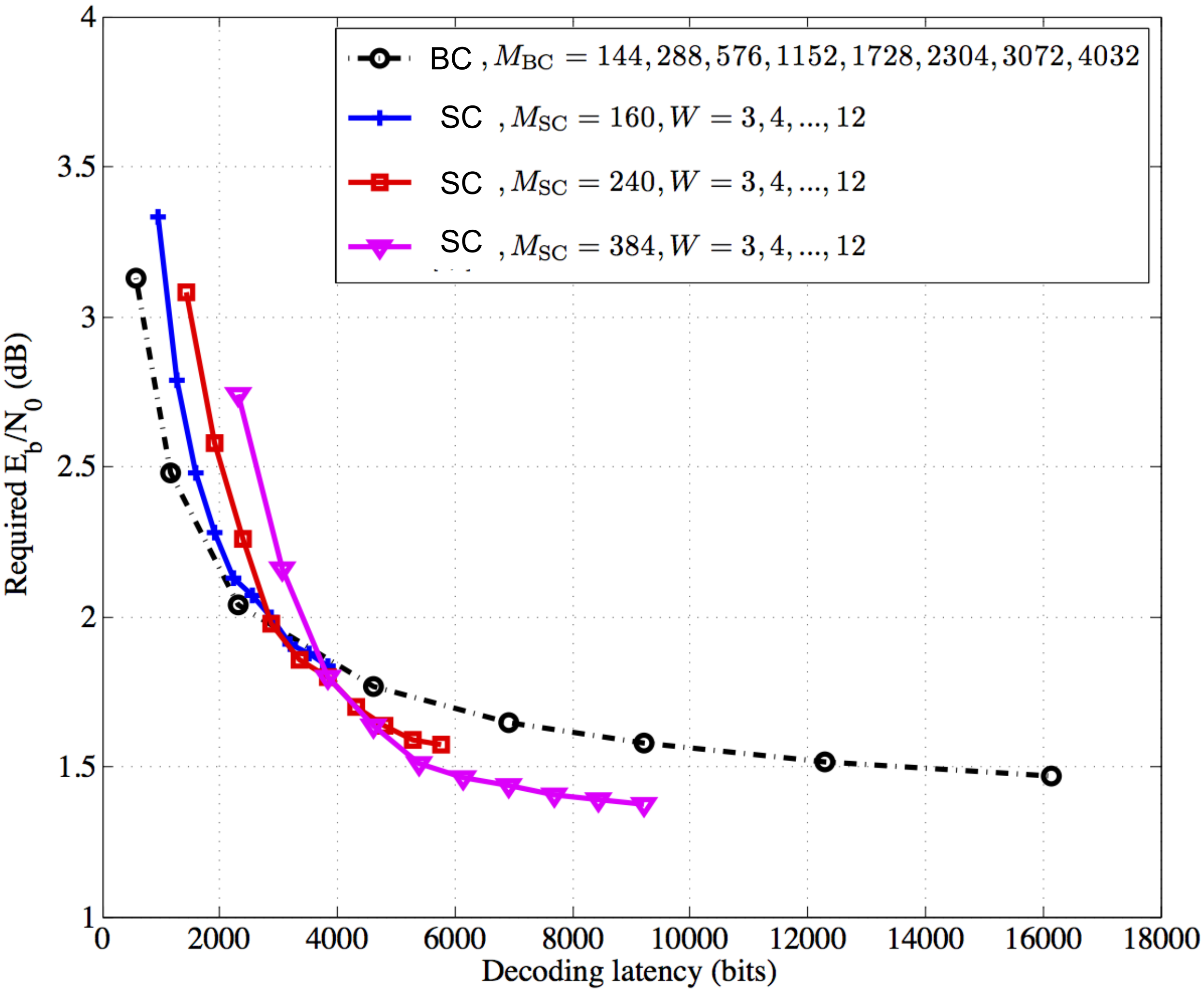

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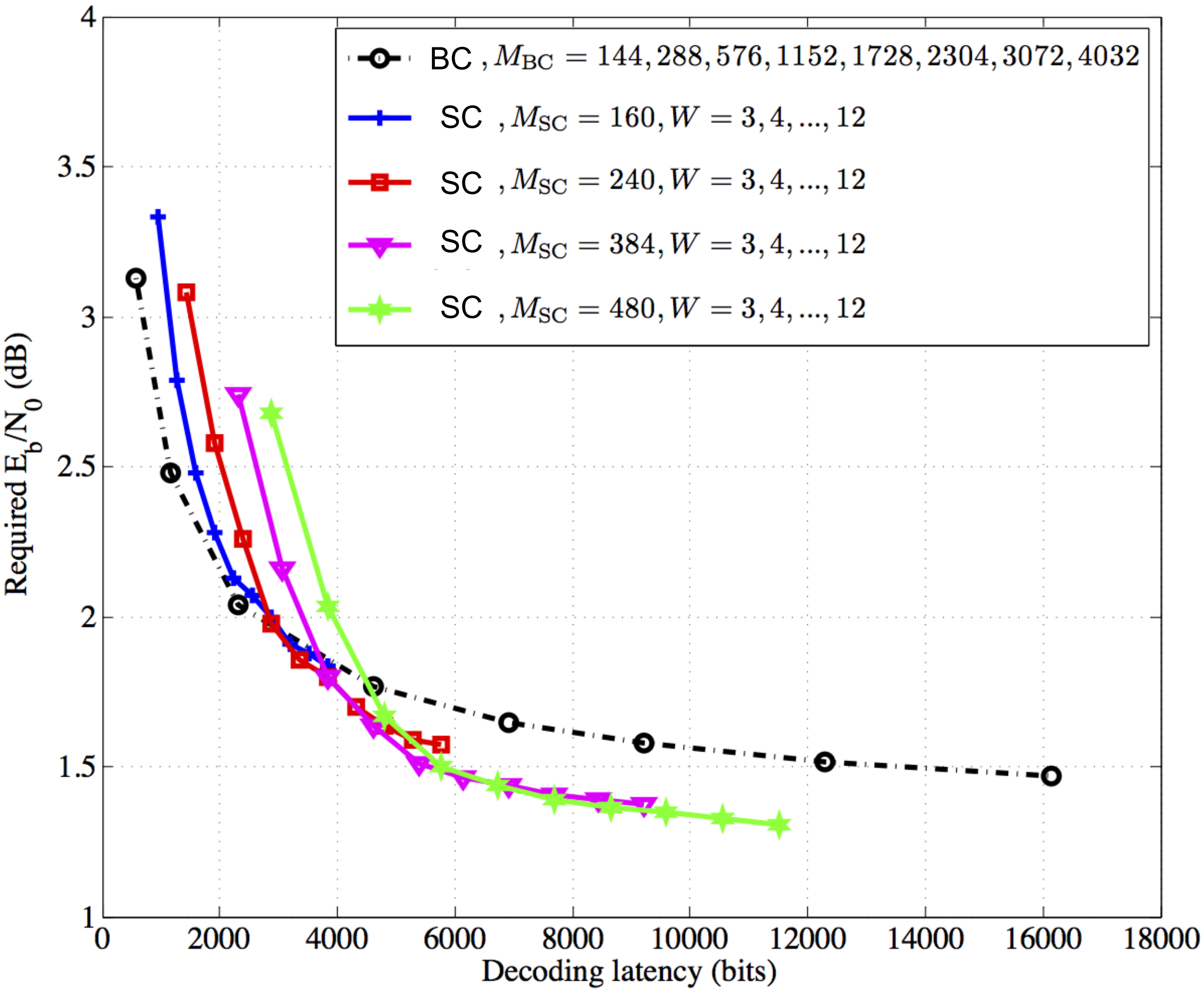

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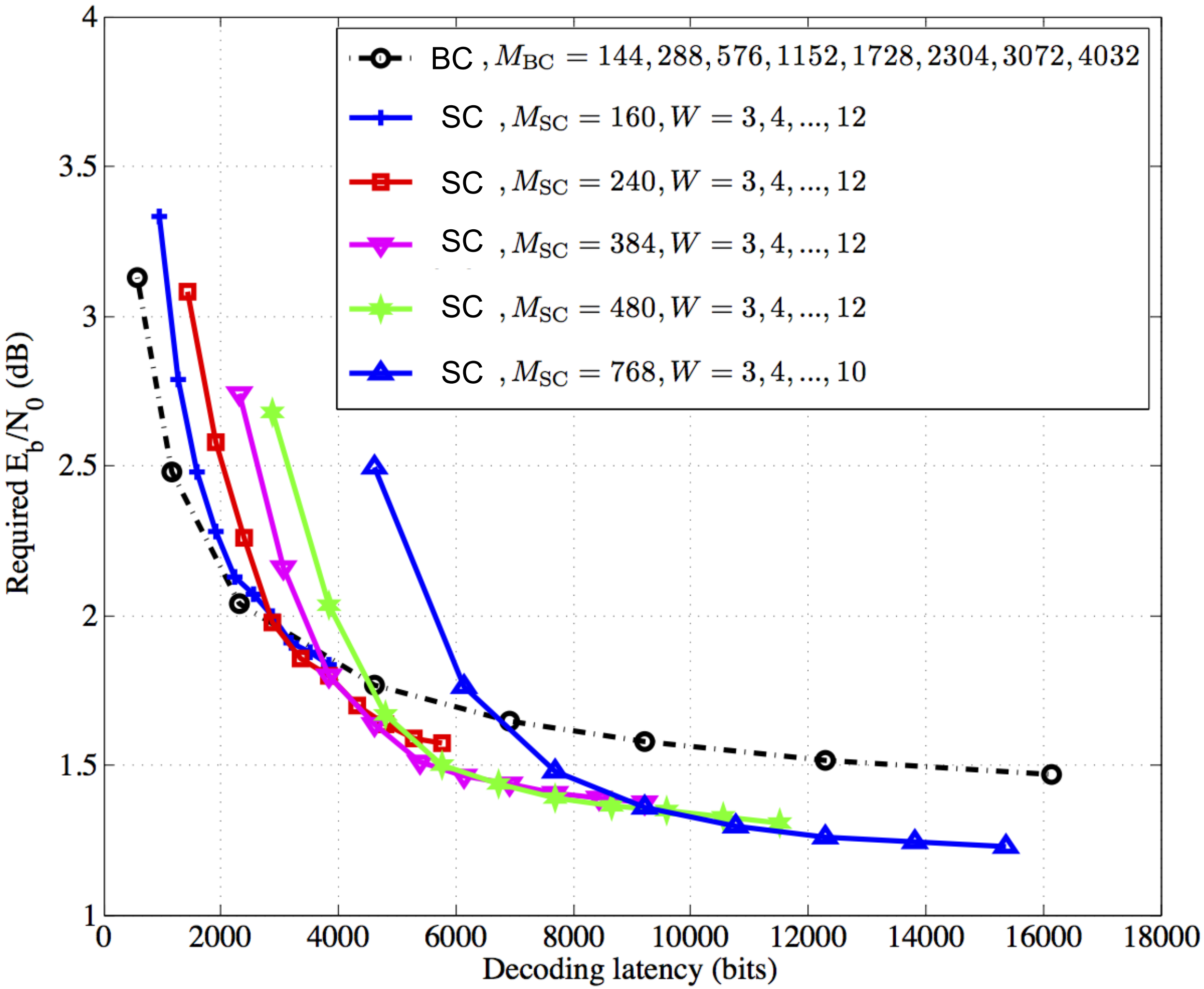

