The little we know about

Multipartite Entanglement

Gilad Gour
University of Calgary
Department of Mathematics and Statistics
&
Institute for Quantum Science and Technology

G. Gour and N. R. Wallach, Physical Review Letters **111**, 060502 (2013)

Notations

$$|0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; ; \; |1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \langle 0| = (1,0) \; ; \; \langle 1| = (0,1)$$

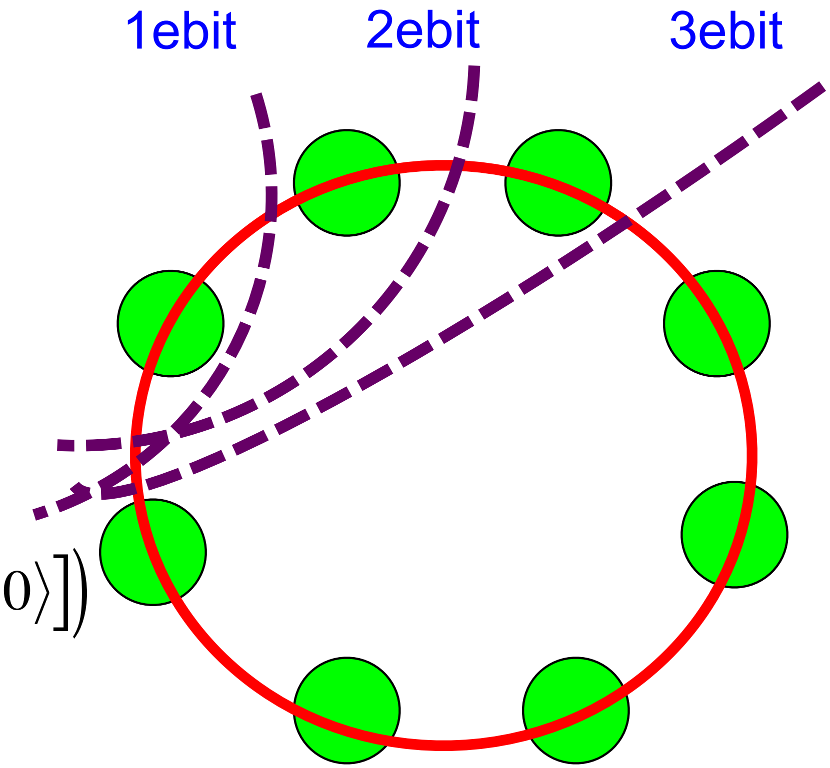
$$|\psi\rangle = a|0\rangle + b|1\rangle \; ; \; \langle\psi| = \bar{a}\langle 0| + \bar{b}\langle 1|$$

Partial Trace

$$|\psi\rangle^{AB} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}|i\rangle|j\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$$

$$\mathrm{Tr}_B|\psi\rangle^{AB}\langle\psi| := A^*A \qquad \mathrm{Tr}_A|\psi\rangle^{AB}\langle\psi| := AA^*$$

Maximally Entangled States



3 qubits: The GHZ state

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

4 qubits: Does not exist

5 qubits: The 5 qubit code state:

$$|C5\rangle = \frac{1}{4}\left(|00000\rangle + \left[|11000\rangle\right] - \left[|10100\rangle\right] - \left[|11110\rangle\right]\right)$$

6 qubits: The state:

$$\psi_6 = \frac{1}{\sqrt{2}}\left(|C5\rangle|0\rangle + |\mathrm{NOT}(\mathrm{C5})\rangle|1\rangle\right)$$

7 qubits: Unknown

n>7 qubits: Does not exist

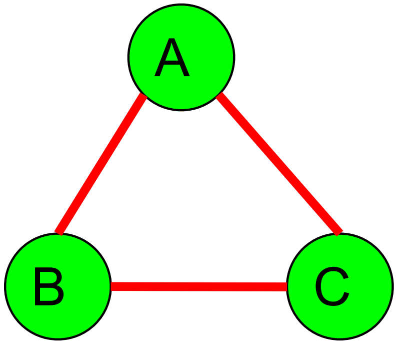
Three Qubits Entanglement

LOCC by three parties is “too restrictive”.

Stochastic LOCC by three parties yields two classes of tripartite entangled states.

The GHZ class:

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$



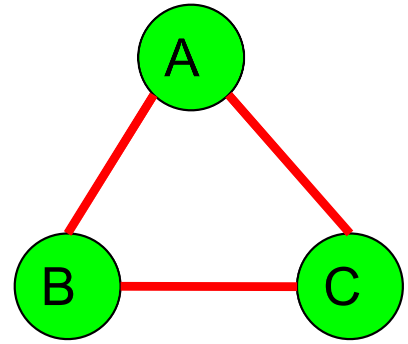
The W class:

$$|\mathrm{W}\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right)$$

Dür, Vidal and Cirac, Physics Review A **62**, 062314 (2000).

