Siegel's Theorem, Edge Coloring, and a Holant Dichotomy

Jin-Yi Cai
(University of Wisconsin-Madison)

Joint with:
Heng Guo and Tyson Williams
(University of Wisconsin-Madison)

Carl L. Siegel

Theorem (Siegel's Theorem)

Any smooth algebraic curve of genus $g > 0$ defined by a polynomial $f(x, y) \in \mathbb{Z}[x, y]$ has only finitely many integer solutions.

Theorem (Siegel's Theorem)

Any smooth algebraic curve of genus $g > 0$ defined by a polynomial $f(x, y) \in \mathbb{Z}[x, y]$ has only finitely many integer solutions.

Theorem (Faltings' Theorem–Mordell Conjecture)

Any smooth algebraic curve of genus $g > 1$ defined by a polynomial $f(x, y) \in \mathbb{Z}[x, y]$ has only finitely many rational solutions.

Pell's Equation (genus 0)

$$x^2 - 61y^2 = 1$$

Diophantine Equations with Enormous Solutions

Pell's Equation (genus 0)

$$x^2 - 61y^2 = 1$$

Smallest solution:

$$(1766319049, 226153980)$$

Diophantine Equations with Enormous Solutions

Pell's Equation (genus 0)

$$x^2 - 61y^2 = 1$$

Smallest solution:

$$(1766319049, 226153980)$$

$$x^2 - 991y^2 = 1$$

Diophantine Equations with Enormous Solutions

Pell's Equation (genus 0)

$$x^2 - 61y^2 = 1$$

Smallest solution:

$$(1766319049, 226153980)$$

$$x^2 - 991y^2 = 1$$

Smallest solution:

$$(379516400906811930638014896080, 12055735790331359447442538767)$$

Diophantine Equations with Enormous Solutions

Pell's Equation (genus 0)

$$x^2 - 61y^2 = 1$$

Smallest solution:

$$(1766319049, 226153980)$$

$$x^2 - 991y^2 = 1$$

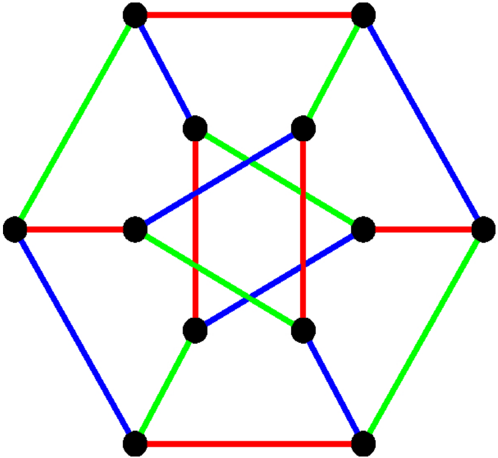
Smallest solution:

$$(379516400906811930638014896080, 12055735790331359447442538767)$$

Next smallest solution:

$$(288065397114519999215772221121510725946342952839946398732799,$$
$$9150698914859994783783151874415159820056535806397752666720)$$

Edge Coloring



Edge Coloring–Decision Problem

Theorem (Vizing's Theorem)

A proper edge coloring using $\Delta(G) + 1$ *colors always exists.*

Edge Coloring–Decision Problem

Theorem (Vizing's Theorem)

A proper edge coloring using $\Delta(G) + 1$ colors always exists.

$\Delta(G)$ is an obvious lower bound.

Edge Coloring–Decision Problem

Tait (1880) showed that, for bridgeless 3-regular planar graphs, the statement that a proper 3-edge coloring always exists is equivalent to the Four Color (Conjecture) Theorem.

Edge Coloring–Decision Problem

Tait (1880) showed that, for bridgeless 3-regular planar graphs, the statement that a proper 3-edge coloring always exists is equivalent to the Four Color (Conjecture) Theorem.

For 3-regular (non-planar) graphs, 3-edge coloring is NP-complete (Holyer (1981)).

Edge Coloring–Decision Problem

Tait (1880) showed that, for bridgeless 3-regular planar graphs, the statement that a proper 3-edge coloring always exists is equivalent to the Four Color (Conjecture) Theorem.

For 3-regular (non-planar) graphs, 3-edge coloring is NP-complete (Holyer (1981)).

But this reduction is not parsimonious (see Welsh).

Edge Coloring–Counting Problem

Problem $\#\kappa$-EdgeColoring:
Input: A graph G.
Output: The number of valid edge colorings of G, using κ colors.

Edge Coloring–Counting Problem

Problem $\#\kappa$-EdgeColoring:
Input: A graph G.
Output: The number of valid edge colorings of G, using κ colors.