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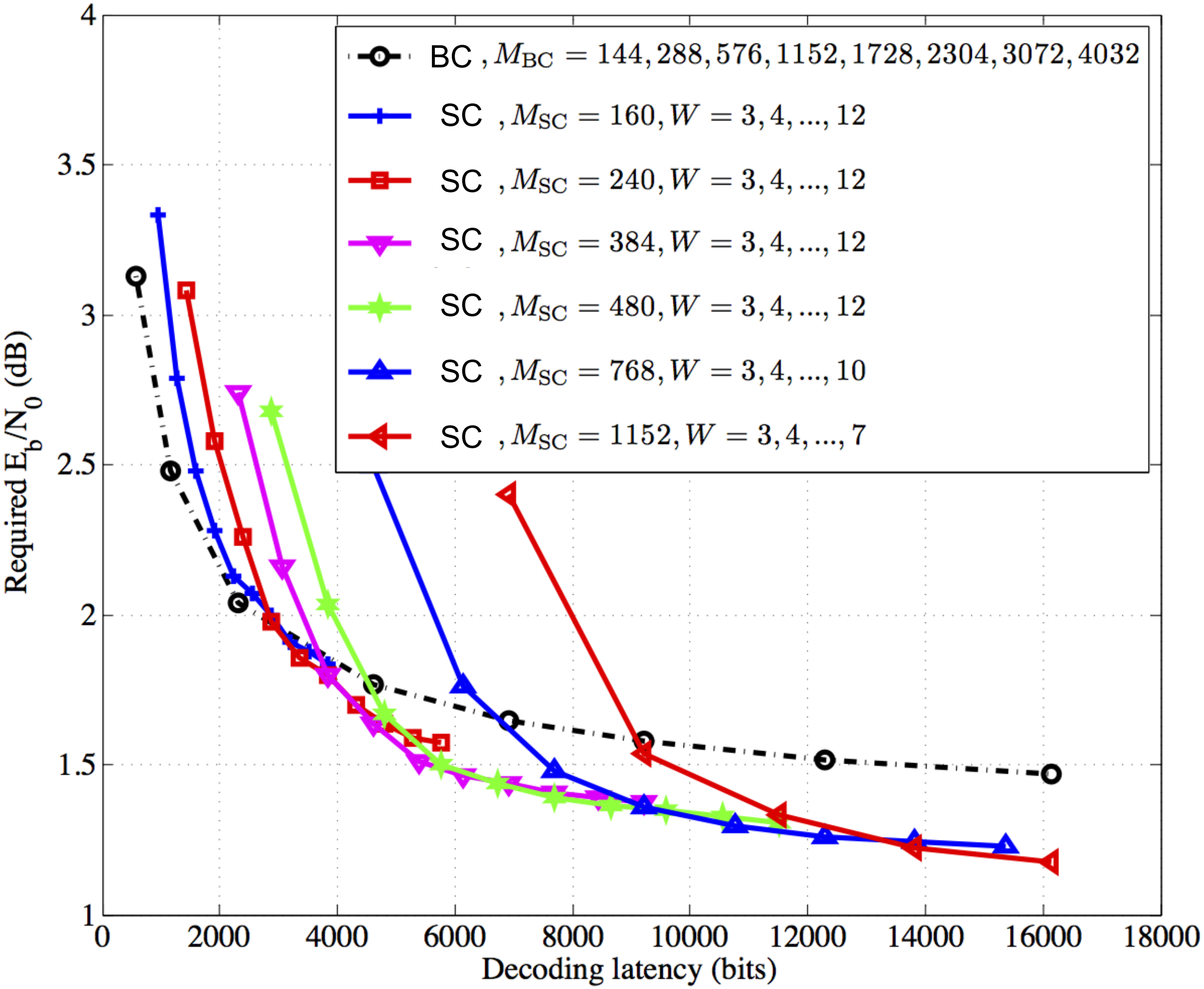

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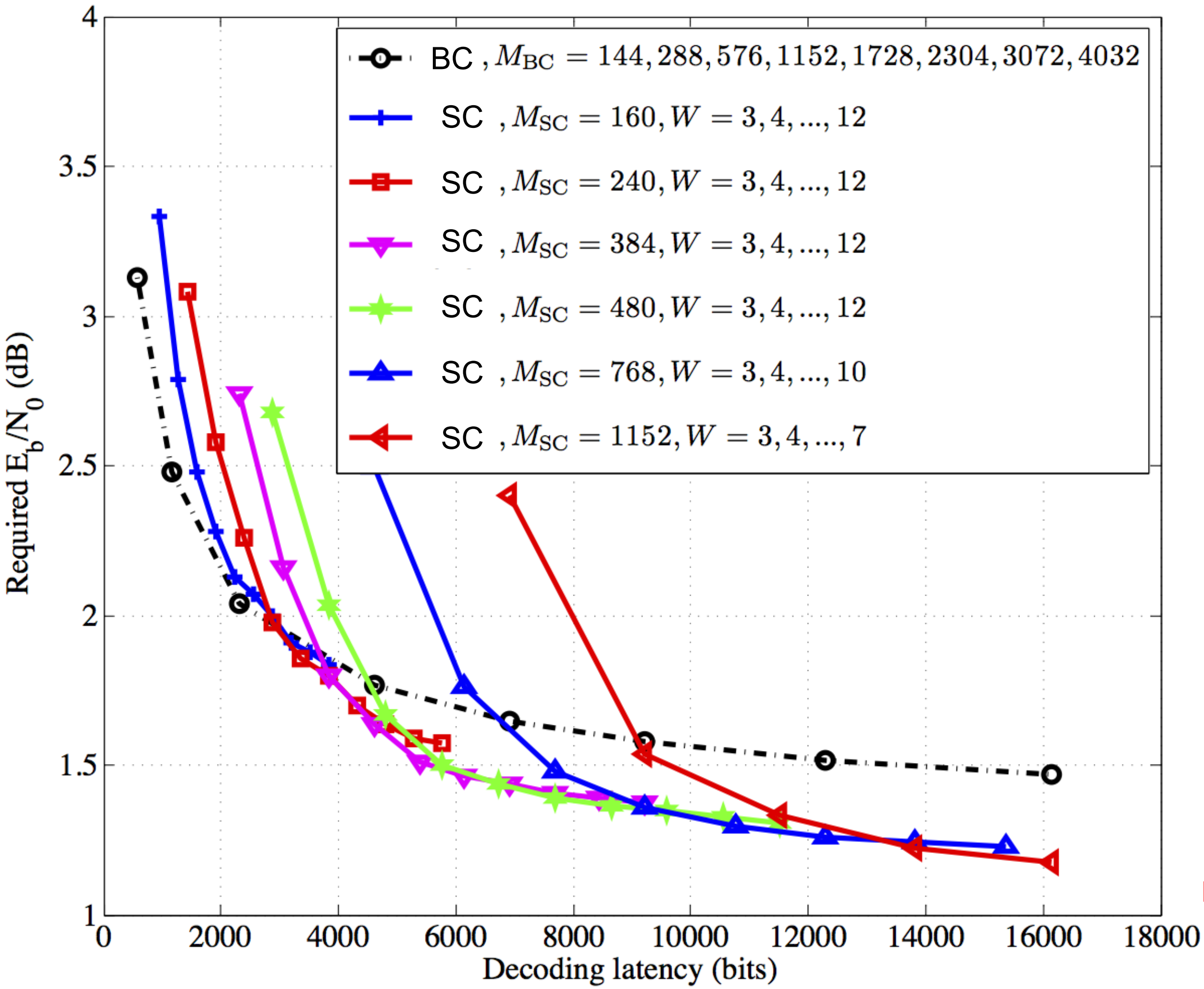

