Kronecker coefficients – computation and bounds

Greta Panova

University of Pennsylvania

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The Kronecker coefficients overview

Irreducible representations of the symmetric group S_n:

(group homomorphisms $S_n \to GL_N(\mathbb{C})$)

— the **Specht modules** $\mathbb{S}_\lambda$, indexed by partitions $\lambda \vdash n$

Tensor product decomposition:

$$\mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_{\nu \vdash n} \mathbb{S}_\nu^{\oplus ??}$$

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$$\mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_{\nu \vdash n} \mathbb{S}_\nu^{\oplus g(\lambda,\mu,\nu)}$$

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$$g(\lambda, \mu, \nu) = \dim \operatorname{Hom}_{S_n}(\mathbb{S}_\nu, \mathbb{S}_\lambda \otimes \mathbb{S}_\mu)$$

In terms of $GL(\mathbb{C}^m)$ modules V_λ, V_μ and $GL(\mathbb{C}^{m^2})$ module V_ν:

$$g(\lambda, \mu, \nu) = \dim \operatorname{Hom}_{GL(\mathbb{C}^m)\times GL(\mathbb{C}^m)}(V_\lambda \otimes V_\mu, V_\nu)$$

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Littlewood-Richardson coefficients $c^\lambda_{\mu\nu}$: Tensor products of GL_N representations:

$$V_\mu \otimes V_\nu = \oplus_{\lambda \vdash |\mu|+|\nu|} V_\lambda^{\oplus c^\lambda_{\mu\nu}}$$

The combinatorial problem

Problem (Murnaghan, 1938)

Find a positive combinatorial interpretation for $g(\lambda,\mu,\nu)$, i.e. a family of combinatorial objects $\mathcal{O}_{\lambda,\mu,\nu}$, s.t. $g(\lambda,\mu,\nu) = \#\mathcal{O}_{\lambda,\mu,\nu}$.

Motivation: Littlewood–Richardson
$c^{\lambda}_{\mu,\nu}$, $\mathcal{O}_{\lambda,\mu,\nu} = \{$ LR tableaux of shape λ/μ, type $\nu\}$
$\lambda = (6,4,3), \mu = (3,1), \nu = (4,3,2)$:

$$\begin{array}{cccccc} & & & 1 & 1 & 1 \\ & 1 & 2 & 2 & & \\ 2 & 3 & 3 & & & \end{array} \qquad \begin{array}{cccccc} & & & 1 & 1 & 1 \\ & 2 & 2 & 2 & & \\ 1 & 3 & 3 & & & \end{array} \Rightarrow c^{\lambda}_{\mu\nu} = 2.$$

Theorem (Murnaghan)

If $|\nu| + |\mu| = |\lambda|$ and $n > |\nu|$, then

$$g((n+|\mu|,\nu),(n+|\nu|,\mu),(n,\lambda)) = c^{\lambda}_{\mu\nu}.$$

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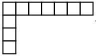
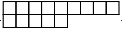
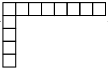
Results since then:

Combinatorial formulas for $g(\lambda, \mu, \nu)$, when:

- μ and ν are hooks (), [*Remmel, 1989*]
- $\nu = (n-k, k)$ () and $\lambda_1 \geq 2k-1$, [*Ballantine–Orellana, 2006*]
- $\nu = (n-k, k)$, $\lambda = (n-r, r)$ [*Remmel–Whitehead*, 1994; *Blasiak–Mulmuley–Sohoni*,2013]
- $\nu = (n-k, 1^k)$ (), [*Blasiak, 2012*]

Kronecker coefficients and GCT

Input: Integers N, ℓ, partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $\mu = (\mu_1, \ldots, \mu_\ell)$, $\nu = (\nu_1, \ldots, \nu_\ell)$, where $0 \leq \lambda_i, \mu_i, \nu_i \leq N$, and $|\lambda| = |\mu| = |\nu|$.
Size(Input)=$O(\ell \log N)$.