Theorem

$\#\kappa$-EdgeColoring is #P-hard over planar r-regular graphs for $\kappa \geq r \geq 3$.

Edge Coloring–Counting Problem

Problem $\#\kappa$-EdgeColoring:
Input: A graph G.
Output: The number of valid edge colorings of G, using κ colors.

Theorem

$\#\kappa$-EdgeColoring is #P-hard over planar r-regular graphs for $\kappa \geq r \geq 3$.

- $\kappa = r$, and
- $\kappa > r$.

Problem $\#\kappa$-EdgeColoring:
Input: A graph G.
Output: The number of valid edge colorings of G, using κ colors.

Theorem

$\#\kappa$-EdgeColoring is #P-hard over planar r-regular graphs for $\kappa \geq r \geq 3$.

- $\kappa = r$, and
- $\kappa > r$.

This is proved in the framework of complexity dichotomy theorems.

Three Frameworks for Counting Problems

1. Graph Homomorphisms
2. Constraint Satisfaction Problems (CSP)
3. Holant Problems

Three Frameworks for Counting Problems

1. Graph Homomorphisms
2. Constraint Satisfaction Problems (CSP)
3. Holant Problems

In each framework, there has been remarkable progress in the classification program of the complexity of counting problems.

Three Frameworks for Counting Problems

1. Graph Homomorphisms
2. Constraint Satisfaction Problems (CSP)
3. Holant Problems

In each framework, there has been remarkable progress in the classification program of the complexity of counting problems.

A signature grid $\Omega = (G, \mathcal{F}, \pi)$ consists of a graph $G = (V, E)$, where π assigns a function $f_v \in \mathcal{F}$ to each $v \in V$.

A signature grid $\Omega = (G, \mathcal{F}, \pi)$ consists of a graph $G = (V, E)$, where π assigns a function $f_v \in \mathcal{F}$ to each $v \in V$.

Over the Boolean domain $\{0, 1\}$, the Holant problem on instance Ω is to evaluate

$$\text{Holant}_\Omega = \sum_{\sigma} \prod_{v \in V} f_v(\sigma \mid_{E(v)}),$$

a sum over all edge assignments $\sigma : E \to \{0, 1\}$.

Constraint Functions

A function f_v can be represented by listing its values in lexicographical order as in a truth table, which is a vector in $\mathbb{C}^{2^n}$, or as a tensor in $(\mathbb{C}^2)^{\otimes n}$.

A function f_v can be represented by listing its values in lexicographical order as in a truth table, which is a vector in $\mathbb{C}^{2^n}$, or as a tensor in $(\mathbb{C}^2)^{\otimes n}$.

Holographic Transformations can change one function to another.
E.g. The n-ary EQUALITY function is

$$\begin{bmatrix}1\\0\end{bmatrix}^{\otimes n} + \begin{bmatrix}0\\1\end{bmatrix}^{\otimes n}.$$

A function f_v can be represented by listing its values in lexicographical order as in a truth table, which is a vector in $\mathbb{C}^{2^n}$, or as a tensor in $(\mathbb{C}^2)^{\otimes n}$.

Holographic Transformations can change one function to another.
E.g. The n-ary EQUALITY function is

$$\begin{bmatrix}1\\0\end{bmatrix}^{\otimes n}+\begin{bmatrix}0\\1\end{bmatrix}^{\otimes n}.$$

Under the Holographic Transformation by $H=\left[\begin{smallmatrix}1&1\\1&-1\end{smallmatrix}\right]$,

$$H^{\otimes n}\left\{\begin{bmatrix}1\\0\end{bmatrix}^{\otimes n}+\begin{bmatrix}0\\1\end{bmatrix}^{\otimes n}\right\}=\begin{bmatrix}1\\1\end{bmatrix}^{\otimes n}+\begin{bmatrix}1\\-1\end{bmatrix}^{\otimes n}$$

is a (constant multiple of) the Parity function.

Equivalence with sl$(2;\mathbb{C})$ representation

sl$(2;\mathbb{C})$ $=$su$(2)_{\mathbb{C}}$.

Equivalence with sl$(2;\mathbb{C})$ representation

sl$(2;\mathbb{C})$ $=$su$(2)_{\mathbb{C}}$.

There is a 1-1 correspondence between representations of sl$(2;\mathbb{C})$ and that of $\mathtt{SU}(2)$.

Equivalence with sl$(2;\mathbb{C})$ representation

sl$(2;\mathbb{C})$ =su$(2)_{\mathbb{C}}$.

There is a 1-1 correspondence between representations of sl$(2;\mathbb{C})$ and that of $\mathtt{SU}(2)$.

Let f be a symmetric constraint function $[f_0, f_1, \ldots, f_n]$.