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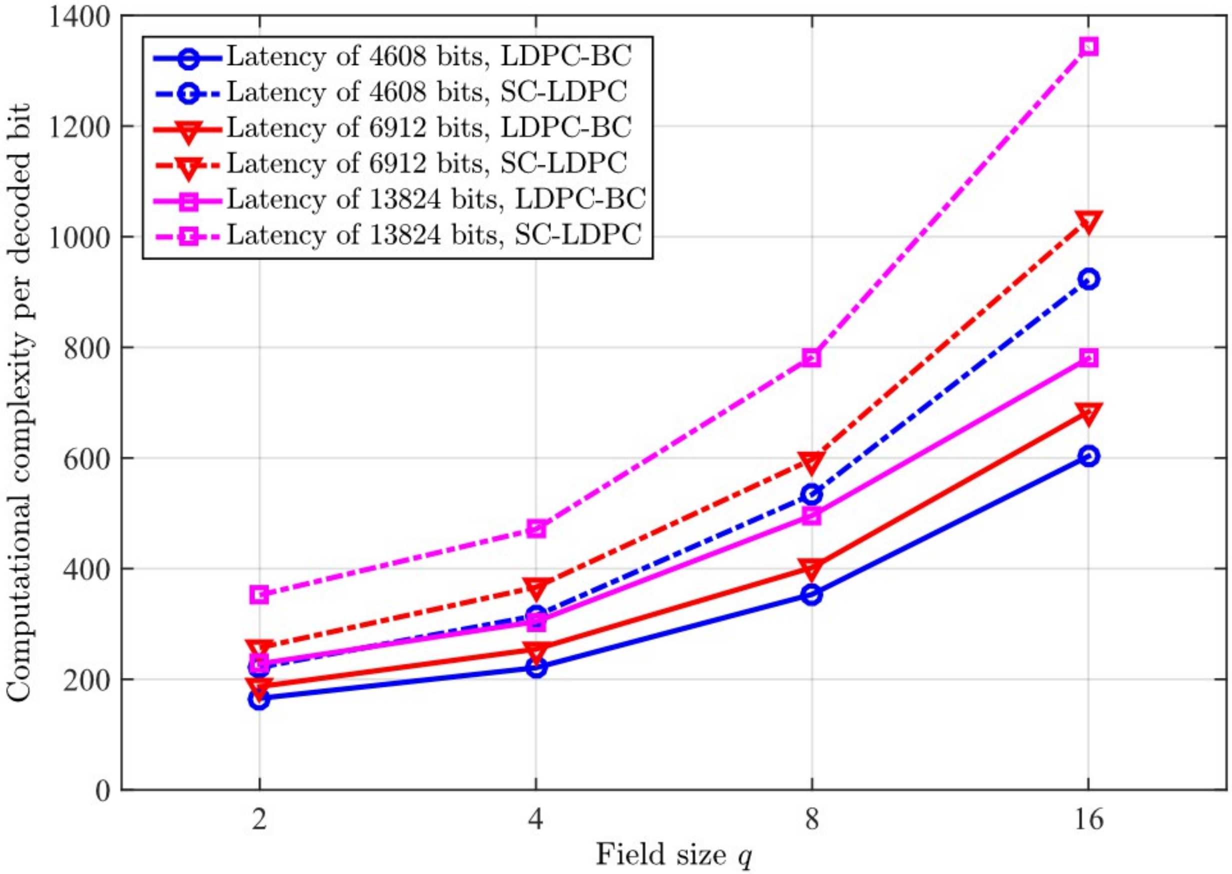
- ➡ decreases as W (and thus the latency) increases.
- ➡ does not decrease significantly beyond a certain W $(W \approx 10)$.
- When choosing parameters:
  - ➡ large M_{SC} improves code performance.
  - ➡ large W improves decoder performance.

Complexity Tradeoffs

- For equal latency, SC-LDPC codes display a performance gain compared to the underlying LDPC-BCs



(including non-binary codes)

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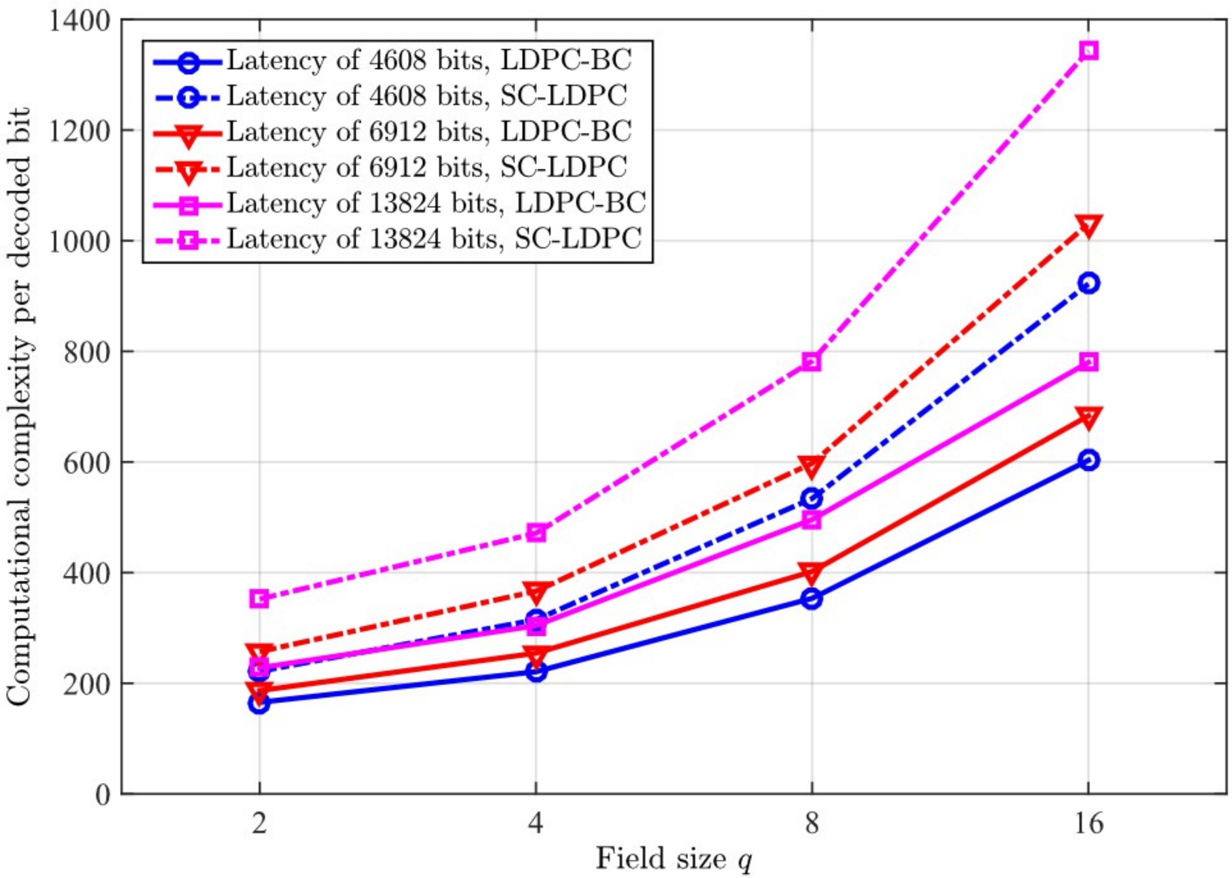

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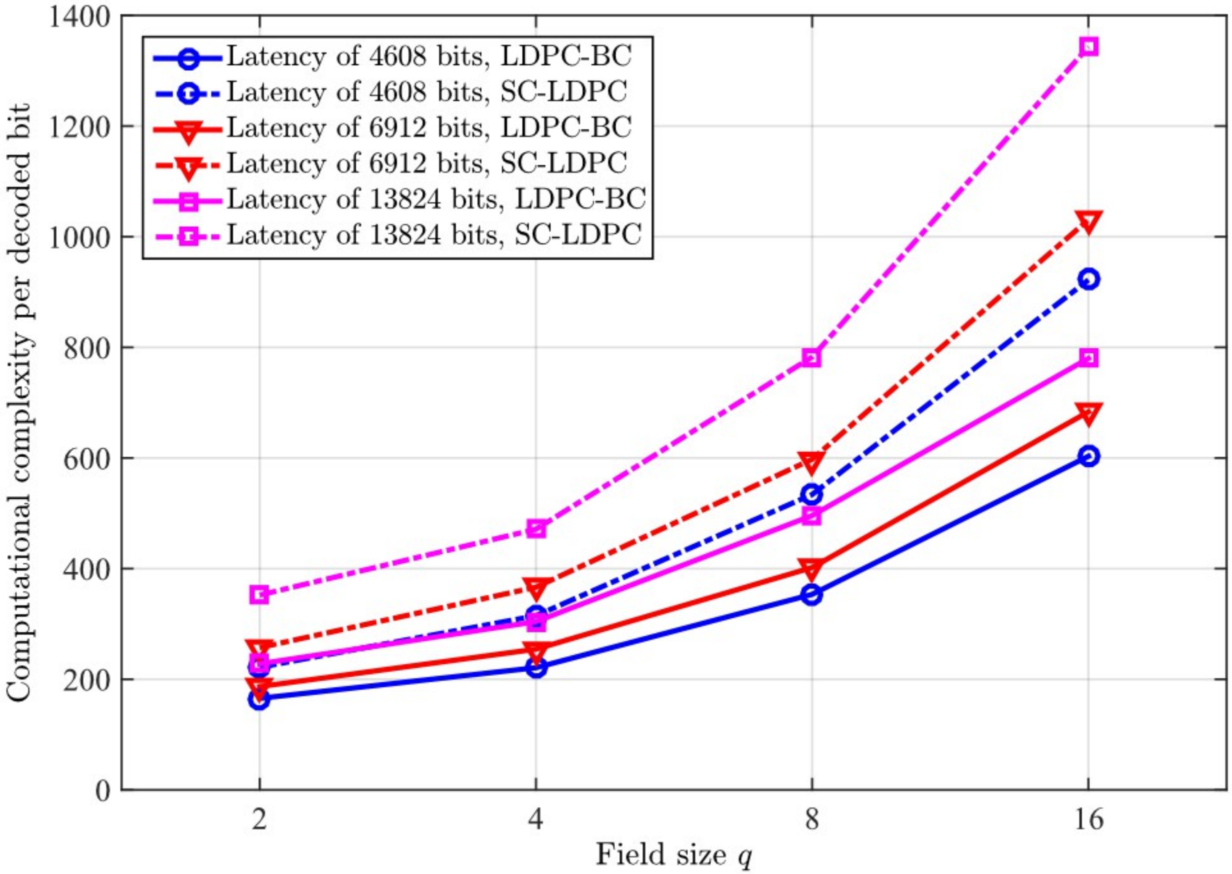