Three Qubits 6 SLOCC CLASSES

$$G \equiv SL(2,C) \otimes SL(2,C) \otimes SL(2,C)$$



$$G|GHZ\rangle$$

$$G|W\rangle$$

$$G|\psi^+\rangle_{AB}|0\rangle_C$$

$$G|\psi^+\rangle_{AC}|0\rangle_B$$

$$G|0\rangle_A|\psi^+\rangle_{BC}$$

$$G|0\rangle_A|0\rangle_B|0\rangle_C$$

Dür, Vidal and Cirac, Physics Review A **62**, 062314 (2000).

Kempf-Ness Theorem

$$\mathcal{H}_n = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$$

$$G \equiv SL(2,\mathbb{C}) \otimes SL(2,\mathbb{C}) \otimes \cdots \otimes SL(2,\mathbb{C})$$

$$K \equiv SU(2) \otimes SU(2) \otimes \cdots \otimes SU(2)$$

$$\mathfrak{g} \equiv Lie(G) \subset End(\mathcal{H}_n)$$

$$Crit(\mathcal{H}_n) = \{\phi \in \mathcal{H}_n | \langle\phi|X|\phi\rangle = 0, X \in \mathfrak{g}\}$$

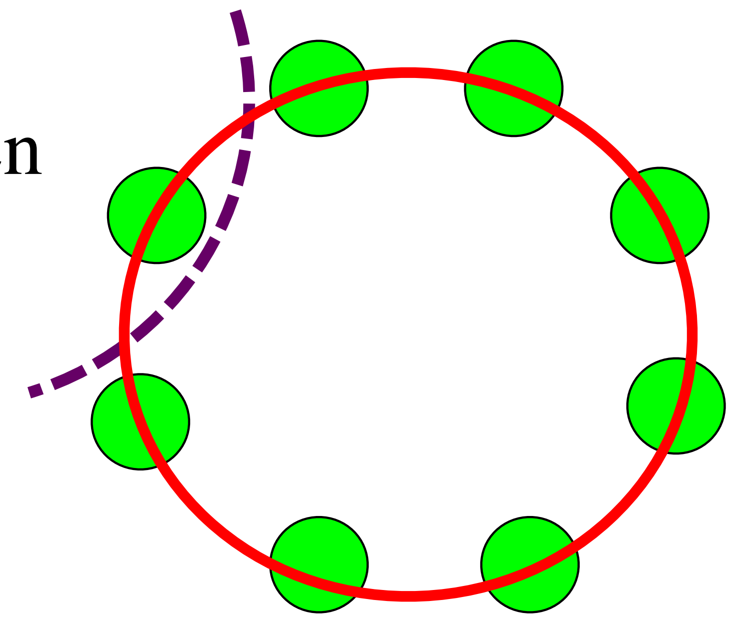
Theorem Let $\phi \in \mathcal{H}_n$.

1. $\phi \in Crit(\mathcal{H}_n), g \in G$ then $\|g\phi\| \geq \|\phi\|$ with equality if and only if $g\phi \in K\phi$.
2. If $\phi \in \mathcal{H}_n$ then $\phi \in Crit(\mathcal{H}_n)$ if and only if $\|g\phi\| \geq \|\phi\|$ for all $g \in G$.
3. If $\phi \in \mathcal{H}_n$ then $G\phi$ is closed in $\mathcal{H}_n$ if and only if $G\phi \cap Crit(\mathcal{H}_n) \neq \emptyset$.

Corollary of Kempf-Ness

Theorem: $$\phi \in Crit(\mathcal{H}_n)$$

$\Leftrightarrow$ Each qubit is maximally entangled with the rest of the qubits.



$$\text{Proof : Let } |\phi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|\varphi_0\rangle + |1\rangle|\varphi_1\rangle\right) \text{ then}$$

for $A \otimes I$ with $A = X, Y, Z \in Lie\left(SU(2)\right)$

we get $\langle\phi|A \otimes I|\phi\rangle = 0$ QED.

Four Qubits

States:

$$\psi \in H_4 \equiv C^2 \otimes C^2 \otimes C^2 \otimes C^2$$

SLOCC Group:

$$G = SL(2,C) \otimes SL(2,C) \otimes SL(2,C) \otimes SL(2,C)$$
$$K = SU(2) \otimes SU(2) \otimes SU(2) \otimes SU(2)$$

The Critical Set: $$Crit(\mathcal{H}_4) = KA$$

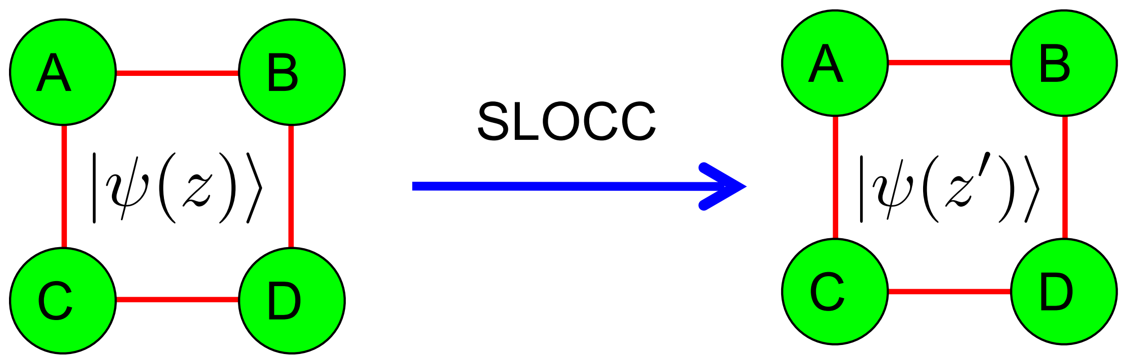
$$A = \left\{ z_0 u_0 + z_1 u_1 + z_2 u_2 + z_3 u_3 \middle| \sum_j |z_j|^2 = 1 \right\}$$