Positivity of Kronecker coefficients (KP):

Decide: whether $g(\lambda, \mu, \nu) > 0$

Kronecker coefficients (Kron):

Compute: $g(\lambda, \mu, \nu)$.

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Conjecture (Mulmuley)

KP *is in* P .

KRONECKER COEFFICIENTS (KRON):

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When the input is binary, KRON *is in* #P .

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Theorem:[Littlewood-Richardson rule/Narayanan + Knutson-Tao]
Conjectures hold for the Littlewood-Richardson coefficients
$c^{\bar\lambda}_{\bar\mu\bar\nu} = g(\lambda, \mu, \nu)$, where $\lambda = (n - |\bar\lambda|, \bar\lambda)$, etc, and $|\bar\lambda| = |\bar\mu| + |\bar\nu|$.

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Theorem[Bürgisser-Ikenmeyer]. KRON is in GapP .
(GapP $= \{F : F = F_1 - F_2, F_1, F_2 \in$ #P$\}$)
Theorem[Narayanan]. KRON is #P –hard.

Bounds on the Kronecker coefficients

A GCT approach to determinant vs permanent:
Find obstruction candidates:
a polynomial $P \in S^d\left(S^n\mathbb{C}^{n^2}\right)$, s.t. $P(\overline{GL_{n^2}\det_n}) = 0$ and $P(\ell^{n-m} perm_m) \neq 0$
(after Bürgisser, Ikenmeyer, Landsberg...)

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Plethysm:
Notation: W – $\mathbb{C}$ vector space. $S^\alpha W$ – irreducible $GL(W)$ module of highest weight α.
Let $|\lambda| = |\mu| * |\nu|$.
The *plethystic coefficients* are defined as (here $V = \mathbb{C}^k$, $k \geq \ell(\lambda)$)

$$a^\lambda_{\mu,\nu} = \dim \mathrm{Hom}_{GL(V)}\left(S^\mu(S^\nu V), S^\lambda V\right)$$

Problem

Find λ, such that $sg(\lambda, (n^d)) < a^\lambda_{(d),(n)}$?

(here $sg(\lambda, \mu) = \dim \mathrm{Hom}_{S_{|\lambda|}}(\mathbb{S}^\lambda, S^2(\mathbb{S}^\mu)) \leq g(\lambda, \mu, \mu)$)
Recipe for P: look for $P \in S^\lambda V$, for λ found in Problem.

Representations of S_n

$\lambda \vdash n$, T – SYT of shape λ. $\sigma \in S_n$ acts on T: $i \to \sigma(i)$
$R(T) := \{\sigma \in S_n \ : \ row_T(\sigma(i)) = row_T(i)\} \simeq S_{\lambda_1} \times S_{\lambda_2} \times \cdots$
– **Young subgroup**.
$C(T) := \{\sigma \in S_n \ : \ col_T(\sigma(i)) = col_T(i)\}$

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 5 \\ \hline 3 & 7 & 8 \\ \cline{1-3} 2 & 9 \\ \cline{1-2}\end{array}\ , \quad R(T) = S_{1,4,5,6} \times S_{3,7,8} \times S_{2,9}$$

Tabloid $\{T\} = \begin{array}{c}\overline{1\ 4\ 6\ 5} \\ \overline{3\ 7\ 8} \\ \overline{2\ 9} \end{array}$ (equiv class: $T \sim T'$ for $R(T) = R(T')$)

$$a_T = \sum_{\sigma \in R(T)} \sigma, \qquad b_T = \sum_{\pi \in C(T)} \mathrm{sgn}(\pi)\pi \in \mathbb{C}[S_n]$$

$c_T := a_T b_T$: **Young symmetrizer**

Young modules $M^\lambda := Span_{\mathbb{C}}\langle \{T\} : sh(T) = \lambda\rangle$, a (left) $\mathbb{C}[S_n]$ module.

Specht modules

$\mathbb{S}^\lambda := Span_{\mathbb{C}}\langle b_T\{T\} \ : \ sh(T) = \lambda\rangle$ – $\mathbb{C}[S_n]$-module.

