Symbolic and Numerical Methods for Tensors and Representation Theory

(open) problems and exercises

November 18, 2014

Warm-up

Exercise 1

Consider the rational normal curve $\mathcal{C} \subset \mathbb{P}^4$, which is the image of the map

$$v_4 : \mathbb{P}^1 \to \mathbb{P}^4 \quad \text{given by} \quad [s:t] \mapsto [s^4 : s^3t : s^2t^2 : st^3 : t^4].$$

(a) Show that the ideal of equations vanishing on the curve $\mathcal{C}$ is generated by the 2×2 minors of any of the matrices H, H', where

$$H = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix}, \qquad H' = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix}.$$

(b) Show that the curve $\mathcal{C}$ is cut out (as an algebraic set) by the three equations

$$\det \begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \end{pmatrix} = \det \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & 0 \end{pmatrix} = \det \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & 0 \\ x_3 & x_4 & 0 & 0 \end{pmatrix} = 0$$

(c) Recall that the first secant variety of $\mathcal{C}$ is the (Zariski closure of the) union of all lines between two points on $\mathcal{C}$. Show that this is cut out by the determinant of the matrix H.

(d) Does every point on the first secant variety of $\mathcal{C}$ come from a unique pair of points on $\mathcal{C}$?

Exercise 2

Consider binary quartics

$$f = a_0s^4 + a_1s^3t + a_2s^2t^2 + a_3st^3 + a_4t^4 \quad \in \quad \mathbb{C}[s,t]_4.$$

(a) What is the variety of binary quartics $f \in \mathbb{C}[s,t]_4$ that can be written as fourth powers of linear forms $f = \ell^4$? What is its dimension, degree, and the generators of its ideal? How does it relate to the curve $\mathcal{C}$ in Exercise 1?

(b) What is the variety of quartics $f \in \mathbb{C}[s,t]_4$ that can be written as sums of two fourth powers of linear forms (or limits of such): $f = \ell_1(s,t)^4 + \ell_2(s,t)^4$ with $\ell_1, \ell_2 \in \mathbb{C}[s,t]_1$?

(c) Show that if $f = \ell_1(s,t)^4 \pm \ell_2(s,t)^4$ where $\ell_1, \ell_2 \in \mathbb{R}[s,t]_1$ are *real* linear forms, then f has at most two distinct real roots.

(d) Find a real quartic $f \in \mathbb{R}[s,t]_4$ that can be written $\ell_1^4 + \ell_2^4$ with $\ell_1, \ell_2 \in \mathbb{C}[s,t]_1$, but not as $\ell_1^4 \pm \ell_2^4$ with $\ell_1, \ell_2 \in \mathbb{R}[s,t]_1$.

Exercise 3

For $d = 3, 4, 5, 6$, find the equations defining the set of binary forms $f \in \mathbb{C}[s,t]_d$ that can be written as $l_1^{d-1} \cdot l_2$, where $l_1, l_2 \in \mathbb{C}[s,t]_1$ are linear forms. What are the minimal free resolutions (the syzygies) of the ideals generated by these equations?

Exercise 4

We let $S = \mathbb{C}[s,t]$ denote the usual ring of polynomials in two variables, and consider the ring $R = \mathbb{C}[\partial_s, \partial_t]$ of differential operators with constant coefficients. Given a binary form $f \in S_d$ we obtain a collection $\psi_0, \cdots, \psi_d$ of linear transformations,

$$\psi_i : R_i \to S_{d-i},$$

defined by $\psi_i(\alpha) = \alpha \cdot f$ (the effect of applying the differential operator α to f).

(a) Try to predict the ranks of the linear transformations ψ_i without writing down any matrices, when $f = s^3 t$ and $f = s^2 t^2$.

(b) Write down matrices representing the linear transformations ψ_i when $f = s^3 t$ and $f = s^2 t^2$. Try to find bases that make these matrices resemble the ones in Exercise 1, part (a)?

Tensor rank and tensor decomposition (Giorgio Ottaviani)

For the following problems, it will be useful to recall the Sylvester criterion for expressing binary forms as sums of powers of linear forms. Let $K = \mathbb{R}$ or $\mathbb{C}$, let $f(x,y) \in \mathrm{Sym}^d K^2$ be a binary form of degree d, and let $(a_i, b_i) \in \mathbb{P}^1_K$ be distinct points, $i = 1, \ldots, r$.