Equivalence with sl$(2;\mathbb{C})$ representation

sl$(2;\mathbb{C})$ =su$(2)_{\mathbb{C}}$.

There is a 1-1 correspondence between representations of sl$(2;\mathbb{C})$ and that of $\mathtt{SU}(2)$.

Let f be a symmetric constraint function $[f_0, f_1, \ldots, f_n]$.

For any $U \in \mathtt{SU}(2)$, $U^{\otimes n} f$ is also a symmetric constraint function

$$U^{\otimes n} f = [f_0', f_1', \ldots, f_n'].$$

Equivalence with $\mathfrak{sl}(2;\mathbb{C})$ representation

$\mathrm{sl}(2;\mathbb{C}) = \mathrm{su}(2)_{\mathbb{C}}$.

There is a 1-1 correspondence between representations of $\mathrm{sl}(2;\mathbb{C})$ and that of $\mathtt{SU}(2)$.

Let f be a symmetric constraint function $[f_0, f_1, \ldots, f_n]$.

For any $U \in \mathtt{SU}(2)$, $U^{\otimes n} f$ is also a symmetric constraint function

$$U^{\otimes n} f = [f'_0, f'_1, \ldots, f'_n].$$

This gives a representation

$$\varphi_n : (f_0, f_1, \ldots, f_n) \mapsto (f'_0, f'_1, \ldots, f'_n).$$

Another representation

Let $p_n(x,y) = \sum_{i=0}^{n} a_i \binom{n}{i} x^{n-i} y^i$. Then

$$q_n(x,y) = p_n((x,y)U) = \sum_{i=0}^{n} a'_i \binom{n}{i} x^{n-i} y^i.$$

This gives another representation

$$\psi_n : (a_0, a_1, \ldots, a_n) \mapsto (a'_0, a'_1, \ldots, a'_n).$$

Another representation

Let $p_n(x,y) = \sum_{i=0}^{n} a_i \binom{n}{i} x^{n-i} y^i$. Then

$$q_n(x,y) = p_n((x,y)U) = \sum_{i=0}^{n} a_i' \binom{n}{i} x^{n-i} y^i.$$

This gives another representation

$$\psi_n : (a_0, a_1, \ldots, a_n) \mapsto (a_0', a_1', \ldots, a_n').$$

Theorem

The two representations φ_n and ψ_n are the same.

Consider a 3-regular graph G.
Let AD_3 denote the following local constraint function

$$\text{AD}_3(x, y, z) = \begin{cases} 1 & \text{if } x, y, z \in [\kappa] \text{ are all distinct} \\ 0 & \text{otherwise} \end{cases}$$

#κ-EdgeColoring as a Holant Problem

Consider a 3-regular graph G.
Let AD_3 denote the following local constraint function

$$\mathrm{AD}_3(x, y, z) = \begin{cases} 1 & \text{if } x, y, z \in [\kappa] \text{ are all distinct} \\ 0 & \text{otherwise} \end{cases}$$

Now place AD_3 at each vertex v, with incident edges x, y, z.
Then we evaluate the sum of product

$$\mathrm{Holant}(G; \mathrm{AD}_3) = \sum_{\sigma : E(G) \to [\kappa]} \prod_{v \in V(G)} \mathrm{AD}_3 \left(\sigma \mid_{E(v)} \right).$$

$\#\kappa$-EdgeColoring as a Holant Problem

Consider a 3-regular graph G.
Let AD_3 denote the following local constraint function

$$\mathrm{AD}_3(x, y, z) = \begin{cases} 1 & \text{if } x, y, z \in [\kappa] \text{ are all distinct} \\ 0 & \text{otherwise} \end{cases}$$

Now place AD_3 at each vertex v, with incident edges x, y, z.
Then we evaluate the sum of product

$$\mathrm{Holant}(G; \mathrm{AD}_3) = \sum_{\sigma : E(G) \to [\kappa]} \prod_{v \in V(G)} \mathrm{AD}_3 \left(\sigma \mid_{E(v)} \right).$$

Clearly $\mathrm{Holant}(G; \mathrm{AD}_3)$ computes $\#\kappa$-EdgeColoring.