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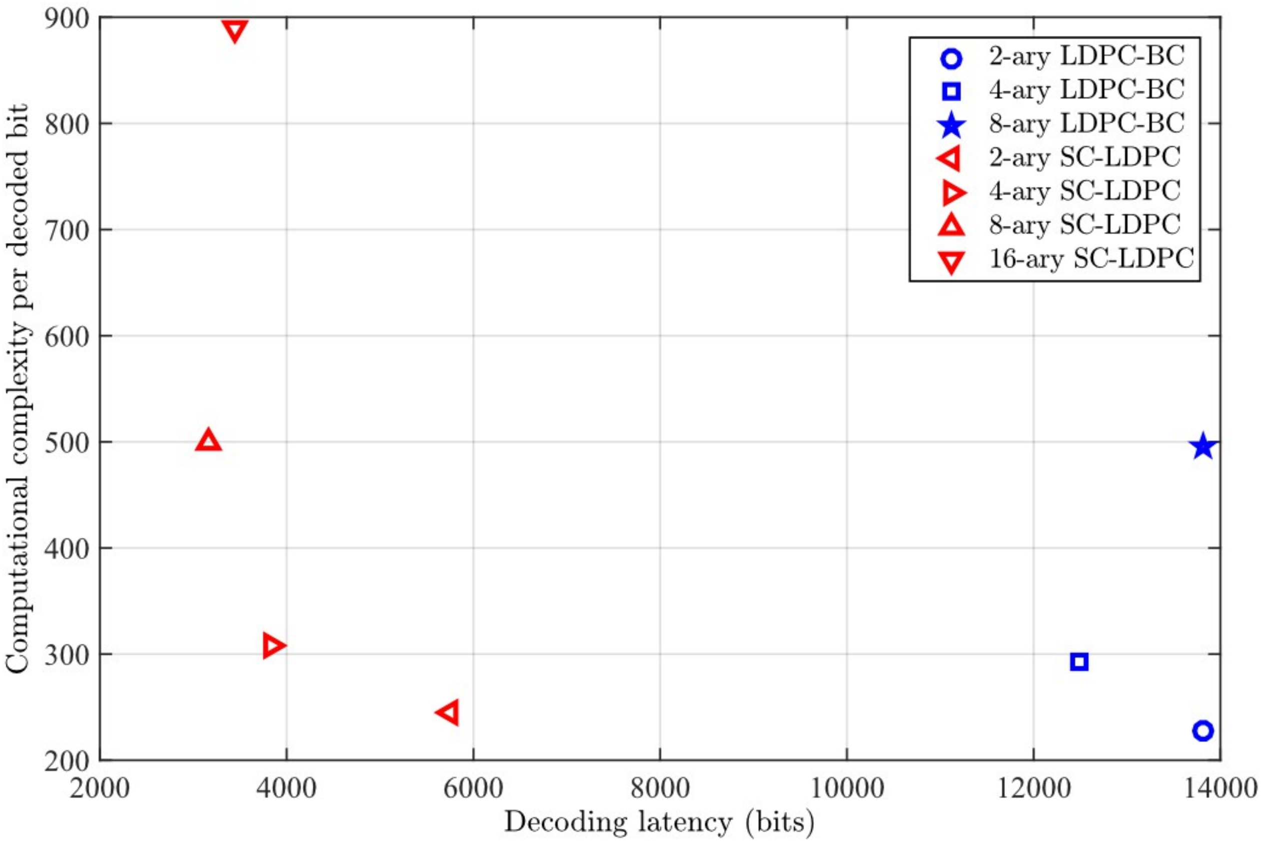
- LDPC-BCs **cannot** achieve equal performance by increasing the number of iterations

Complexity/Latency Tradeoffs

- For equal performance (BER of 10^{-5} at 1.5dB), SC-LDPC codes display a large reduction in latency compared to LDPC-BCs for similar complexity



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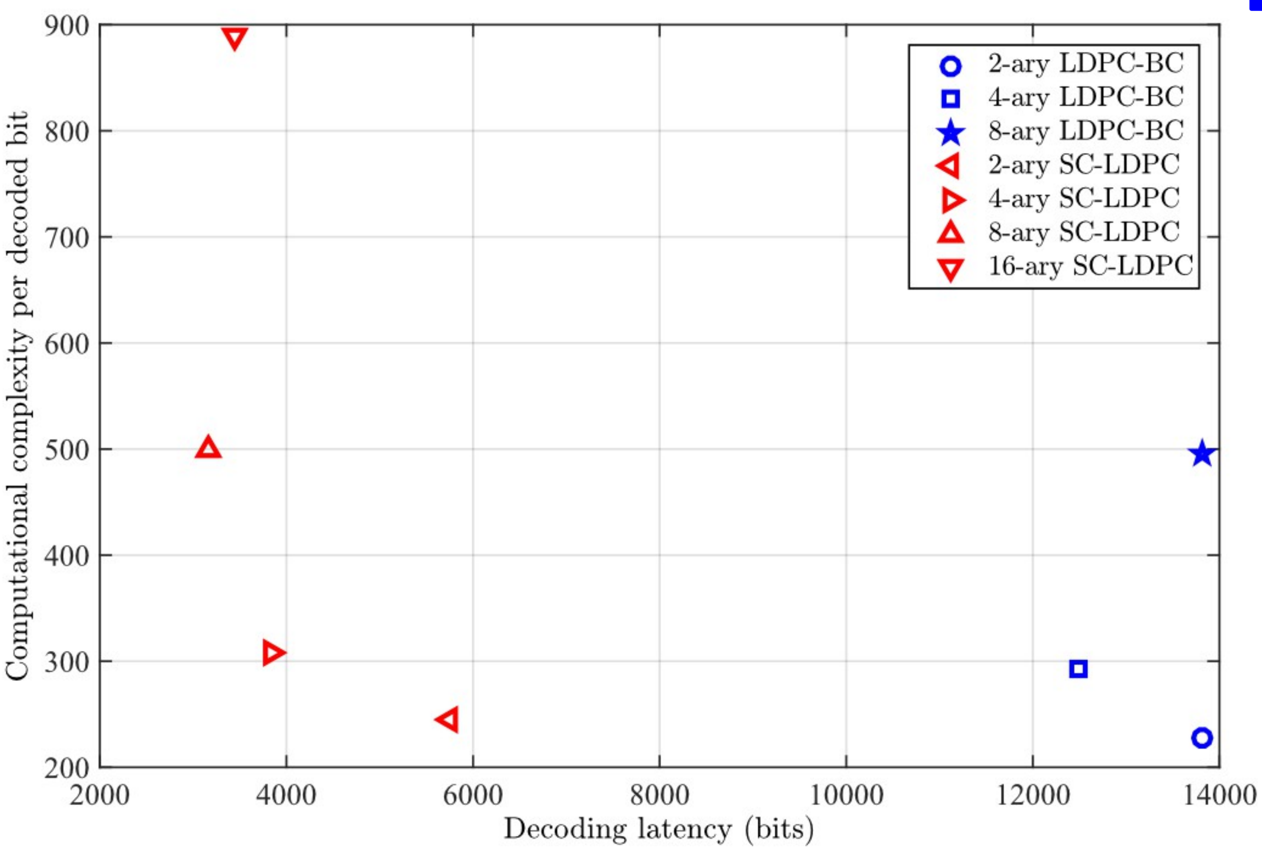

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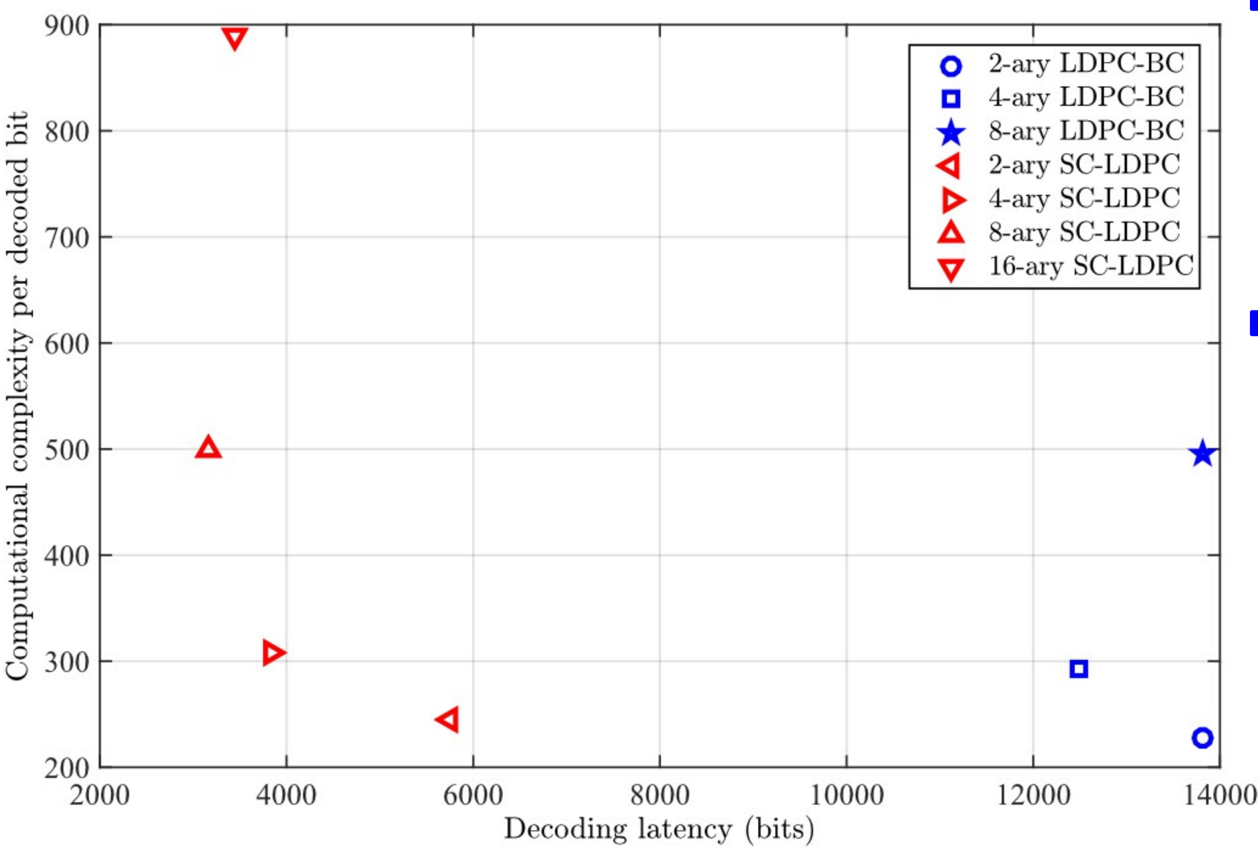