Theorem

The **Specht modules** $\mathbb{S}^\lambda$ *for* $\lambda \vdash n$ *form a complete set of irreducible representations of* S_n*. For each* λ*, the set*

$$\left\{ \underbrace{\left(\sum_{\pi \in C(T)} \mathrm{sgn}(\pi)\pi \right)}_{b_T} \{T\} \ : \ T - SYT \text{ of shape } \lambda \right\}$$

forms a basis for $\mathbb{S}^\lambda$*.*

Young's rule:

$$M^\mu = \oplus_{\lambda \vdash n} \underbrace{K_{\lambda\mu}}_{\text{Kostka number}} \mathbb{S}^\lambda$$

Characters of S_n

characters: Trace $M_{\mathbb{S}^\lambda}(w) = \chi^\lambda[w] : S_n \to \mathbb{C}$

$\chi^\lambda[\alpha]$ – character value at any permutation of cycle type $\alpha = (\alpha_1, \alpha_2, \ldots)$

$$g(\lambda, \mu, \nu) = \langle \chi^\lambda \otimes \chi^\mu, \chi^\nu \rangle$$

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Murnaghan–Nakayama rule:

$$\chi^\lambda[\alpha] = \sum_{T \,:\, \text{MN tableaux, shape } \lambda, \text{ content } \alpha} (-1)^{ht(T)}$$

1	1	1	3	3	4	4
1	2	2	3	4	4	
2	2	3	3	4		

— a M-N tableau T of shape $\lambda = (7, 6, 5)$, content $\alpha = (4, 4, 5, 5)$, $ht(T) = (2-1)+(2-1)+(3-1)+(3-1) = 6$.

Examples: $\chi^{(n)}[\alpha] = 1$, $\chi^{(1^n)}[\alpha] = (-1)^{n-\ell(\alpha)}$

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A formula: $g(\lambda, \mu, \nu) = \dfrac{1}{n!} \sum_{w \in S_n} \chi^\lambda[w]\chi^\mu[w]\chi^\nu[w].$

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Another "formula":

$$g(\lambda, \mu, \nu) = \dim \mathrm{Hom}(\mathbb{1}, \mathbb{S}^\lambda \otimes \mathbb{S}^\mu \otimes \mathbb{S}^\nu) = \dim \left(\mathbb{S}^\lambda \otimes \mathbb{S}^\mu \otimes \mathbb{S}^\nu\right)^{S_n}$$

The Schur-Weyl duality

Let V^λ be the irreducible representations of $GL(V)$.
S_n and $GL(V)$ act on $V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n}$, where

$$\sigma(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}.$$

$$A(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = Av_1 \otimes Av_2 \otimes \cdots \otimes Av_n, \quad A \in GL(V).$$

Theorem (Schur-Weyl duality)

Under the joint action of the groups S_n and $GL(V)$, the tensor space decomposes into a direct sum of tensor products of irreducible modules for these two groups that determine each other:

$$V \otimes V \otimes \cdots \otimes V = \sum_{\lambda \vdash n} \mathbb{S}^\lambda \otimes V^\lambda.$$

Representations of $GL(V)$

Let $V = \mathbb{C}^k$, then the irreducible rational representations of $GL(V)$ are V^λ, where $\ell(\lambda) \leq k$.
Tensor products of $GL(V)$ representations:

$$V_\mu \otimes V_\nu = \oplus_{\lambda \vdash |\mu|+|\nu|} V_\lambda^{\oplus c^\lambda_{\mu\nu}}$$

$c^\lambda_{\mu\nu}$ – **Littlewood-Richardson coefficients**
The LR rule: $c^\lambda_{\mu,\nu} = \#\{$ LR tableaux of shape λ/μ, type $\nu\}$
$\lambda = (6,4,3), \mu = (3,1), \nu = (4,3,2)$:

			1	1	1
	1	2	2		
2	3	3			

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$\Rightarrow c^\lambda_{\mu\nu} = 2.$