Holant Problems

In general, we consider all local constraint functions

$$f(x,y,z) = \begin{cases} a & \text{if } x = y = z \in [\kappa] \\ b & \text{if } |\{x,y,z\}| = 2 \\ c & \text{if } |\{x,y,z\}| = 3 \end{cases}$$

And the Holant problem is to compute

$$\text{Holant}(G; f) = \sum_{\sigma: E(G) \to [\kappa]} \prod_{v \in V(G)} f\left(\sigma \mid_{E(v)}\right).$$

In general, we consider all local constraint functions

$$f(x,y,z) = \begin{cases} a & \text{if } x = y = z \in [\kappa] \\ b & \text{if } |\{x,y,z\}| = 2 \\ c & \text{if } |\{x,y,z\}| = 3 \end{cases}$$

And the Holant problem is to compute

$$\text{Holant}(G; f) = \sum_{\sigma : E(G) \to [\kappa]} \prod_{v \in V(G)} f\left(\sigma \mid_{E(v)}\right).$$

Succinct signatures $f = \langle a, b, c \rangle$, where $a, b, c \in \mathbb{C}$.

In general, we consider all local constraint functions

$$f(x,y,z) = \begin{cases} a & \text{if } x = y = z \in [\kappa] \\ b & \text{if } |\{x,y,z\}| = 2 \\ c & \text{if } |\{x,y,z\}| = 3 \end{cases}$$

And the Holant problem is to compute

$$\text{Holant}(G; f) = \sum_{\sigma: E(G) \to [\kappa]} \prod_{v \in V(G)} f\left(\sigma \mid_{E(v)}\right).$$

Succinct signatures $f = \langle a, b, c \rangle$, where $a, b, c \in \mathbb{C}$.
Thus $\text{AD}_3 = \langle 0, 0, 1 \rangle$.

L. Lovász:

Operations with structures, Acta Math. Hung. 18 (1967), 321-328.

http://www.cs.elte.hu/~lovasz/hom-paper.html

L. Lovász:
Operations with structures, Acta Math. Hung. 18 (1967), 321-328.

http://www.cs.elte.hu/~lovasz/hom-paper.html

Let $\mathbf{A} = (A_{i,j}) \in \mathbb{C}^{\kappa \times \kappa}$ be a symmetric complex matrix.

The graph homomorphism problem is:
INPUT: An undirected graph $G = (V, E)$.
OUTPUT:

$$Z_{\mathbf{A}}(G) = \sum_{\xi: V \to [\kappa]} \prod_{(u,v) \in E} A_{\xi(u),\xi(v)}.$$

Examples of Graph Homomorphism

Let

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

then $Z_{\mathbf{A}}(G)$ counts the number of VERTEX COVERS in G.

Let

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

then $Z_{\mathbf{A}}(G)$ counts the number of VERTEX COVERS in G.

Let

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix}$$

then $Z_{\mathbf{A}}(G)$ counts the number of vertex κ-COLORINGS in G.

Dichotomy Theorem for Graph Homomorphism

Theorem (C., Xi Chen and Pinyan Lu)

There is a complexity dichotomy for $Z_{\mathbf{A}}(\cdot)$*:*
For any symmetric complex valued matrix $\mathbf{A} \in \mathbb{C}^{\kappa \times \kappa}$*, the problem of computing* $Z_{\mathbf{A}}(G)$*, for any input* G*, is either in P or #P-hard.*
Given $\mathbf{A}$*, whether* $Z_{\mathbf{A}}(\cdot)$ *is in P or #P-hard can be decided in polynomial time in the size of* $\mathbf{A}$*.*

SIAM J. Comput. 42(3): 924-1029 (2013) (106 pages)

Dichotomy Theorem for Graph Homomorphism

Theorem (C., Xi Chen and Pinyan Lu)

There is a complexity dichotomy for $Z_{\mathbf{A}}(\cdot)$:
For any symmetric complex valued matrix $\mathbf{A} \in \mathbb{C}^{\kappa \times \kappa}$, the problem of computing $Z_{\mathbf{A}}(G)$, for any input G, is either in P or #P-hard.
Given $\mathbf{A}$, whether $Z_{\mathbf{A}}(\cdot)$ is in P or #P-hard can be decided in polynomial time in the size of $\mathbf{A}$.

SIAM J. Comput. 42(3): 924-1029 (2013) (106 pages)

Further generalized to all counting CSP.

Theorem (C., Xi Chen)

Every finite set $\mathcal{F}$ of complex valued constraint functions on any finite domain set $[\kappa]$ defines a counting CSP problem #CSP($\mathcal{F}$) that is either computable in P or #P-hard.

The decision version of this is open (Feder-Vardi Dichotomy Conjecture).

Theorem (Main Theorem)

For any κ, any 3-regular graph G and any $f = \langle a, b, c \rangle$, the problem Holant$(G; f)$ *is either computable in polynomial time or is #P-hard.*

Dichotomy Theorem for Holant$(G; f)$

Theorem (Main Theorem)

For any κ, any 3-regular graph G and any $f = \langle a, b, c \rangle$, the problem Holant$(G; f)$ *is either computable in polynomial time or is #P-hard.*

$\#\kappa$-EdgeColoring is the special case for $f = \langle 0, 0, 1 \rangle$.