where $u_0 = |\psi^+\rangle|\psi^+\rangle$, $u_1 = |\psi^-\rangle|\psi^-\rangle$, $u_2 = |\phi^+\rangle|\phi^+\rangle$, $u_3 = |\phi^-\rangle|\phi^-\rangle$

Theorem: The set GA is open and dense in H_4.

Uncountable SLOCC inequivalent classes

SLOCC by four parties



$$|\psi(z)\rangle = z_1|\psi^+\rangle|\psi^+\rangle + z_2|\psi^-\rangle|\psi^-\rangle + z_3|\phi^+\rangle|\phi^+\rangle + z_4|\phi^-\rangle|\phi^-\rangle$$

$$|\psi(z')\rangle = z'_1|\psi+\rangle|\psi^+\rangle + z'_2|\psi-\rangle|\psi^-\rangle + z'_3|\phi+\rangle|\phi^+\rangle + z'_4|\phi-\rangle|\phi^-\rangle$$

The transformation $|\psi(z)\rangle \rightarrow |\psi(z')\rangle$ is not possible if $z_i \neq \pm z'_j$

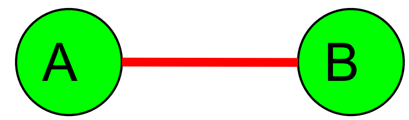
SL-Invariant Polynomials

Definition:

Let $H_n = \left(\mathbb{C}^2\right)^{\otimes n}$ and $G = SL(2,\mathbb{C})^{\otimes n}$. Then, a polynomial $f : H_n \to \mathbb{C}$ is SL-invariant if $f(g\psi) = f(\psi)$for all $g \in G$.

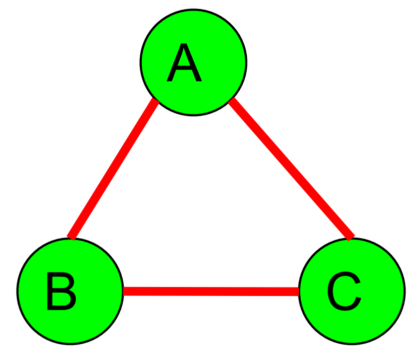
For two qubits:

$$f(\psi) = \langle\tilde{\psi}|\psi\rangle \equiv \langle\psi^*|\sigma_y\otimes\sigma_y|\psi\rangle$$



For three qubits:

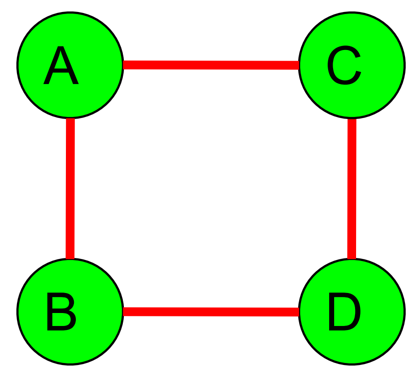
$$\Psi \equiv |0\rangle|\psi_0\rangle + |1\rangle|\psi_1\rangle \qquad f(\Psi) = \det\begin{pmatrix} \langle\tilde{\psi}_0|\psi_0\rangle & \langle\tilde{\psi}_0|\psi_1\rangle \\ \langle\tilde{\psi}_1|\psi_0\rangle & \langle\tilde{\psi}_1|\psi_1\rangle \end{pmatrix}$$



For four qubits:

$$f(\psi) = \langle\tilde{\psi}|\psi\rangle \equiv \langle\psi^*|\sigma_y\otimes\sigma_y\otimes\sigma_y\otimes\sigma_y|\psi\rangle$$

$$\tau_{ABCD} = |f(\psi)|^2 = |z_0^2+z_1^2+z_2^2+z_3^2|^2 \text{ for a critical state.}$$



Evolution of Multipartite Entanglement

Proposition:

Let $f(\psi)$ be an homogenous SL - invariant polynomial of degree k.

Then, $E_{SL}(\psi) \equiv |f(\psi)|^{1/k}$ is an entanglement monotone.