Symmetric functions:

$\Lambda(x) = \{f \in \mathbb{Q}[[x_1, x_2, \ldots]]\ ,\ f(x_1, x_2, \ldots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots)$ for all permutations $\sigma\}$
(class functions on $GL_{\ldots}$)
Schur functions $s_\lambda(x)$ – characters of V^λ:

$$s_\lambda(x_1, \ldots, x_n) = \mathrm{Trace} M_{V^\lambda}(A)\ ,\quad \text{where } A = P.\ \mathrm{diag}(x_1, \ldots, x_n).P^{-1}$$

– (orthonormal basis) for Λ: $\langle s_\lambda, s_\mu \rangle = \langle V^\lambda, V^\mu \rangle = \delta_{\lambda\mu}$
Weyl character formula:

$$s_\lambda(x_1, \ldots, x_n) = \frac{\det \left[x_i^{\lambda_j + n - j} \right]_{i,j=1}^n}{\det \Delta(x_1, \ldots, x_n)}$$

Combinatorial: **Semi-Standard Young tableaux** of shape λ :

$$s_{(2,2)}(x_1, x_2, x_3) = s_{\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline\end{array}}(x_1, x_2, x_3) =$$

$$= \underset{\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline\end{array}}{x_1^2 x_2^2} + \underset{\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & 3 \\ \hline\end{array}}{x_1^2 x_3^2} + \underset{\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 3 \\ \hline\end{array}}{x_2^2 x_3^2} + \underset{\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline\end{array}}{x_1^2 x_2 x_3} + \underset{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline\end{array}}{x_1 x_2^2 x_3} + \underset{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 3 \\ \hline\end{array}}{x_1 x_2 x_3^2}.$$

Schur functions in action

generalized Cauchy:

$$\sum_{\lambda,\mu,\nu} g(\lambda,\mu,\nu) s_\lambda(x) s_\mu(y) s_\nu(z) = \prod_{i,j,k} \frac{1}{1 - x_i y_j z_k}$$

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Classic Cauchy:

$$\sum_{\alpha} s_\alpha(u) s_\alpha(v) = \prod_{i,j} \frac{1}{1 - u_i v_j}$$

“plethystic” version: coefficients in the expansion into Schur functions in x and y:

$$g(\lambda,\mu,\nu) = [s_\lambda(x) s_\mu(y)] s_\nu(xy) \ \text{ where } xy = (x_1y_1, x_1y_2, \ldots, x_2y_1, x_2y_2, \ldots)$$

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Kronecker product $*$ on Λ, defined as:

$$\lambda, \mu \vdash n : \qquad s_\lambda * s_\mu = \sum_{\nu \vdash n} g(\lambda,\mu,\nu) s_\nu$$

$$\langle f(xy), g(x)h(y) \rangle = \langle f, g * h \rangle$$

Formulas for Kronecker coefficients I

Special cases:

$$g(\lambda,\mu,(n)) = \delta_{\lambda,\mu} \qquad g(\lambda,\mu,(1^n)) = \delta_{\lambda\mu'}$$

$$\Longrightarrow h_n(xy) = \sum_{\alpha\vdash n} s_\alpha(x)s_\alpha(y); \qquad e_n(xy) = \sum_{\alpha\vdash n} s_\alpha(x)s_{\alpha'}(y) \qquad (\diamond)$$

Via (multi-) LR coefficients: [1] $(k=\ell(\nu))$

$$s_\nu(xy) = \sum_{\sigma\in S_k} \operatorname{sgn}(\sigma)\prod_i h_{\nu_i - i+\sigma_i}(xy) = ..(\diamond)..$$

$$\Longrightarrow g(\lambda,\mu,\nu) = \sum_{\sigma\in S_k}\operatorname{sgn}(\sigma) \underbrace{\sum_{\alpha^i\vdash \nu_i - i+\sigma_i, i=1\ldots k} c^{\lambda}_{\alpha^1,\ldots,\alpha^k} c^{\mu}_{\alpha^1,\ldots,\alpha^k}}_{\#(P_{\nu-\delta+\sigma,\mu-\delta+\sigma}\cap\mathbb{N}^d)}$$

$P_{a,b}$ – polytope, a "product" of k LR–polytopes, dimension $d = k^2(k-1)/2$.