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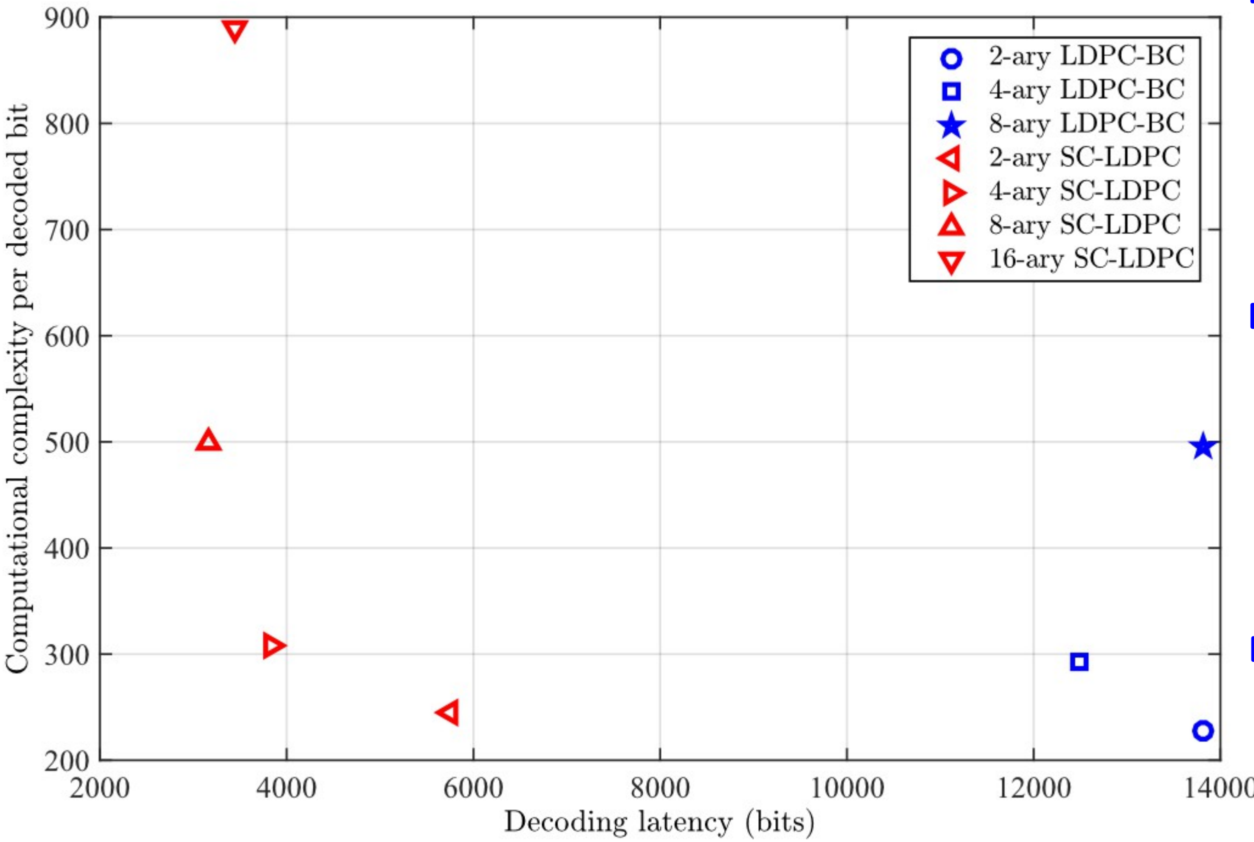

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- With increasing (small) field sizes q, latency decreases for increasing complexity
- For larger q, both latency **and** complexity increase (for both SC-LDPC codes and LDPC-BCs)!
- SC-LDPC codes over GF(4) offer a good balance between complexity and latency

Regular SC-LDPC Codes vs. Irregular LDPC-BCs

- Consider a comparison of a (3,6)-regular SC-LDPC code vs. an optimized irregular LDPC code with degree distribution

$$\lambda(x) = 0.409x + 0.202x^2 + 0.0768x^3 + 0.1971x^6 + 0.1151x^7$$
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- We will compare this to a (3,6)-regular SC-LDPC code with L=50 and R=0.49. The corresponding window decoding thresholds are $\epsilon^* \approx 0.4758$ and $(E_b/N_0)^* \approx 0.5925$ dB.

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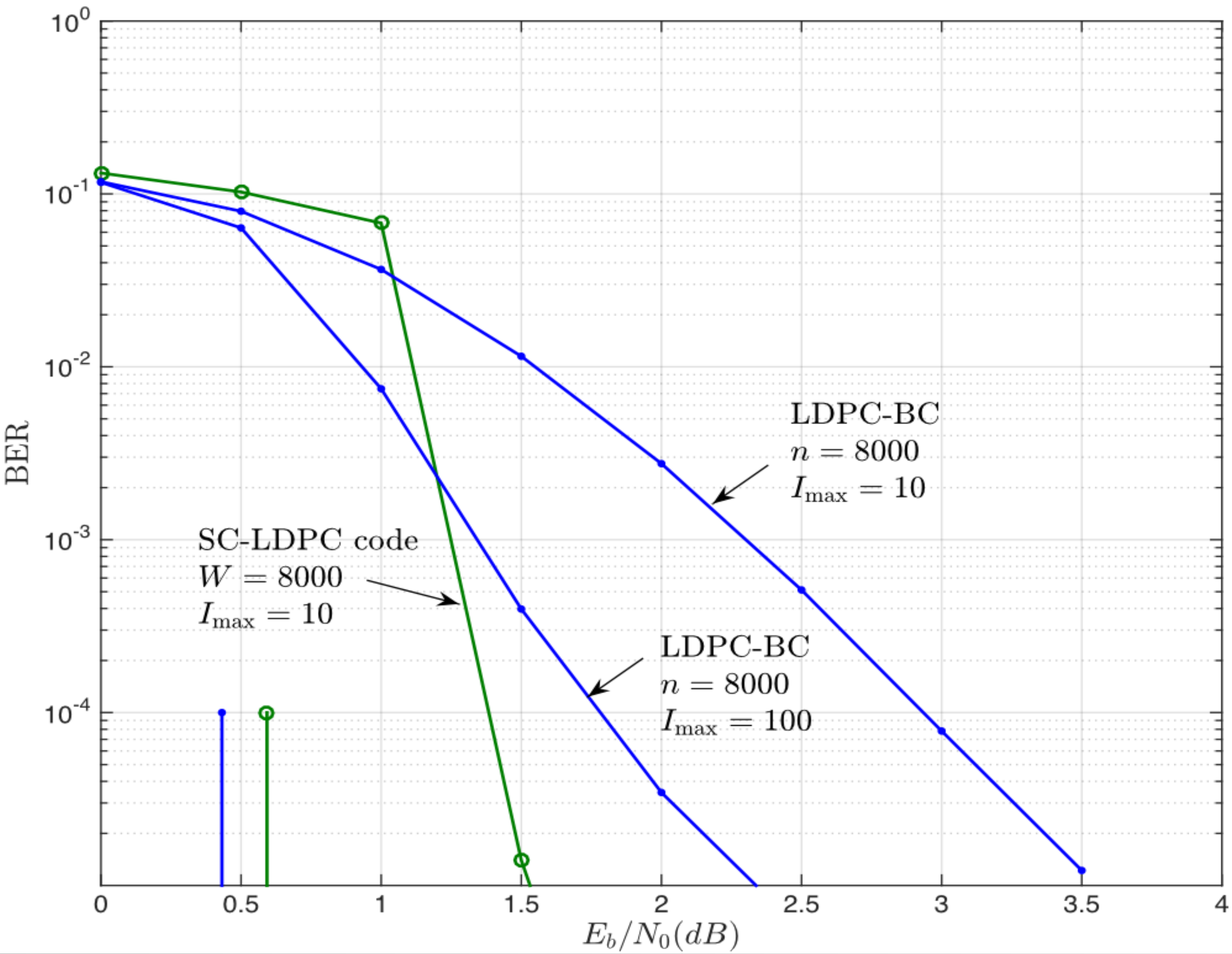

- On an equal latency basis, the regular SC-LDPC code outperforms the irregular LDPC-BC at BERs below 10^{-3}