Sylvester's criterion:
There exist $\lambda_i \in K$ such that $f(x,y) = \sum_{i=1}^r \lambda_i (a_i x + b_i y)^d$ if and only if

$$\left[\prod_{i=1}^r (-b_i \partial_x + a_i \partial_y) \right] \cdot f = 0.$$

Problem 1

Consider the quintic binary form

$$f_a(x, y) = x^5 + 10ax^3y^2 + y^5.$$

(a) Find the complex (symmetric) rank of f_a, for any $a \in \mathbb{C}$. For which $a \in \mathbb{C}$ is the Waring decomposition of f_a (as a sum of 5-th powers of linear forms) unique?

(b) Find a Waring decomposition of f_a for $a = \frac{1}{2}$. (*Hint:* in order to solve numerically a cubic equation, compute the eigenvalues of the companion matrix, by the command *eigenvalues*).

(c) Find the real (symmetric) rank of f_a, for any $a \in \mathbb{R}$. Which are the cases when the real rank is strictly bigger than the complex rank?

Problem 2

Let A, B, C be vector spaces of dimension 2 over $K = \mathbb{R}$ or $\mathbb{C}$, with respective bases $\{a_0, a_1\}, \{b_0, b_1\}, \{c_0, c_1\}$. Consider the following tensor in $A \otimes B \otimes C$

$$S_t = a_0 \otimes b_0 \otimes c_0 + a_1 \otimes b_1 \otimes c_1 + t \cdot (a_0 + a_1) \otimes (b_0 + b_1) \otimes (c_0 + c_1).$$

(a) Find the complex rank of S_t for any $t \in \mathbb{C}$.

(b) Find a tensor decomposition of S_2.

(c) Find the real rank of S_t for any $t \in \mathbb{R}$.

(d) Note that there are isomorphisms $B \simeq A$, $C \simeq A$ such that, under these identifications, $S_t \in \mathrm{Sym}^3 A$. Is this property true for the *general* tensor $S \in A \otimes B \otimes C$? Is the property true for *every* tensor $S \in A \otimes B \otimes C$? What happens if A, B, C have dimension $n > 2$?

Symmetric bivariate tensors (Greg Blekherman)

The problems in this section are (most likely) open. Greg's opinion is however that it is reasonable to expect that progress can be made on many if not all of them. The following fact, explained in Greg's talk, may be useful:

Real bivariate forms of degree d have typical ranks between $\lfloor \frac{d+2}{2} \rfloor$ and d. Moreover, every rank between $\lfloor \frac{d+2}{2} \rfloor$ and d is typical.

Problem 1

Find the degrees of the three irreducible components of the algebraic boundary of the set of bivariate forms of degree d and rank $\lfloor \frac{d+2}{2} \rfloor$ (minimal typical rank). The three components will be described in Greg's talk.

Problem 2

Find an efficient algorithm to find the real rank of a bivariate form. More generally, find the rank and the minimal decomposition.

Problem 3

Describe the algebraic boundary of the set of bivariate forms of degree d and typical rank k. Find the number of irreducible components, as well as their degrees.

Problem 4 (Potentially harder)

Determine the maximal and/or minimal typical real rank for ternary quartics.

Schemes associated with eigenvectors (Hirotachi Abo)

Problem 1

Let $\{e_0, e_1\}$ be the standard basis for $\mathbb{C}^2$. Find the eigenvectors of the following tensors in $(\mathbb{C}^2)^{\otimes 3}$:

(a) $e_0 \otimes e_0 \otimes e_1 + e_1 \otimes e_0 \otimes e_0 + e_0 \otimes e_1 \otimes e_1 + e_1 \otimes e_0 \otimes e_1$

(b) $e_0 \otimes e_0 \otimes e_0 + e_0 \otimes e_1 \otimes e_1 + e_1 \otimes e_0 \otimes e_0 + e_1 \otimes e_0 \otimes e_1 + e_1 \otimes e_1 \otimes e_0 + e_1 \otimes e_1 \otimes e_1$

Write an approximation if necessary.

Problem 2

Write down all the 2×2 symmetric matrices with entries in $\mathbb{C}$ that are not diagonalizable.