[1] [Robinson-Taulbe, Vallejo, Ballantine, Orellana, Rosas etc]

Formulas for Kronecker coefficients II

Schur expansion via Weyl's character formula:

$$\langle s_\lambda, f\rangle = [s_\lambda(x)]f(x) = [x_1^{\lambda_1+k-1}\cdots x_k^{\lambda_k}]\Delta(x_1,\ldots,x_k)f(x_1,\ldots,x_k)$$

Combine with generalized Cauchy:

$$\sum_{\lambda,\mu,\nu} g(\lambda,\mu,\nu)s_\lambda(x)s_\mu(y)s_\nu(z) = \prod_{i,j,k}\frac{1}{1-x_iy_jz_k}$$

[2][Christandl-Doran-Walter, Pak-Panova]

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Via Contingency Arrays: [2]

$$g(\alpha,\beta,\gamma) = \sum_{\sigma^1,\sigma^2,\sigma^3\in S_\ell} \mathrm{sgn}(\sigma^1\sigma^2\sigma^3)\, CA(\alpha+1-\sigma^1,\beta+1-\sigma^2,\gamma+1-\sigma^3)\,,$$

$CA(u,v,w) =$ is # of $\ell\times\ell\times\ell$ contingency arrays $[A_{i,j,k}]\in\mathbb{N}^{k\times k\times k}$:

$$\sum_{j,k}A_{i,j,k}=u_i,\quad \sum_{i,k}A_{i,j,k}=v_j,\quad \sum_{i,j}A_{i,j,k}=w_k$$

[2] [Christandl-Doran-Walter, Pak-Panova]

When $\nu = (n-k, k)$ – two rows

$\ell(\nu) = 2$:

$$g(\lambda,\mu,\nu) = \sum_{\sigma \in S_2} \mathrm{sgn}(\sigma) \sum_{\alpha^i \vdash \nu_i - i + \sigma_i, i=1,2} c^{\lambda}_{\alpha^1\alpha^2} c^{\mu}_{\alpha^1\alpha^2}$$

$$= \underbrace{\sum_{\alpha \vdash k, \beta \vdash n-k} c^{\lambda}_{\alpha\beta} c^{\mu}_{\alpha\beta}}_{a_k(\lambda,\mu)} - \underbrace{\sum_{\alpha \vdash k-1, \beta \vdash n-k+1} c^{\lambda}_{\alpha\beta} c^{\mu}_{\alpha\beta}}_{a_{k-1}(\lambda,\mu)}$$

Corollary (Pak-P, Vallejo)

The sequence $a_0(\lambda,\mu), a_1(\lambda,\mu), \ldots, a_n(\lambda,\mu)$ is unimodal for all $\lambda, \mu \vdash n$, i.e.

$$a_0(\lambda,\mu) \le a_1(\lambda,\mu) \le \ldots \le a_{\lfloor n/2 \rfloor}(\lambda,\mu) \ge \ldots \ge a_n(\lambda,\mu).$$

When $\nu = (n-k, k)$ – two rows II

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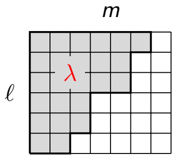

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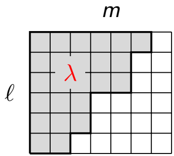

Theorem (Sylvester 1878, Cayley's conjecture 1856)

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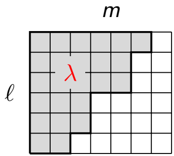

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Proof via Kronecker:[Pak-P]

$$a_k(m^\ell, m^\ell) = \sum_{\alpha \vdash k, \beta \vdash m\ell - k} 1(\beta_i = m - \alpha_{\ell+1-i}, i = 1 \ldots \ell) = p_k(\ell, m)$$