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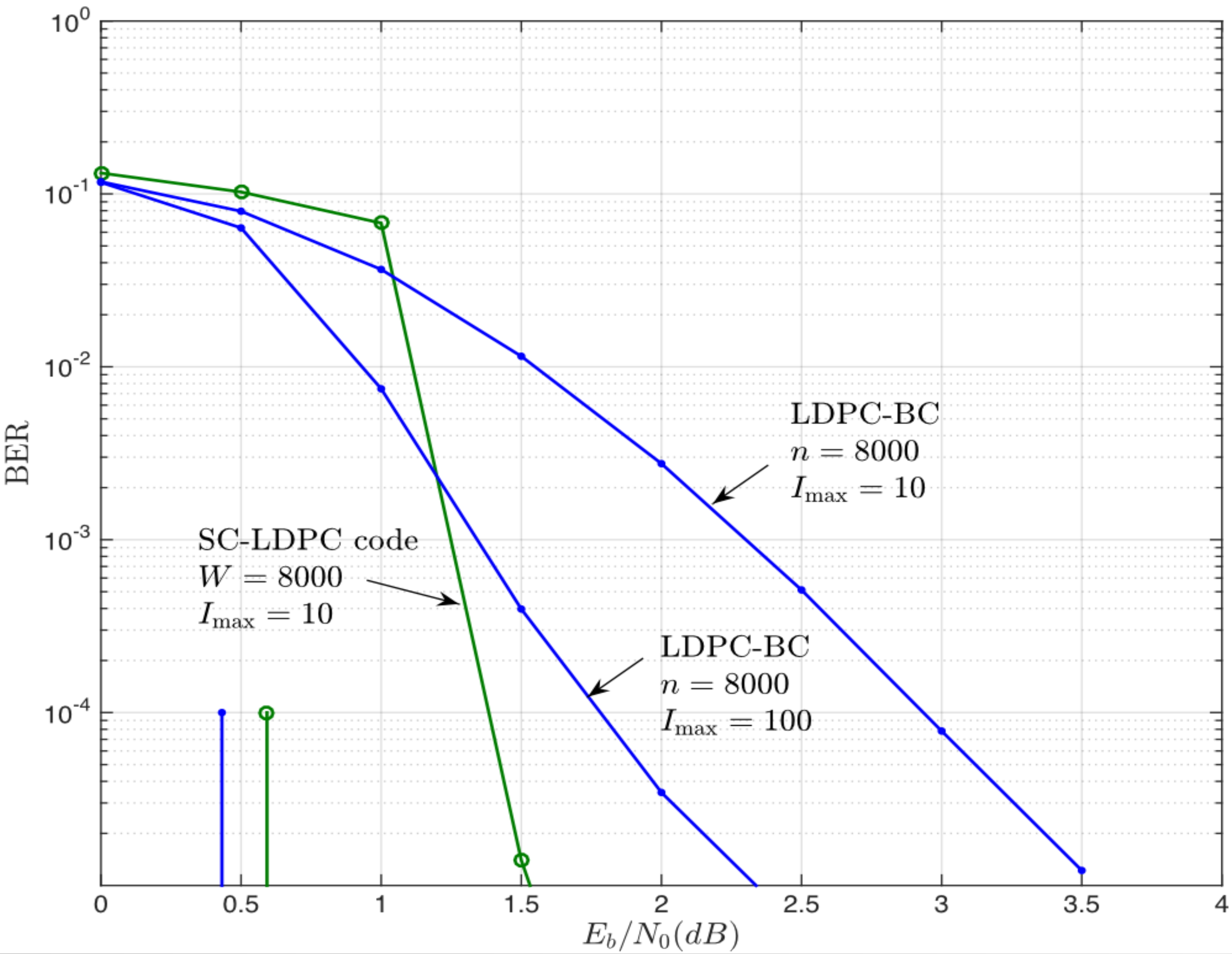

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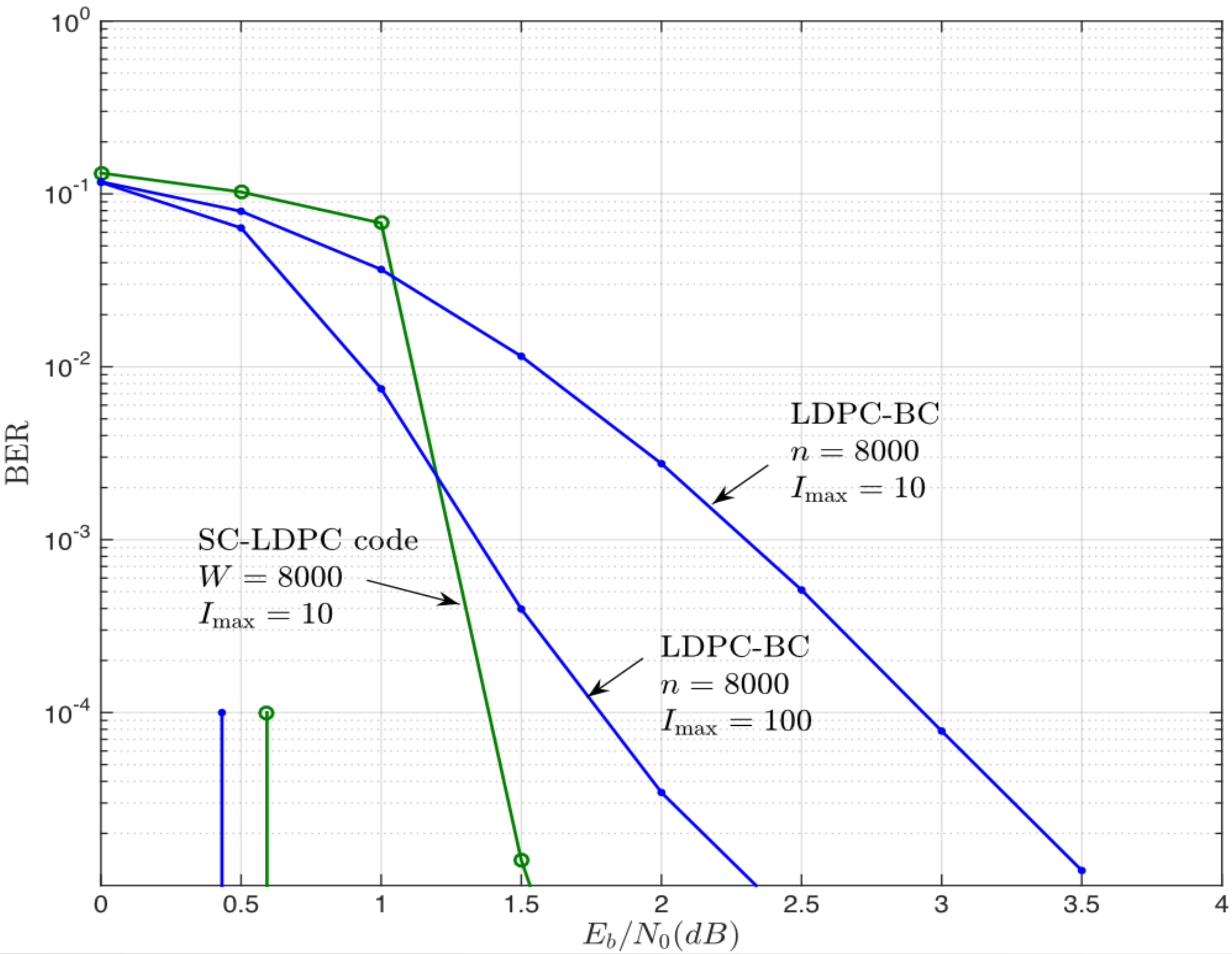
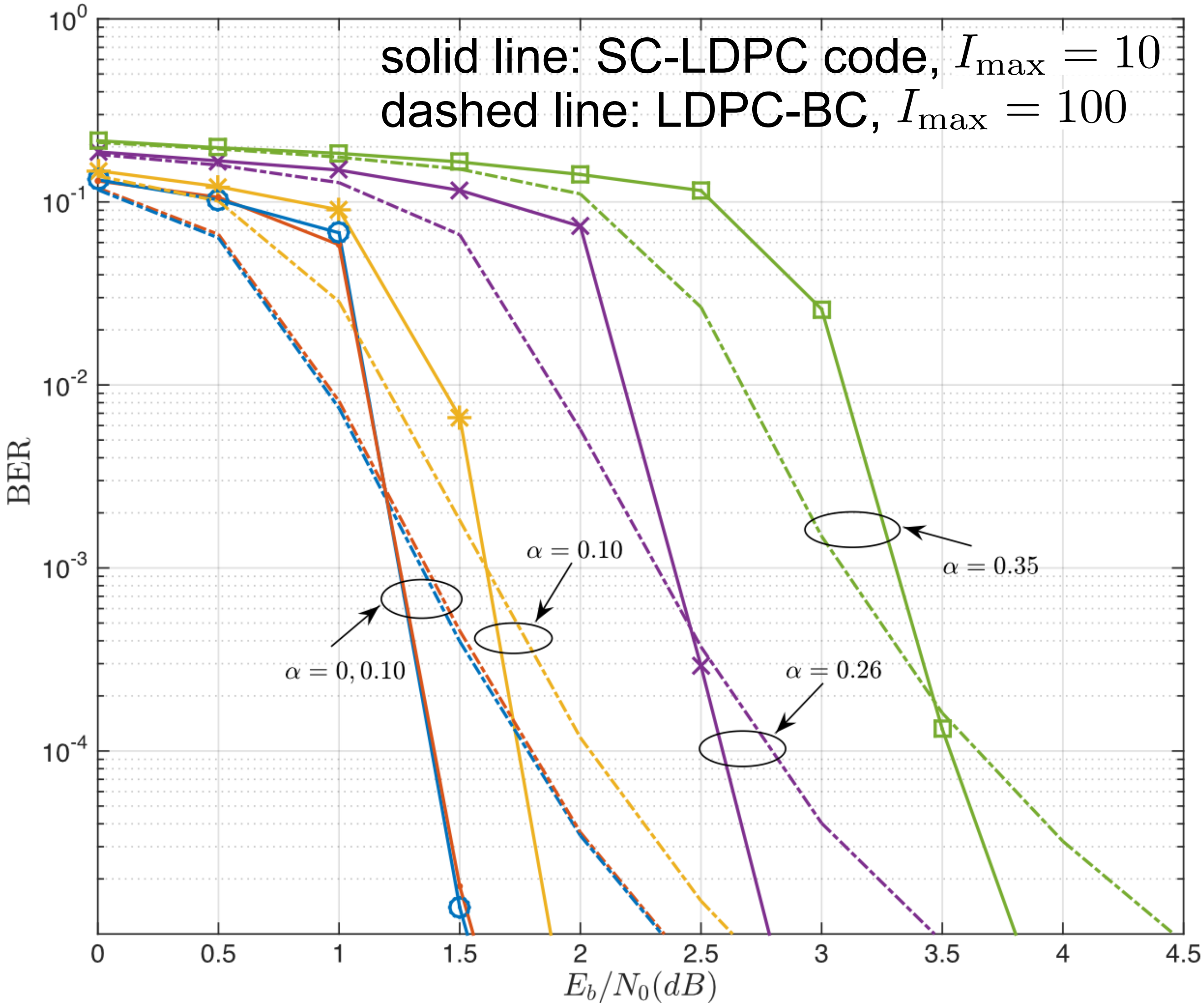