Evolution: Consider $\psi_i \to \psi_f \equiv \Lambda(\psi_i)$ where $\Lambda = \Lambda_1 \otimes I \otimes \cdots \otimes I$

Theorem:

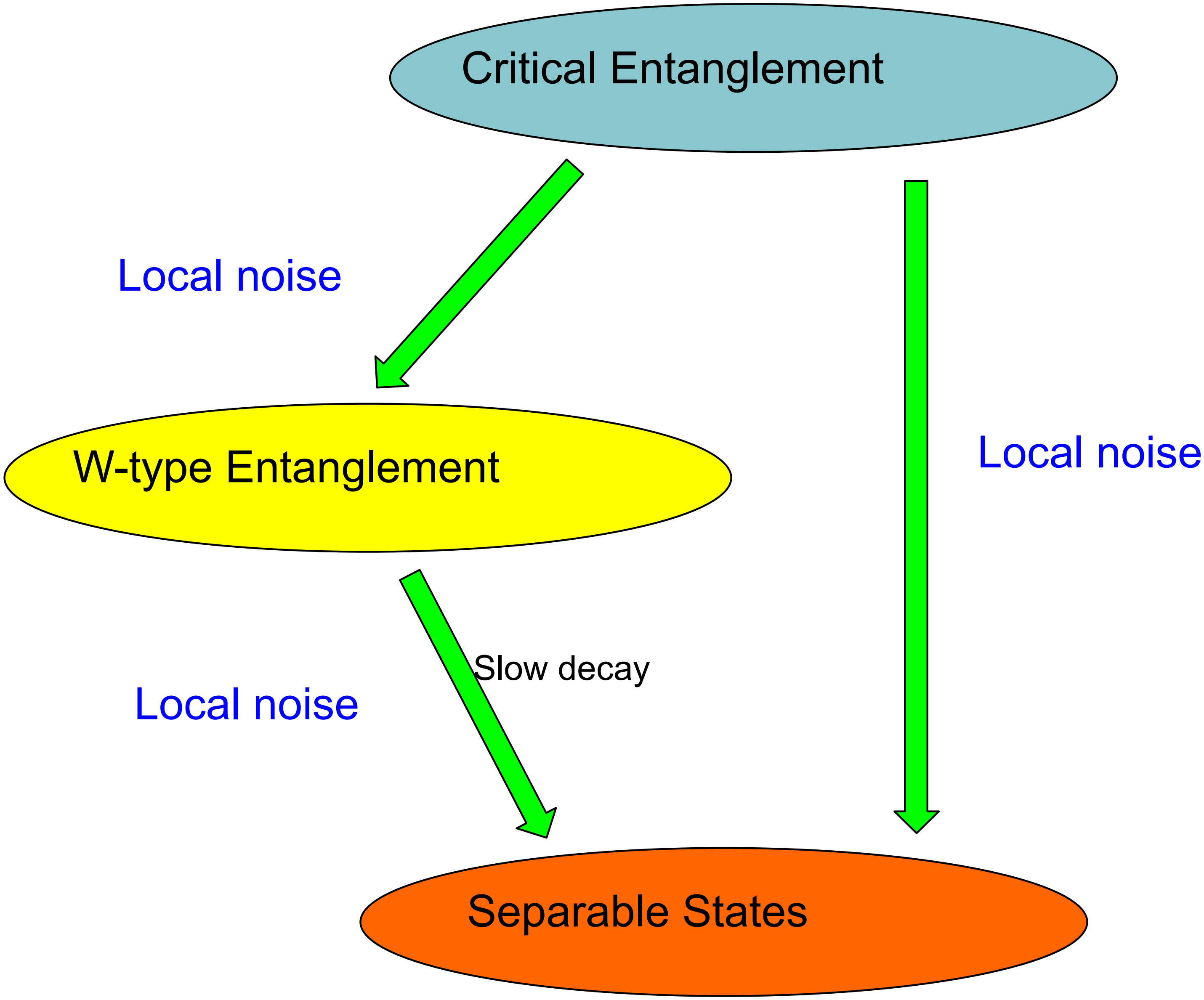
$$\frac{E_{SL}(\psi_f)}{E_{SL}(\psi_i)} = F(\Lambda_1) = \min \sum_j |\det K_j|^{2/d} = C\left(\Lambda_1 \otimes I(\psi^+)\right)$$

If $\Lambda = \Lambda_1 \otimes \Lambda_2 \otimes \cdots \otimes \Lambda_n$ then

$$\frac{E_{SL}(\rho_f)}{E_{SL}(\rho_i)} \leq \prod_{k=1}^{n} F(\Lambda_k)$$

GG, PRL **105**, 190504 (2010).

Exponential Multipartite Entanglement Decay



Distinguishing SLOCC Orbits

$$\mathcal{H}_n \equiv \mathbb{C}^{m_1} \otimes \mathbb{C}^{m_2} \otimes \cdots \otimes \mathbb{C}^{m_n}$$

$$|\psi\rangle = e^{i\theta} \frac{g|\phi\rangle}{\|g|\phi\rangle\|} \quad g \in G \equiv \mathrm{SL}(m_1, \mathbb{C}) \otimes \cdots \mathrm{SL}(m_n, \mathbb{C})$$

For homogenous polynomial f_k

$$f_k(|\psi\rangle) = \frac{e^{i\theta k}}{\|g|\phi\rangle\|^k} f_k(|\phi\rangle)$$

For two homogenous polynomials:

$$\frac{f_k(|\psi\rangle)}{h_k(|\psi\rangle)} = \frac{f_k(|\phi\rangle)}{h_k(|\phi\rangle)}$$

GG and Nolan Wallach, PRL 111, 060502 (2013).