Problem 3 (Sturmfels/Oeding)

Find all the eigenvectors of $x_1 x_2 \cdots x_n \in \mathbb{C}[x_1, \ldots, x_n]$.

Problem 4

Let A and B be $n \times n$ matrices with entries in $\mathbb{C}$. A non-zero vector $\boldsymbol{v} \in \mathbb{C}^n$ is called a *generalized eigenvector* of A and B if there is a non-zero $\lambda \in \mathbb{C}$ such that

$$A\boldsymbol{v} = \lambda B\boldsymbol{v}$$

Describe the locus of pairs (A, B) of 2×2 matrices with entries in $\mathbb{C}$ such that there are no two linearly independent generalized eigenvectors of A and B.

Problem 5

Let $d \in \mathbb{N}$ and let Z be a set of distinct $d^2 - d + 1$ points in the projective plane $\mathbb{P}^2$. Determine whether the following statement is true or false: if Z is the set of eigenvectors of some tensor $T \in (\mathbb{C}^3)^{\otimes d}$, then no $(d+1)$ points of Z are collinear.

Tensors in algebraic statistics (Elizabeth Gross)

Problem 1

The independence model for 3 binary random variables can be parameterized as follows:

$$\begin{array}{ll} p_{000} = \alpha_0\beta_0\gamma_0 & p_{010} = \alpha_0\beta_1\gamma_0 \\ p_{100} = \alpha_1\beta_0\gamma_0 & p_{110} = \alpha_1\beta_1\gamma_0 \\ p_{001} = \alpha_0\beta_0\gamma_1 & p_{011} = \alpha_0\beta_1\gamma_1 \\ p_{101} = \alpha_1\beta_0\gamma_1 & p_{111} = \alpha_1\beta_1\gamma_1 \end{array}$$

Use `Macaulay2` to find implicit equations for the model variety.

Problem 2

Let $\mathcal{P}$ be the independence model of 3 ternary random variables (i.e. $\overline{\mathcal{P}}$ = variety of all rank one $3 \times 3 \times 3$ tensors). Find the ideal of the mixture model $\mathrm{Mixt}^2(P)$.

Problem 3

Let T be your favorite 6-leaf trivalent tree and consider the general Markov model on T where each random variable has two possible states. Find the ideal of this model.

Parliaments of Polytopes and Toric Vector Bundles (Greg Smith)

Problem 1

Describe the parliament of polytopes for the tangent bundle $\mathcal{T}_X$ on $X := \mathbb{P}^1 \times \mathbb{P}^1$. Show that this toric vector bundle is nef and globally generated, but not ample.

Problem 2

Describe the parliament of polytopes for the tangent bundle $\mathcal{T}_Y$ on the first Hirzebruch surface $Y := \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)\big)$. Show that this toric vector bundle is neither nef nor globally generated.

Problem 3

Let $\mathbf{v}_1 := (1,0)$, $\mathbf{v}_2 := (0,1)$, $\mathbf{v}_3 := (-1,-1)$ be the primitive lattice points on the rays in the fan associated to $\mathbb{P}^2$, and let $\sigma_1 := \mathrm{pos}(\mathbf{v}_2, \mathbf{v}_3)$, $\sigma_2 := \mathrm{pos}(\mathbf{v}_1, \mathbf{v}_3)$, $\sigma_3 := \mathrm{pos}(\mathbf{v}_1, \mathbf{v}_2)$ be the maximal cones in the same fan. If $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ denotes the standard basis of $E := \mathbb{C}^3$, then consider the toric vector bundle $\mathcal{E}$ defined by the following decreasing filtrations:

$$E^{\mathbf{v}_1}(j) = \begin{cases} E & \text{if} \quad j \leq -1 \\ \operatorname{Span}(\mathbf{e}_1, \mathbf{e}_2) & \text{if} \quad -1 < j \leq 0 \\ \operatorname{Span}(\mathbf{e}_1) & \text{if} \quad 0 < j \leq 4 \\ 0 & \text{if} \quad 4 < j \end{cases}$$