- On an equal latency basis, the regular SC-LDPC code outperforms the irregular LDPC-BC at BERs below 10^{-3}
- The asymptotically good regular SC-LDPC code shows no sign of an error floor
- The regular structure has implementation advantages

Randomly Punctured LDPC Codes



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Randomly Punctured LDPC Codes

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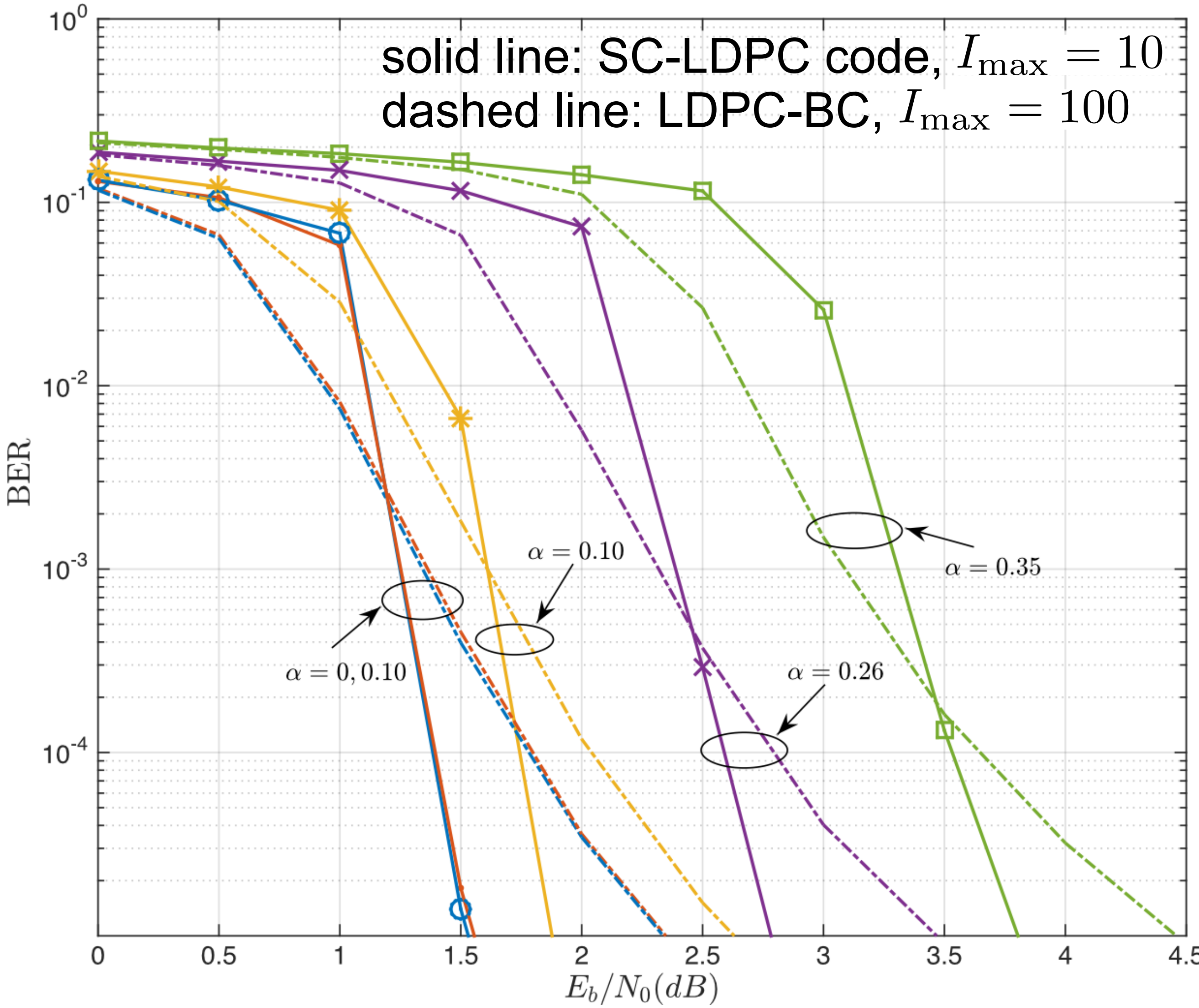

- Random puncturing can be applied to LDPC code ensembles to increase the rate
- Equal latency performance comparisons are consistent for higher rate ensembles

➡ Regular SC-LDPC codes display robust decoding performance compared to irregular LDPC-BCs

An Irregular Example

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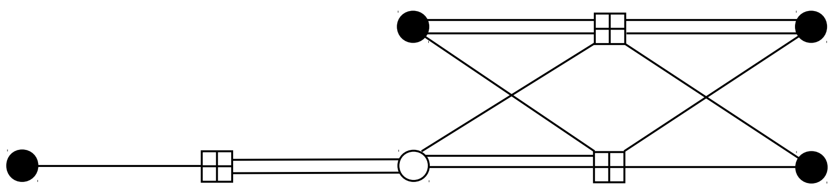
- Alternatively, we can couple irregular codes to construct an irregular SC-LDPC code ensemble. Consider the ARJA LDPC-BC protograph:



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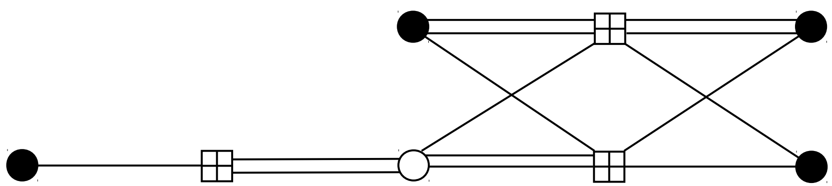

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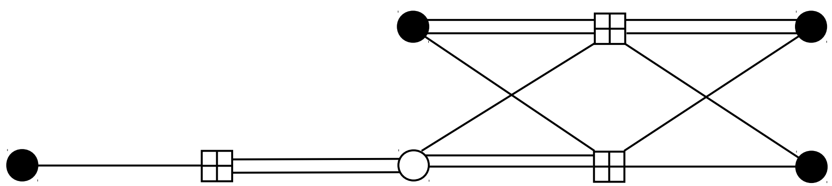

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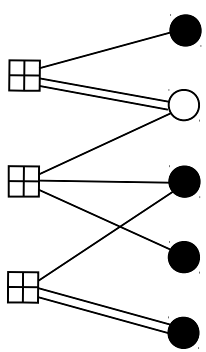

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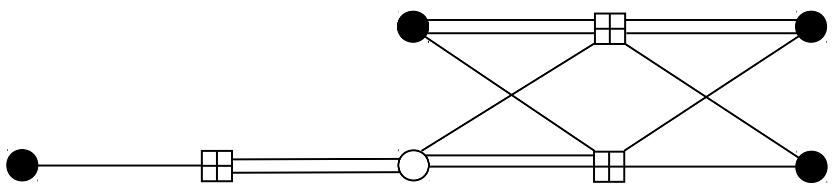

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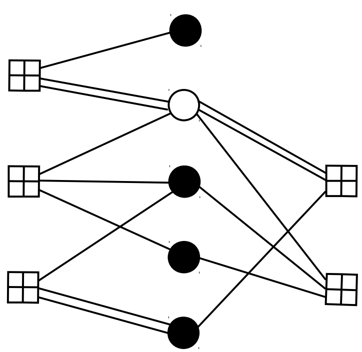

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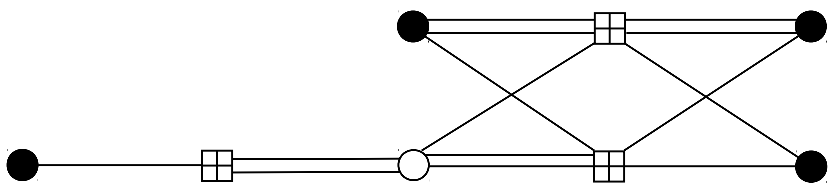

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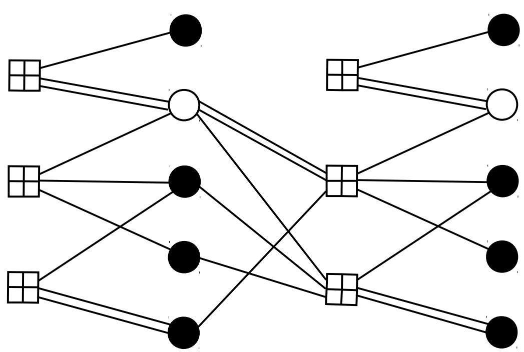

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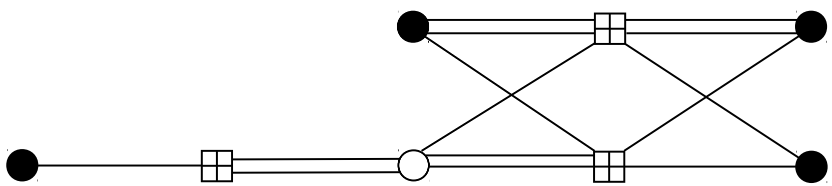

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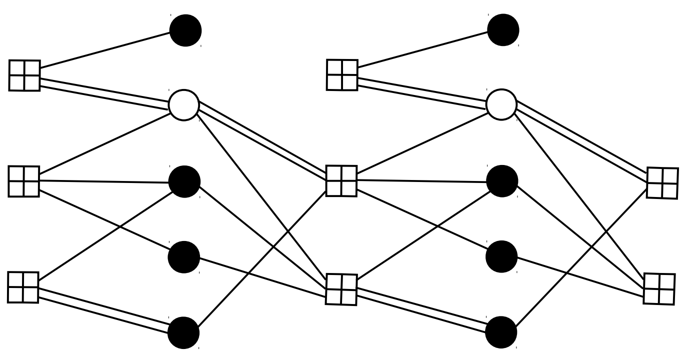

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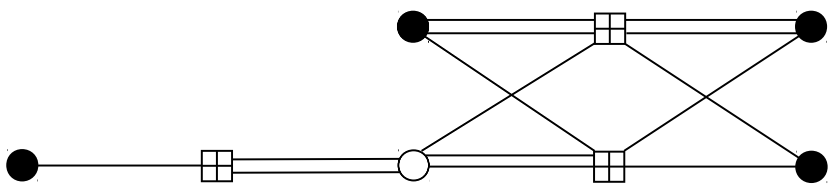