The Degree of Homogenous SLIPs

$$\text{If } \zeta^{m_j} = 1 \text{ for some } j = 1, .., n \text{ then } \zeta I \in G$$

$$\Downarrow$$

$$f_k(|\psi\rangle) = f_k(\zeta|\psi\rangle) = \zeta^k f_k(|\psi\rangle)$$

$$\Downarrow$$

$$k = qr \quad \text{where} \quad r = lcm(m_1, ..., m_n) \quad \text{and} \quad q \in \mathbb{N}$$

GG and Nolan Wallach, PRL 111, 060502 (2013).

Construction of ALL SLIPs

$\mathcal{H}_n \equiv \mathbb{C}^{m_1} \otimes \mathbb{C}^{m_2} \otimes \cdots \otimes \mathbb{C}^{m_n}$ $G \equiv \mathrm{SL}(m_1, \mathbb{C}) \otimes \cdots \mathrm{SL}(m_n, \mathbb{C})$

Let $|v\rangle \in \otimes^k \mathcal{H}_n$ and $|x\rangle \in \mathcal{H}_n$

$$f_v(x) \equiv \langle v| \otimes^k x\rangle$$

If $(\otimes^k g)\,|v\rangle = |v\rangle$ for all $g \in G$, then the polynomial $f_v(x)$ is G-invariant.

⇨ The set of all SLIPs of degree k: $\left\{ f_v \mid v \in \left(\otimes^k \mathcal{H}_n\right)^G \right\}$

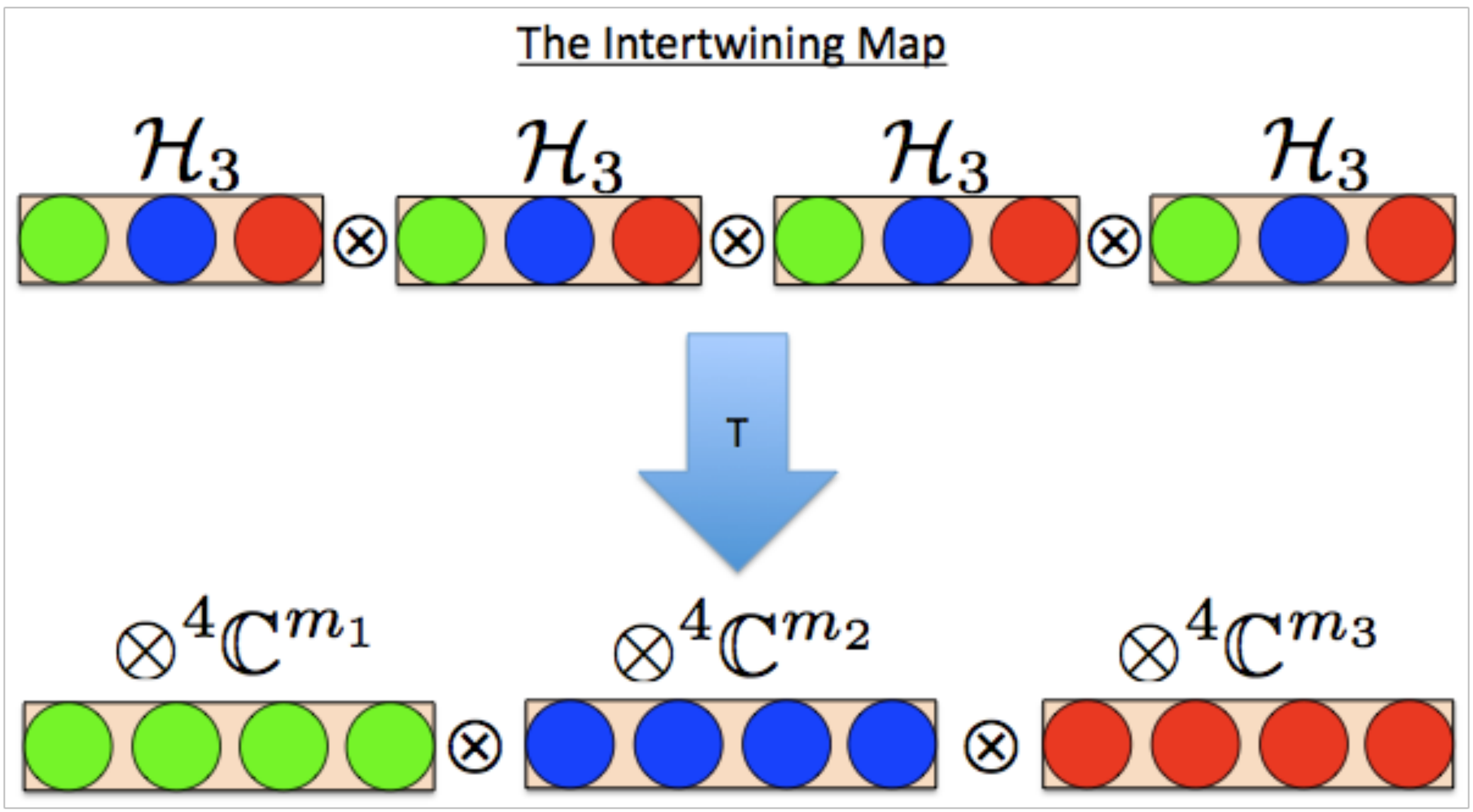
GG and Nolan Wallach, PRL 111, 060502 (2013).

