Algebraic Fitness

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1. How can you tell whether a given 4×4-matrix can be written in the form $A_1 \otimes B_1 + A_2 \otimes B_2 + A_3 \otimes B_3$, where A_i and B_j are 2×2-matrices?

2. Compute the *secant variety* of the curve $\{(s^{10} : s^7t^3 : s^5t^5 : s^4t^6 : t^{10})\}$ in projective 4-space. How many terms does its defining polynomial have?

3. Find the image of the unit ball $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ under the map $\mathbb{R}^3 \to \mathbb{R}^2$, $(x, y, z) \mapsto (\, xy + xz + yz \,,\, xyz \,)$. Boundary curve?

4. True or false: every 4×4-matrix with entries in the non-zero complex numbers is the *Hadamard product* of two 4×4-matrices of rank ≤ 2? What happens when we restrict to the space of 4×4 *Toeplitz matrices*?

5. Multiplication of two $2{\times}2$-matrices is represented by a tensor of format $4{\times}4{\times}4$. Compute all *eigenvectors* of that tensor. Draw this set in $\mathbb{P}^3$.

6. How many *facets* can a simplicial 4-polytope with 8 vertices have ?

7. Consider the $2{\times}2{\times}2$-tensor with entries $u_{ijk} = (i{+}j)(i{+}k{+}1)(j{+}k{+}2)$ for $1 \leq i, j, k \leq 2$. Determine its complex rank, real rank, *nonnegative rank*, eigenvalues, eigenvectors, singular values, and singular vectors.

8. Determine the (piecewise quadratic) formula for the number of nonnegative integer 2×3-matrices with row sums $r = (r_1, r_2)$ and $c = (c_1, c_2, c_3)$. Fix four pairs (r, c) and draw the corresponding *parliament*.

9. Let $f = (x_{11}x_{22} - x_{12}x_{21})^{-2}$ and Df be the left module it generates over the *Weyl algebra* $D = \mathbb{C}\langle x_{11}, x_{12}, x_{21}, x_{22}, \partial_{11}, \partial_{12}, \partial_{21}, \partial_{22}\rangle$. Find a left ideal $I \subset D$ such that $Df \simeq D/I$. Why is this D-module *holonomic*?

10. Statisticians like to compute *p-values*. Why? Can they use `Macaulay2`?

11. The *cotangent bundle* of $\mathbb{P}^2$ is a sheaf on $\mathbb{P}^2$. Write this sheaf as the *sheafification* of an explicit graded $\mathbb{C}[x, y, z]$-module M. Is M unique? Next, consider the analogous question for $\mathbb{P}^1 \times \mathbb{P}^1$. Other toric surfaces?

12. Examine the $3 \times 3 \times 3$-tensors $x = (x_{ijk})$ that admit a decomposition

$$x_{ijk} \;=\; \sum_{r=1}^{2}\sum_{s=1}^{2}\sum_{t=1}^{2} a_{ri}b_{rj}\, c_{sj}d_{sk}\, e_{tk}f_{ti} \qquad\qquad (1 \leq i, j, k \leq 3)$$

where a, b, c, d, e, f are 2×3-matrices of parameters. What is the dimension of this variety? Meaning for *statistics* or *physics*? Equations?

13. What is the dimension of the space of polynomials of degree 16 in the entries of a $4 \times 4 \times 4$-tensor that are $\mathrm{SL}(4) \times \mathrm{SL}(4) \times \mathrm{SL}(4)$-*invariant*?

14. Consider the homogeneous radical ideal $I \subset \mathbb{Q}[x, y, z]$ that defines the six points (1:0:0), (0:1:0), (0:0:1), (1:1:1), (1:2:3), (1:4:9) in the plane $\mathbb{P}^2$. Find a 3×4-matrix of linear forms whose maximal minors generate I. Construct the corresponding *cubic surface* in $\mathbb{P}^3$, and list its 27 lines.

15. The set of $2{\times}2{\times}2{\times}2{\times}2$-tensors of rank ≤ 5 defines a variety in $\mathbb{P}^{31}$. What can you say about dimension, degree, equations, singularities, and the fibers of the natural parametrization (= *space of explanations*)?

16. Locate the *character table* of the symmetric group S_6. For any three rows of this $11{\times}11$ table, compute the weighted average of the entries in their Hadamard product. For how many triples do we get 2 or more?

17. Define a notion of nonnegative rank for *Hankel matrices*. Which $3{\times}3$-Hankel matrices have nonnegative rank 2? How about $4{\times}4$ and rank 3?

18. Write C_{d,e_1,e_2,e_3,e_4} for the dimension of the space of homogeneous polynomials of degree d in x, y, z that vanish to orders e_1, e_2, e_3, e_4 at four general points in $\mathbb{P}^2$. Show that $\sum_{d,e} C_{d,e_1,e_2,e_3,e_4} x_0^d x_1^{e_1} x_2^{e_2} x_3^{e_3} x_4^{e_4}$ is a *rational generating function*. Compute this rational function explicitly.

19. There are two trivalent trees on six leaves: *snowflake* and *caterpillar*. Their *phylogenetic varieties* in $\mathbb{C}^{2\times2\times2\times2\times2\times2}$ are defined by the three split flattenings having rank ≤ 2. What are their dimensions? Degrees?

20. For a matrix $A \in \mathbb{C}^{3\times3}$, consider the ideal of $2{\times}2$-minors of the $3{\times}2$-matrix $[x \mid Ax]$, where $x = (x_1, x_2, x_3)^T$. For which A is this not radical?