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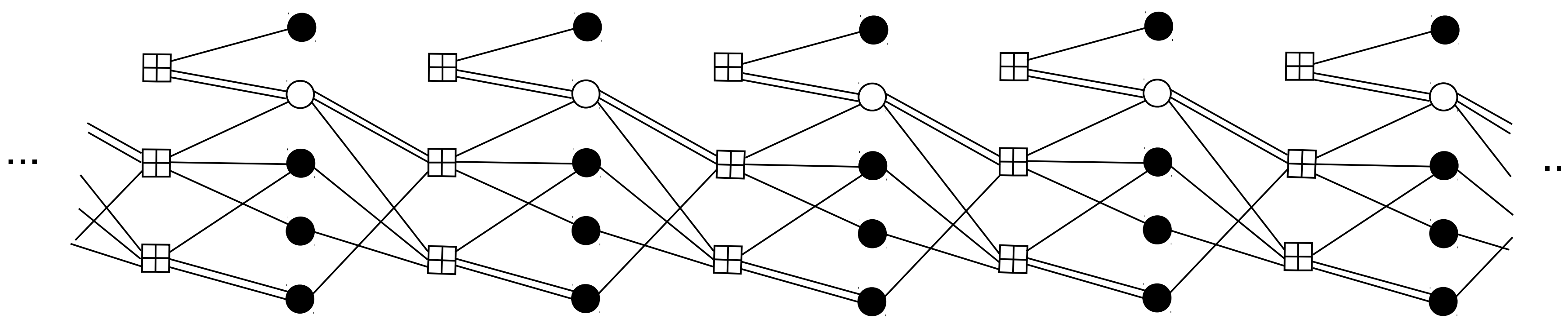
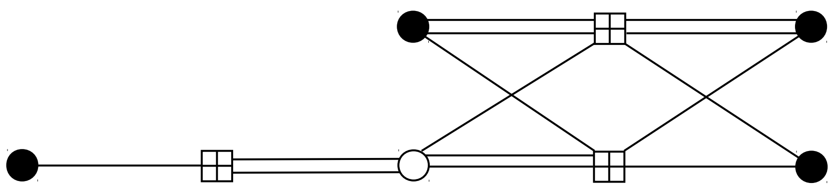


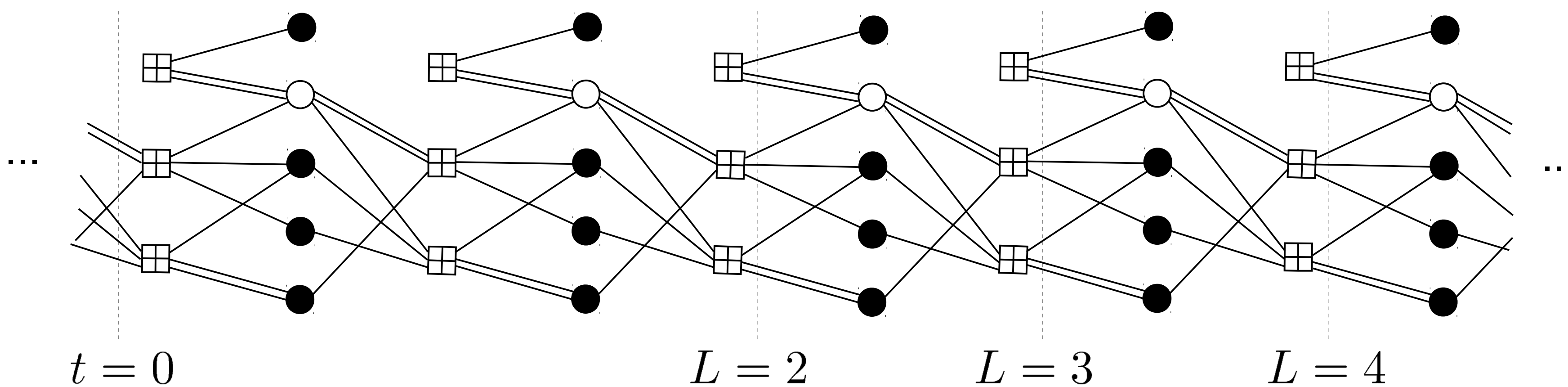
An Irregular Example

- Alternatively, we can couple irregular codes to construct an irregular SC-LDPC code ensemble. Consider the ARJA LDPC-BC protograph:



$$\mathbf{B} = \begin{array}{c} \begin{array}{ccccc} & \mathsf{x} & & & \end{array} \\ \left[\begin{array}{ccccc} 1 & 2 & 0 & 0 & 0 \\ 0 & 3 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & 2 \end{array} \right] \end{array}$$

- A spatially coupled version can be created by edge spreading:



AWGNC Thresholds vs Distance Growth

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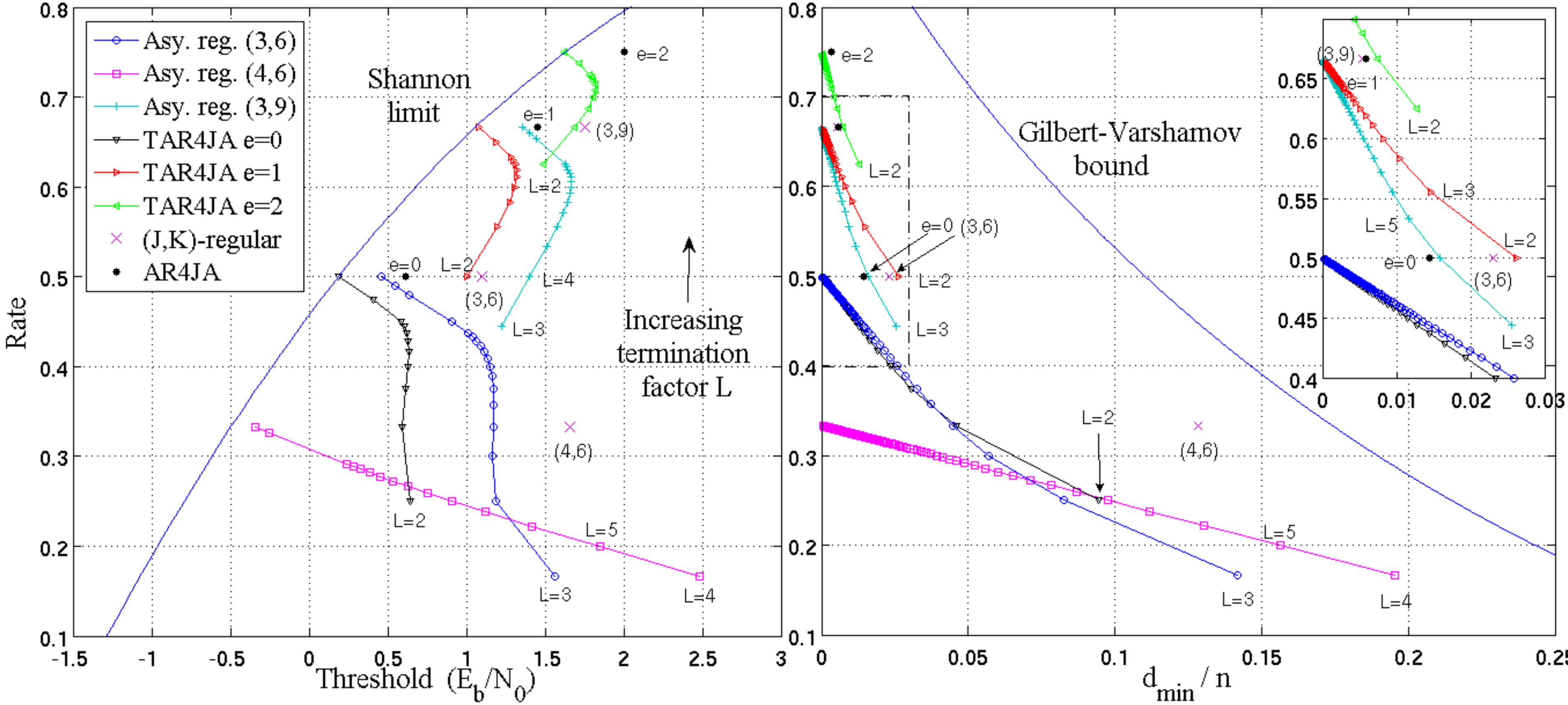
- Irregular SC-LDPC code ensembles also display excellent properties



[MLC10] D. G. M. Mitchell, M. Lentmaier and D. J. Costello, Jr., “AWGN Channel Analysis of Terminated LDPC Convolutional Codes”, *Proc. Information Theory and Applications Workshop*, San Diego, Feb. 2011.

Implementation Aspects

- As a result of their excellent performance and simple structure, regular SC-LDPC codes may be attractive for future coding standards. Several key features will require further investigation:
  - Hardware advantages of QC designs obtained by circulant liftings
  - Hardware advantages of the 'asymptotically-regular' structure
  - Design advantages of the flexible frame length feature obtained by varying L
  - Flexible rate feature obtained by puncturing

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  - ➡ Hardware advantages of the 'asymptotically-regular' structure
  - ➡ Design advantages of the flexible frame length feature obtained by varying L
  - ➡ Flexible rate feature obtained by puncturing
- Of particular importance for applications requiring extremely low decoded bit error rates (*e.g.*, optical communication, data storage) is an investigation of error floor issues related to **stopping sets**, **trapping sets**, and **absorbing sets**.

Conclusions

- Spatially coupled LDPC code ensembles achieve **threshold saturation**, i.e., their iterative decoding thresholds (for large L and M) approach the MAP decoding thresholds of the underlying LDPC block code ensembles.

- The threshold saturation and linear minimum distance growth properties of (J,K)-regular SC-LDPC codes combine the best asymptotic features of both regular and irregular LDPC-BCs.

- With window decoding, SC-LDPC codes also compare favorably to LDPC-BCs in the finite-length regime, providing flexible tradeoffs between BER performance, decoding latency, and decoder complexity.