Construction of ALL SLIPs

$$\mathcal{H}_n \equiv \mathbb{C}^{m_1} \otimes \mathbb{C}^{m_2} \otimes \cdots \otimes \mathbb{C}^{m_n} \quad G \equiv \mathrm{SL}(m_1, \mathbb{C}) \otimes \cdots \mathrm{SL}(m_n, \mathbb{C})$$

Key Observation:

$$\otimes^k \mathcal{H}_n \cong \left(\otimes^k \mathbb{C}^{m_1}\right) \otimes \left(\otimes^k \mathbb{C}^{m_2}\right) \otimes \cdots \otimes \left(\otimes^k \mathbb{C}^{m_n}\right)$$



Construction of ALL SLIPs

$$\mathcal{H}_n \equiv \mathbb{C}^{m_1} \otimes \mathbb{C}^{m_2} \otimes \cdots \otimes \mathbb{C}^{m_n} \quad G \equiv \mathrm{SL}(m_1, \mathbb{C}) \otimes \cdots \mathrm{SL}(m_n, \mathbb{C})$$

Key Observation:

$$\bigotimes^k \mathcal{H}_n \cong \left(\bigotimes^k \mathbb{C}^{m_1}\right) \otimes \left(\bigotimes^k \mathbb{C}^{m_2}\right) \otimes \cdots \otimes \left(\bigotimes^k \mathbb{C}^{m_n}\right)$$

For: $g = g_1 \bigotimes g_2 \bigotimes \cdots \bigotimes g_n$

Under the intertwining map:

$$\bigotimes^k g \xrightarrow{T} \bigotimes^k g_1 \bigotimes^k g_2 \bigotimes^k \cdots \bigotimes^k g_n$$

Instead of calculating $\left(\bigotimes^k \mathcal{H}_n\right)^G$ it is enough to find out:

$$\left(\bigotimes^k \mathbb{C}^m\right)^{SL(m,\mathbb{C})} \equiv \{|a\rangle \in \bigotimes^k \mathbb{C}^m \mid \bigotimes^k h|a\rangle = |a\rangle \ \ \forall h \in \mathrm{SL}(m, \mathbb{C})\}$$

Construction of ALL SLIPs

$\mathcal{H}_n \equiv \mathbb{C}^{m_1} \otimes \mathbb{C}^{m_2} \otimes \cdots \otimes \mathbb{C}^{m_n}$ $G \equiv \mathrm{SL}(m_1, \mathbb{C}) \otimes \cdots \mathrm{SL}(m_n, \mathbb{C})$

To summarize:

$$\left(\otimes^k \mathcal{H}_n\right)^G \cong \left(\otimes^k \mathbb{C}^{m_1}\right)^{\mathrm{SL}(m_1,\mathbb{C})} \otimes \cdots \otimes \left(\otimes^k \mathbb{C}^{m_n}\right)^{\mathrm{SL}(m_n,\mathbb{C})}$$

The orthogonal projection of $\otimes^k \mathbb{C}^m$ onto $\left(\otimes^k \mathbb{C}^m\right)^{SL(m,\mathbb{C})}$ (which is 0 unless k is divisible by m) :

$$P_{m,k} = \frac{d_\lambda}{k!} \sum_{\sigma \in S_k} \chi_\lambda(\sigma)\sigma$$

where $k = mr$ for some $r \in \mathbb{N}$, χ_λ is the character of S_k corresponding to the partition of k given by $\lambda = (r, r, ..., r)$ (m r's), and d_λ is the dimension of the irrep corresponding to the partition λ.

GG and Nolan Wallach, PRL 111, 060502 (2013).

Dimensions

The dimension, $d(k,n)$, of the space of SLIPs of degree k in n qubits:

$$d(2,n) = \tfrac{1}{2}(1+(-1)^n) \quad , \quad d(4,n) = \frac{2^{n-1}+(-1)^n}{3} \; ,$$

$$d(6,n) = \frac{1}{144}\left(36 + 44\cdot(-1)^n + 8\cdot 2^n + 3\cdot(-3)^n + 5^{n-1}\right)$$

$$d(10,n) =$$

$$272160 + 28448(-3)^n + 766080(-1)^n + 338751(2)^n + 14175(-1)^n 2^{2+n} +$$

$$11200(3)^{n+1} + 35(2)^{n+1}3^{n+2} + 315(-1)^n 4^{n+3} + 189(-2)^n 5^{n+1} + +45(14)^n + 42^n$$

$d(k,n)$ is exponential in both n and k !