Non-trivial Examples of Tractable Holant Problems

- On domain size $\kappa = 3$, Holant$(G; \langle 5, 2, -4 \rangle)$ is computable in P.

- On domain size $\kappa = 3$, Holant$(G; \langle 5, 2, -4 \rangle)$ is computable in P.

$f = \langle 5, 2, -4 \rangle = \frac{1}{3}\left[(-1, 2, 2)^{\otimes 3} + (2, -1, 2)^{\otimes 3} + (2, 2, -1)^{\otimes 3}\right]$.

Holographic transformation by the orthogonal matrix $T = \frac{1}{3}\left[\begin{smallmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{smallmatrix}\right]$.

Non-trivial Examples of Tractable Holant Problems

- On domain size $\kappa = 3$, Holant$(G; \langle 5, 2, -4 \rangle)$ is computable in P.

$$f = \langle 5, 2, -4 \rangle = \frac{1}{3}\left[(-1,2,2)^{\otimes 3} + (2,-1,2)^{\otimes 3} + (2,2,-1)^{\otimes 3}\right].$$

Holographic transformation by the orthogonal matrix $T = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$.

- In general Holant$(G; \langle \kappa^2 - 6\kappa + 4, -2(\kappa - 2), 4 \rangle)$ is computable in P.

Non-trivial Examples of Tractable Holant Problems

- On domain size $\kappa = 3$, Holant$(G; \langle 5, 2, -4 \rangle)$ is computable in P.

$f = \langle 5, 2, -4 \rangle = \frac{1}{3}\left[(-1, 2, 2)^{\otimes 3} + (2, -1, 2)^{\otimes 3} + (2, 2, -1)^{\otimes 3}\right]$.
Holographic transformation by the orthogonal matrix $T = \frac{1}{3}\left[\begin{smallmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{smallmatrix}\right]$.

- In general Holant$(G; \langle \kappa^2 - 6\kappa + 4, -2(\kappa - 2), 4 \rangle)$ is computable in P.

- Suppose $\kappa = 4$. For any $\lambda \in \mathbb{C}$,

$$\text{Holant}(G; \lambda\langle -3 - 4i, 1, -1 + 2i \rangle)$$

is computable in P.

Tutte Polynomial

Definition

For an undirected graph $G = (V, E)$, the Tutte polynomial of G is

$$T(G; x, y) = \sum_{A \subseteq E} (x - 1)^{\kappa(A) - \kappa(E)} (y - 1)^{\kappa(A) + |A| - |V|},$$

where $\kappa(A)$ denotes the number of connected components of the spanning subgraph (V, A).

Tutte Polynomial

Definition

For an undirected graph $G = (V, E)$, the Tutte polynomial of G is

$$T(G; x, y) = \sum_{A \subseteq E} (x-1)^{\kappa(A)-\kappa(E)}(y-1)^{\kappa(A)+|A|-|V|},$$

where $\kappa(A)$ denotes the number of connected components of the spanning subgraph (V, A).

The chromatic polynomial is

$$\chi(G; \lambda) = (-1)^{|V|-k(G)}\lambda^{k(G)}\,\mathrm{T}(G; 1-\lambda, 0), \tag{1}$$

Tutte Polynomial

Definition

For an undirected graph $G = (V, E)$, the Tutte polynomial of G is

$$T(G; x, y) = \sum_{A \subseteq E} (x-1)^{\kappa(A)-\kappa(E)} (y-1)^{\kappa(A)+|A|-|V|},$$

where $\kappa(A)$ denotes the number of connected components of the spanning subgraph (V, A).

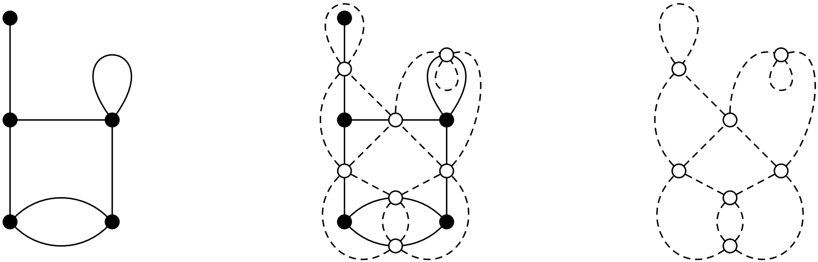
The chromatic polynomial is

$$\chi(G; \lambda) = (-1)^{|V|-k(G)} \lambda^{k(G)} \mathrm{T}(G; 1-\lambda, 0), \quad (1)$$

Theorem (Vertigan)

For any $x, y \in \mathbb{C}$, the problem of computing the Tutte polynomial at (x, y) over planar graphs is #P-hard unless $(x-1)(y-1) \in \{1, 2\}$ or $(x, y) \in \{(1,1), (-1,-1), (\omega, \omega^2), (\omega^2, \omega)\}$, where $\omega = e^{2\pi i/3}$. In each of these exceptional cases, the computation can be done in polynomial time.