+Corollary – $a_k(\lambda,\mu)$ unimodal

Character Lemma and Stanley's theorem

Lemma (Pak–P)

If $\mu = \mu'$, then

$$g(\lambda, \mu, \mu) \geq |\chi^\lambda [(2\mu_1 - 1, 2\mu_2 - 3, \ldots)]|.$$

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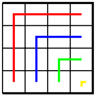
$\mu =$ 

$$\chi^\lambda = \chi^{(n^2-k)\circ(k)} - \chi^{(n^2-k+1)\circ(k-1)}$$

$$|\chi^\mu[(1,3,5,\ldots)]| = |b_k(n) - b_{k-1}(n)|,$$

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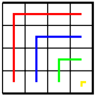

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$p_k(n,n) - p_{k-1}(n,n) = g(\lambda,\mu,\mu) \geq |\chi^\lambda[\widehat{\mu}]| = |b_k(n) - b_{k-1}(n)|$

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Theorem (Stanley, 1982)

The following polynomial in q is symmetric and unimodal

$$\binom{2n}{n}_q - \prod_{i=1}^{n} \left(1 + q^{2i-1}\right).$$

Effective bounds on $p_k(m,\ell) - p_{k-1}(m,\ell)$

Theorem (Pak-P, 2014+)

For all $m \geq \ell \geq 8$ *and* $2 \leq k \leq \ell m/2$*, we have:*

$$p_k(\ell,m) - p_{k-1}(\ell,m) \; > \; A\, \frac{2^{\sqrt{s}}}{s^{9/4}}\,, \quad \text{where} \quad s = \min\{2k, \ell^2\}\,,$$

and $A = 0.00449$ *is an universal constant.*

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– Stanley's Theorem: $p_k(n,n) - p_{k-1}(n,n) \geq b_k(n) - b_{k-1}(n)$, and asymptotics on b_k.

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Part II – for $m \neq \ell$:

Semigroup/monotonicity property[Manivel] : If $g(\alpha,\beta,\gamma) > 0$, then for all λ,μ,ν:

$g(\lambda+\alpha,\mu+\beta,\nu+\gamma) \; \geq \; g(\lambda,\mu,\nu)$.

$\Longrightarrow g(m^\ell, m^\ell, (m\ell - k, k)) \geq g(\ell^\ell, \ell^\ell, (\ell^2 - s, s)),\; (s = \min\{k, \ell^2/2\})$

NP and #P from combinatorics

Theorem (Pak-P,2014+)

Let r be fixed and $\lambda = (m^\ell, 1^r)$ and $\mu = (m + r, m^{\ell-1})$. Then $g(\lambda, \mu, (m\ell + r - k, k))$ is equal to the number of certain trees with local conditions of depth $O(\log \ell)$, width $O(\ell)$, and entries $O(m\ell)$. Thus computing $g(\lambda, \mu, (m\ell + r - k, k))$ is in #P (input size is $O(\ell \log m)$).

Proof: formulas in terms of q-binomial coefficients (partitions inside rectangle) + O'Hara's combinatorial proof of Sylvester's theorem.

Theorem (Pak-P, corollary of Blasiak's combinatorial interpretation)

When ν is a hook, $\mathrm{KP} \in \mathrm{NP}$ *and* $\textsc{Kron} \in \#\mathrm{P}$.

Complexity of KRON and KP

Theorem (Pak-P)

Let $\lambda, \mu, \nu \vdash n$ be partitions with lengths $\ell(\lambda), \ell(\mu)$, $\ell(\nu) \leq \ell$, the largest parts $\lambda_1, \mu_1, \nu_1 \leq N$, and $\nu_2 \leq M$. Then the Kronecker coefficients $g(\lambda, \mu, \nu)$ can be computed in time

$$O(\ell \log N) + (\ell \log M)^{O(\ell^3 \log \ell)}.$$

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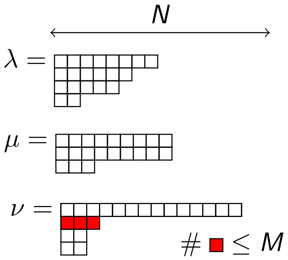
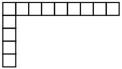


Corollary. *Suppose*

$$\log M, \ell = O\left(\frac{(\log \log N)^{1/3}}{(\log \log \log N)^{2/3}}\right).$$

Then there is a **polynomial time** *algorithm to compute $g(\lambda, \mu, \nu)$.*

Example: ℓ small and $\nu =$ .