Simple Construction of SLIPs

$$J \equiv -i\sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Rightarrow \quad A^T J A = J \qquad \forall\, A \in \mathrm{SL}(2,\mathbb{C})$$

Define: $$(\psi,\phi)_n \equiv \langle \psi^* | J \otimes \cdots \otimes J | \phi \rangle$$

$$\Downarrow$$

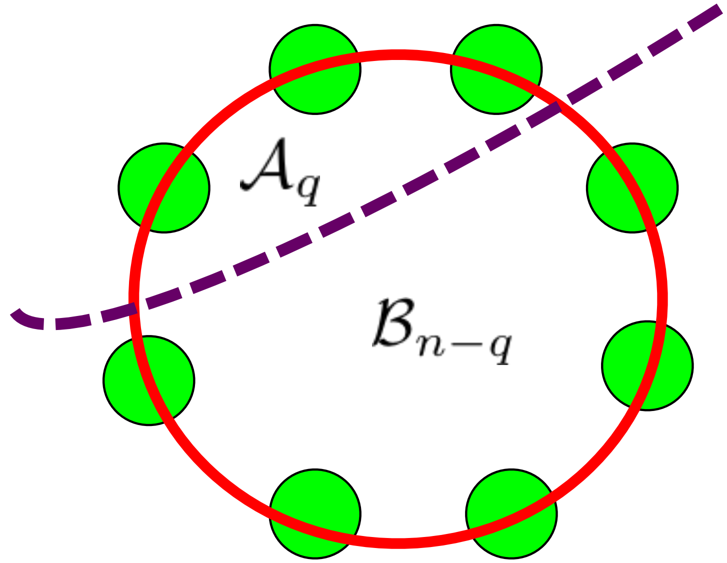
$$(g\psi, g\phi)_n = (\psi,\phi)_n \quad \forall\, g \in G \equiv \mathrm{SL}(2,\mathbb{C})^n$$

$$\Rightarrow \quad (\psi,\psi)_n \text{ is SLIP of degree 2}$$

GG and Nolan Wallach, PRL 111, 060502 (2013).

Simple Construction of SLIPs

A Bipartite Cut $\mathcal{H}_n \cong \mathcal{A}_q \otimes \mathcal{B}_{n-q}$



$$|\psi\rangle = \sum_{j=1}^{2^q} \sum_{k=1}^{2^{n-q}} a_{jk} |u_j\rangle |v_k\rangle$$

A class of SL-Invariant polynomials:

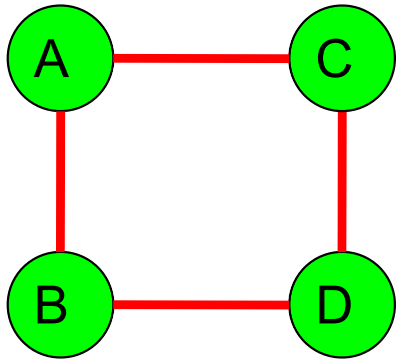
$$f_\ell^{\mathcal{A}_q|\mathcal{B}_{n-q}}(\psi) \equiv \mathrm{Tr}\left[\left(UAVA^T\right)^\ell\right]$$

$$U_{jj'} = (u_j, u_{j'})_q \text{ and } V_{kk'} = (v_k, v_{k'})_{n-q}$$

GG and Nolan Wallach, PRL 111, 060502 (2013).

Four Invariants for Four Qubits

Consider a four qubits state:



$$|\psi\rangle = \sum_{k,k'=0,1,2,3} T_{kk'} |k\rangle_{AB} |k'\rangle_{CD}$$

Four invariants:

$$\varepsilon_m(|\psi\rangle) = \begin{cases} \mathrm{Det}(T_\psi) & \text{for } m = 0 \\ \mathrm{Tr}\left[\left(T_\psi J T_\psi^T J\right)^m\right] & \text{for } m = 1,2,3 \end{cases} \quad \text{where} \quad J = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The 4-tangle:

$$\tau_{ABCD} = \left|\langle\psi|\tilde{\psi}\rangle\right|^2 \equiv \left|\langle\psi|\sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y|\psi^*\rangle\right|^2 = |\mathcal{E}_2(|\psi\rangle)|^2$$

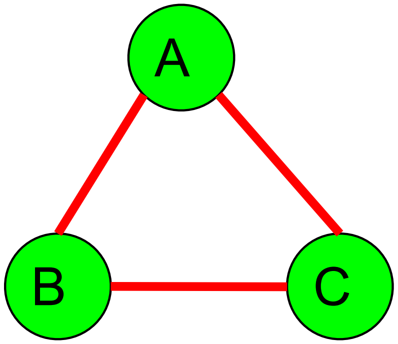
Monogamy of Entanglement

The Tangle:

$$\tau\left(\psi^{AB}\right) = 2\left(1 - \mathrm{Tr}\left(\rho_r^2\right)\right)$$

For three qubits:

$$E_{SL}\left(\psi^{ABC}\right) = \tau_{ABC} = \tau_{A(BC)} - \tau_{AB} - \tau_{AC}$$



Coffman, Kundu, Wootters (2000); Osborne, Verstraete (2005)