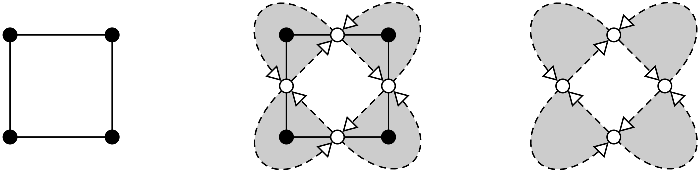
Medial Graph



(a) (b) (c)

A plane graph (a), its medial graph (c), and the two graphs superimposed (b).

Directed Medial Graph



(a) (b) (c)

A plane graph (a), its directed medial graph (c), and the two graphs superimposed (b).

Eulerian Subgraphs

A graph is Eulerian if every vertex has an even degree.

Eulerian Subgraphs

A graph is Eulerian if every vertex has an even degree.

A directed graph is Eulerian if $\mathsf{deg}_{\mathrm{in}}(v) = \mathsf{deg}_{\mathrm{out}}(v)$, at every vertex v.

Eulerian Subgraphs

A graph is Eulerian if every vertex has an even degree.

A directed graph is Eulerian if $\deg_{\mathrm{in}}(v) = \deg_{\mathrm{out}}(v)$, at every vertex v.

We don't require connectedness.

Eulerian Subgraphs

A graph is Eulerian if every vertex has an even degree.

A directed graph is Eulerian if $\deg_{\mathrm{in}}(v) = \deg_{\mathrm{out}}(v)$, at every vertex v.

We don't require connectedness.

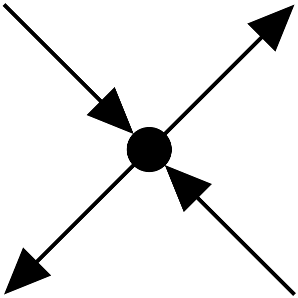
Suppose G is a connected plane graph and $\vec{G}_m$ its directed medial graph. For any κ, the Eulerian partitions $\pi(\vec{G}_m)$ are κ-labelings of edges of $\vec{G}_m$, such that each color set forms an Eulerian digraph.

Theorem (Ellis-Monaghan)

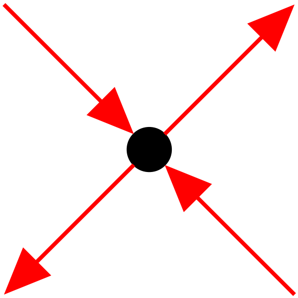
$$\kappa \,\mathrm{T}(G; \kappa + 1, \kappa + 1) = \sum_{c \in \pi(\vec{G}_m)} 2^{\mu(c)},$$

where $\mu(c)$ is the number of monochromatic vertices in the coloring c.

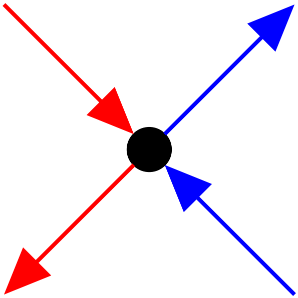
Directed Medial Graph Local Configuration

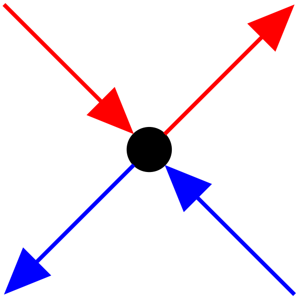


Eulerian Local Configuration



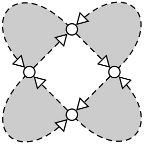
Directed Medial Graph Local Configuration





The Eulerian Signature

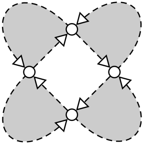
The sum $\sum_{c \in \pi(\vec{G}_m)} 2^{\mu(c)}$ can be expressed as a Holant problem:



$$\mathcal{E}\left(\begin{smallmatrix} w & z \\ x & y \end{smallmatrix}\right) = \begin{cases} 2 & \text{if } w = x = y = z \in [\kappa] \\ 1 & \text{if } w = x \neq y = z \in [\kappa] \\ 0 & \text{if } w = y \neq x = z \in [\kappa] \\ 1 & \text{if } w = z \neq x = y \in [\kappa] \\ 0 & \text{all other cases.} \end{cases}$$

Denote by $\mathcal{E} = \langle 2, 1, 0, 1, 0 \rangle$.