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Corollary (Christandl-Doran-Walter)

When ℓ is fixed, the Kronecker coefficients can be computed in polynomial time, i.e. KRON $\in$ FP (this case: Mulmuley's conjecture ✓)

Theorem (Pak-P)

When the number of parts (ℓ) is fixed, there exists a **linear time** *algorithm to decide whether $g(\lambda, \mu, \nu) > 0$ (i.e. solve KP).*

Proofs I: the Reduction Lemma

Lemma (Pak-P)

Let $\lambda, \mu, \nu \vdash n$ and $\ell(\lambda), \ell(\mu), \ell(\nu) \leq \ell$. Set $s = n - \nu_1$. Then:

(i) *If $|\lambda_i - \mu_i| > s$ for some i, then $g(\lambda, \mu, \nu) = 0$,*

(ii) *If $|\lambda_i - \mu_i| \leq s$ for all i, $1 \leq i \leq \ell$, there $\exists$ an $r \leq 2s\ell^2$, s.t.*

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$+$ **Semigroup property** [Manivel, Brion]:

$$g(\alpha, \beta, \gamma) > 0, g(\lambda, \mu, \nu) > 0 \Rightarrow g(\alpha + \lambda, \beta + \mu, \gamma + \nu) \geq g(\lambda, \mu, \nu)$$

Corollary ([3])

For any m and partition $\alpha \vdash m$, we have that

$$g(\lambda + n\alpha, \mu + n\alpha, \nu + (nm))$$

is bounded and increasing as a function of $n \in \mathbb{N}$, i.e. **stable**.

[3] [Pak-P], indep in [Vallejo], [Stembridge]

Proofs II: Explicit bounds on KRON complexity

Lemma: [4]

$$g(\alpha,\beta,\gamma) = \sum_{\sigma^1,\sigma^2,\sigma^3 \in S_\ell} \operatorname{sgn}(\sigma^1\sigma^2\sigma^3)\, C(\alpha+1-\sigma^1, \beta+1-\sigma^2, \gamma+1-\sigma^3)\,,$$

where $C(u,v,w)$ is the number of $\ell \times \ell \times \ell$ contingency arrays $[A_{i,j,k}]$:

$$\sum_{j,k} A_{i,j,k} = u_i, \quad \sum_{i,k} A_{i,j,k} = v_j, \quad \sum_{i,j} A_{i,j,k} = w_k$$

[4] In [Christandl-Doran-Walter],[Pak-P]

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Let $\alpha,\beta,\gamma \vdash n$ be partitions of the same size, such that $\alpha_1,\beta_1,\gamma_1 \le m$ and $\ell(\alpha),\ell(\beta),\ell(\gamma) \le \ell$. Then $g(\alpha,\beta,\gamma)$ can be computed in time $(\ell \log m)^{O(\ell^3 \log \ell)}$.

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The theorem

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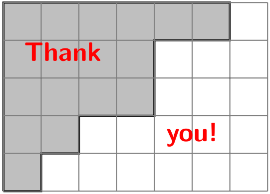
The analogous question for characters

Input: Integers N, ℓ, partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $\mu = (\mu_1, \ldots, \mu_\ell)$, where $0 \leq \lambda_i, \mu_i \leq N$, and $|\lambda| = |\mu|$.

Decide: whether $\chi^\lambda[\mu] = 0$

Proposition (Pak-P)

This problem is NP *–hard.*


Thank
you!