$$E^{\mathbf{v}_2}(j) = \begin{cases} E & \text{if} \quad j \leq -2 \\ \operatorname{Span}(\mathbf{e}_2, \mathbf{e}_3) & \text{if} \quad -2 < j \leq 0 \\ \operatorname{Span}(\mathbf{e}_3) & \text{if} \quad 0 < j \leq 3 \\ 0 & \text{if} \quad 3 < j \end{cases}$$

$$E^{\mathbf{v}_3}(j) = \begin{cases} E & \text{if} \quad j \leq -1 \\ \operatorname{Span}(\mathbf{e}_3 - \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_2) & \text{if} \quad -1 < j \leq 2 \\ \operatorname{Span}(\mathbf{e}_1 - \mathbf{e}_2) & \text{if} \quad 2 < j \leq 3 \\ 0 & \text{if} \quad 3 < j \ . \end{cases}$$

Describe the parliament of polytopes for $\mathcal{E}$. Determine the smallest $m \in \mathbb{N}$ such that $\operatorname{Sym}^m(\mathcal{E})$ is globally generated.

Hint: For Problems 1 and 2, compare with Example 2.7 in [DJS]. For Problem 3, compare with Example 2.9 in [DJS].

[DJS] Sandra Di Rocco, Kelly Jabbusch, and Gregory G. Smith, Toric vector bundles and parliaments of polytopes, available at `arXiv:1409.3109v1 [math.AG]`.

Tensors, secant varieties and interpolation (Elisa Postinghel)

Problem 1 (Hilbert functions of ideals of fat points)

(a) Use `Macaulay2` to construct the ideal I of s random points in the projective space $\mathbb{P}^2$ over the rational numbers $\mathbb{Q}$. Compute the Hilbert function of I, in particular verify that $\deg(I) = s$.

(b) Fix integers $m_1, \dots, m_s$ and compute the Hilbert function of the ideal of s random fat points with multiplicities $m_1, \dots, m_s$ respectively.

(You may choose your favourite $s, m_1, \dots, m_s$, but make sure they're not too large!)

Problem 2 (Polynomial interpolation problems)

If $f(x, y, z)$ is a homogeneous polynomial of degree d in three variables, we say that f interpolates a point $q \in \mathbb{P}^2$ with multiplicity m if all partial derivatives of f of order $m - 1$ vanish at q.

(a) For a fixed fat point q of multiplicity m, what is the dimension of the space of all homogeneous polynomials interpolating q with multiplicity m?

(b) Denote by $\mathcal{L}_d(m_1, \ldots, m_s)$ the vector space of all degree d homogeneous polynomials interpolating distinct points $q_1, \ldots, q_s \in \mathbb{P}^2$ with multiplicities $m_1, \ldots, m_s$. What is its expected dimension?

(c) Compute with the help of `Macaulay2` the dimension of the following spaces: $\mathcal{L}_4(2)$, $\mathcal{L}_4(2,2)$, $\mathcal{L}_4(2,2,2)$, $\mathcal{L}_4(2,2,2,2)$, $\mathcal{L}_4(2,2,2,2,2)$, and $\mathcal{L}_6(4,4)$, $\mathcal{L}_6(4,4,4)$. Compare the result with the expected dimension.

You may want to work over a finite field (such as $\mathbb{F}_{32003}$).

Problem 3 (Secant varieties to Veronese, Segre and Segre-Veronese varieties)

Terracini's Lemma is a classical result due to the Italian school of Algebraic Geometry of the beginning of last century. One of its implications is the following:

Lemma. The codimension of the s-secant variety to the d-th Veronese embedding of $\mathbb{P}^2$ equals the dimension of $\mathcal{L}_d(2, \ldots, 2)$ for s general double points.

It follows that in fact in the previous problem you computed dimensions of secant varieties to Veronese embeddings of $\mathbb{P}^2$!

(a) Write a program with `Macaulay2` to compute the dimension of the s-secant varieties to the *Segre embedding* of $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ in $\mathbb{P}^{26}$. Compare the result with the corresponding expected dimensions.

(b) Write a program with `Macaulay2` to compute the dimension of the s-secant varieties to the *Segre-Veronese embedding* of $\mathbb{P}^2 \times \mathbb{P}^2$ of order $(2,2)$ in $\mathbb{P}^{35}$, namely the composition of the Segre embedding with the 2nd Veronese embedding of two copies of $\mathbb{P}^2$. Compare the result with the expected dimension.