The sum $\sum_{c \in \pi(\vec{G}_m)} 2^{\mu(c)}$ can be expressed as a Holant problem:

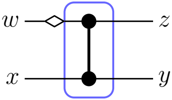


$$\mathcal{E}\left(\begin{smallmatrix} w & z \\ x & y \end{smallmatrix}\right) = \begin{cases} 2 & \text{if } w = x = y = z \in [\kappa] \\ 1 & \text{if } w = x \neq y = z \in [\kappa] \\ 0 & \text{if } w = y \neq x = z \in [\kappa] \\ 1 & \text{if } w = z \neq x = y \in [\kappa] \\ 0 & \text{all other cases.} \end{cases}$$

Denote by $\mathcal{E} = \langle 2, 1, 0, 1, 0 \rangle$.
To be Eulerian, at every vertex $v \in V(G)$, either it is monochromatic, or RRBB cyclically, since the local orientation in $\vec{G}_m$ is "in, out, in, out".
Then

$$\sum_{c \in \pi(\vec{G}_m)} 2^{\mu(c)} = \text{Holant}_{G_m}(\langle 2, 1, 0, 1, 0 \rangle)$$

An Arity 4 Gadget

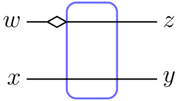


Quaternary gadget f. All vertices are assigned the AD_κ signature.

Think of $\kappa = 3$.

$$f\left(\begin{smallmatrix} w & z \\ x & y \end{smallmatrix}\right) = \begin{cases} 0 & \text{if } w = x = y = z \in [\kappa] \\ 1 & \text{if } w = x \neq y = z \in [\kappa] \\ 1 & \text{if } w = y \neq x = z \in [\kappa] \\ 0 & \text{if } w = z \neq x = y \in [\kappa] \\ 0 & \text{all other cases.} \end{cases}$$

Denote by $f = \langle 0, 1, 1, 0, 0 \rangle$.



A gadget with two parallel edges

Again think of $\kappa = 3$.

$$f_0\left(\begin{smallmatrix} w & z \\ x & y \end{smallmatrix}\right) = \begin{cases} 1 & \text{if } w = x = y = z \in [\kappa] \\ 0 & \text{if } w = x \neq y = z \in [\kappa] \\ 0 & \text{if } w = y \neq x = z \in [\kappa] \\ 1 & \text{if } w = z \neq x = y \in [\kappa] \\ 0 & \text{all other cases.} \end{cases}$$

Denote by $f_0 = \langle 1, 0, 0, 1, 0 \rangle$.

#P-hardness of #κ-EdgeColoring

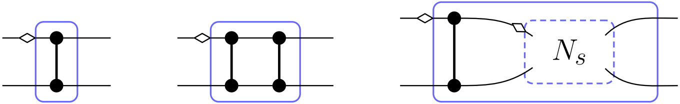
Theorem

#κ-EdgeColoring is #P-hard over planar κ-regular graphs for $\kappa \geq 3$.

Theorem

#κ-EdgeColoring is #P-hard over planar κ-regular graphs for $\kappa \geq 3$.

We reduce Pl-Holant($\langle 2, 1, 0, 1, 0 \rangle$) to Pl-Holant($AD_3$).

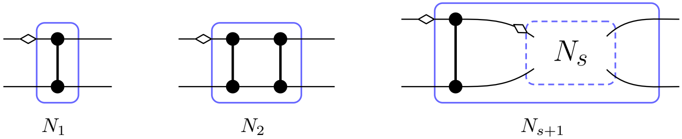


N_1 N_2 N_{s+1}

Theorem

#κ-EdgeColoring is #P-hard over planar κ-regular graphs for $\kappa \geq 3$.

We reduce Pl-Holant($\langle 2, 1, 0, 1, 0 \rangle$) to Pl-Holant($AD_3$).



Let f_s be the signature for the sth gadget. Then $f_s = M^s f_0$, where

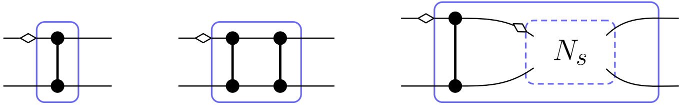
$$M = \begin{bmatrix} 0 & \kappa-1 & 0 & 0 & 0 \\ 1 & \kappa-2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and $f_0 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \end{bmatrix}^{\mathtt{T}}$.

Theorem

#κ-EdgeColoring is #P-hard over planar κ-regular graphs for $\kappa \geq 3$.

We reduce Pl-Holant($\langle 2, 1, 0, 1, 0 \rangle$) to Pl-Holant($AD_3$).