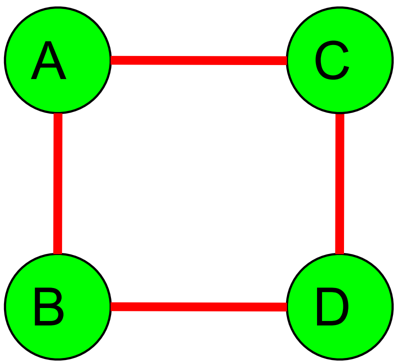
Optimizing The Average Tangle In For Qubits

The Tangle: $\tau\left(\psi^{AB}\right) = 2\left(1 - \mathrm{Tr}\left(\rho_r^2\right)\right)$

For four qubits:

$$\tau_1 = \frac{1}{4}\left(\tau_{A(BCD)} + \tau_{B(ACD)} + \tau_{C(ABD)} + \tau_{D(ABC)}\right)$$

$$\tau_2 = \frac{1}{3}\left(\tau_{(AB)(CD)} + \tau_{(AC)(BD)} + \tau_{(AD)(BC)}\right)$$



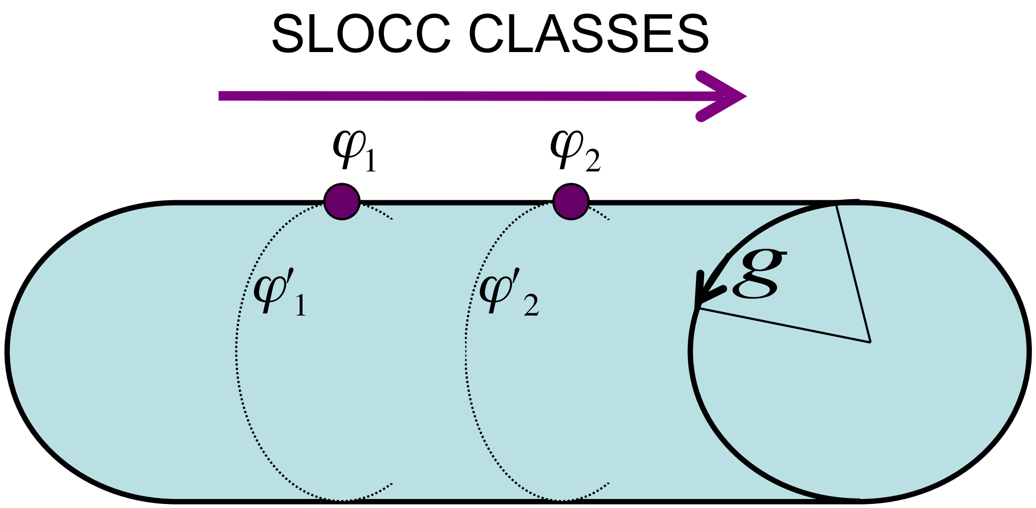
Kempf-Ness: $\tau_1\left(\psi\right) = 1 \quad \text{iff} \quad \psi \in A$

Theorem: $\tau_{ABCD}\left(\psi\right) = 4\tau_1\left(\psi\right) - 3\tau_2\left(\psi\right)$

GG and Nolan R. Wallach, JMP **51**, 112201 (2010).

Manipulation of Multipartite Entanglement

The space $\mathcal{H}_n = \mathbb{C}^{m_1} \otimes \cdots \otimes \mathbb{C}^{m_n}$



When it is possible to convert $|\varphi_1\rangle$ to $|\varphi_1'\rangle$ by LOCC?

Manipulation of Entanglement

Bipartite Entanglement:

$$\left|\psi^{AB}\right\rangle \xrightarrow{LOCC} \left|\phi^{AB}\right\rangle$$

⇕

$$\rho^A = \sum_i p_i U_i \sigma^A U_i^*$$

⇕

$$\rho^A \prec \sigma^A$$

Nielsen, PRL **83**, 436 (1999).

Multipartite Entanglement:

$$\psi_j = \frac{g_j|\varphi\rangle}{\|g_j|\varphi\rangle\|} \text{ and } \rho_j \equiv \frac{g_j^\dagger g_j}{\|g_j|\varphi\rangle\|^2}\ ,\ (j=1,2)$$

$$\psi_1 \xrightarrow{SEP} \psi_2$$

⇕

$$\rho_1 = \sum_{i=1}^m p_i U_i \rho_2 U_i^* \qquad \mathrm{Stab}(|\varphi\rangle) = \{U_i\}_{i=1}^m$$

⇓

$$\rho_1 \prec \rho_2$$

For SEP: GG and Nolan Wallach, NJP 13 073013 (2011).
For LOCC: Vicente, Spee, Kraus, PRL 111, 110502 (2013)

Summary and Conclusions

- SLIPs can be used to classify multipartite entanglement.
- All SLIPs in a given degree can be computed using the characters of the permutation group.
- The number of SLIPs grows exponentially both in n and k.
- SLIPs appear in monogamy relations
- Critical entanglement decay quickly under local noisy
- Analog to Nielsen theorem in the multipartite case
- LOCC and SEP are very strong restrictions

Thank You For Your Attention!!