N_1 N_2 N_{s+1}

Let f_s be the signature for the sth gadget. Then $f_s = M^s f_0$, where

$$M = \begin{bmatrix} 0 & \kappa - 1 & 0 & 0 & 0 \\ 1 & \kappa - 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and $f_0 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \end{bmatrix}^{\mathtt{T}}$. One can easily verify that $f_1 = f$.

Eigenvalues and Eigenvectors

By the spectural decomposition $M = P\Lambda P^{-1}$, where

$$P = \begin{bmatrix} 1 & 1-\kappa & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \kappa-1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Eigenvalues and Eigenvectors

By the spectural decomposition $M = P\Lambda P^{-1}$, where

$$P = \begin{bmatrix} 1 & 1-\kappa & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \kappa-1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let $x = (\kappa-1)^{2s}$, then

$$f_{2s} = P\Lambda^{2s}P^{-1}f_0 = P \begin{bmatrix} x & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} P^{-1}f_0 = \begin{bmatrix} \frac{x-1}{\kappa}+1 \\ \frac{x-1}{\kappa} \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Eigenvalues and Eigenvectors

By the spectural decomposition $M = P\Lambda P^{-1}$, where

$$P = \begin{bmatrix} 1 & 1-\kappa & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \kappa-1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let $x = (\kappa - 1)^{2s}$, then

$$f_{2s} = P\Lambda^{2s}P^{-1}f_0 = P \begin{bmatrix} x & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} P^{-1}f_0 = \begin{bmatrix} \frac{x-1}{\kappa} + 1 \\ \frac{x-1}{\kappa} \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Note that if $x = 1 + \kappa$, then it is the Eulerian Signature $\mathcal{E} = \langle 2, 1, 0, 1, 0 \rangle$.

An Interpolation

Consider an instance Ω of Pl-Holant($\langle 2, 1, 0, 1, 0 \rangle$) on domain size κ.

An Interpolation

Consider an instance Ω of PI-Holant($\langle 2, 1, 0, 1, 0 \rangle$) on domain size κ.

Suppose $\langle 2, 1, 0, 1, 0 \rangle$ appears n times in Ω.

Consider an instance Ω of PI-Holant($\langle 2,1,0,1,0 \rangle$) on domain size κ.

Suppose $\langle 2,1,0,1,0 \rangle$ appears n times in Ω.

We construct from Ω a sequence of instances Ω_{2s} of PI-Holant(AD_κ) indexed by $s \geq 0$, by replacing each occurrence of $\langle 2,1,0,1,0 \rangle$ with the gadget f_{2s}.

An Interpolation

Consider an instance Ω of PI-Holant$(\langle 2, 1, 0, 1, 0\rangle)$ on domain size κ.

Suppose $\langle 2, 1, 0, 1, 0\rangle$ appears n times in Ω.

We construct from Ω a sequence of instances Ω_{2s} of PI-Holant(AD_κ) indexed by $s \geq 0$, by replacing each occurrence of $\langle 2, 1, 0, 1, 0\rangle$ with the gadget f_{2s}.

As a polynomial in $x = (\kappa - 1)^{2s}$, PI-Holant$_{\Omega_{2s}}$ is independent of s and has degree at most n with integer coefficients.

An Interpolation

Consider an instance Ω of PI-Holant($\langle 2, 1, 0, 1, 0 \rangle$) on domain size κ.

Suppose $\langle 2, 1, 0, 1, 0 \rangle$ appears n times in Ω.

We construct from Ω a sequence of instances Ω_{2s} of PI-Holant(AD_κ) indexed by $s \geq 0$, by replacing each occurrence of $\langle 2, 1, 0, 1, 0 \rangle$ with the gadget f_{2s}.

As a polynomial in $x = (\kappa - 1)^{2s}$, $\text{PI-Holant}_{\Omega_{2s}}$ is independent of s and has degree at most n with integer coefficients.

Using our oracle for PI-Holant(AD_κ), we can evaluate this polynomial at $n + 1$ distinct points $x = (\kappa - 1)^{2s}$ for $0 \leq s \leq n$. Then via polynomial interpolation, we can recover the coefficients of this polynomial efficiently.

An Interpolation

Consider an instance Ω of PI-Holant($\langle 2, 1, 0, 1, 0\rangle$) on domain size κ.

Suppose $\langle 2, 1, 0, 1, 0\rangle$ appears n times in Ω.

We construct from Ω a sequence of instances Ω_{2s} of PI-Holant(AD_κ) indexed by $s \geq 0$, by replacing each occurrence of $\langle 2, 1, 0, 1, 0\rangle$ with the gadget f_{2s}.

As a polynomial in $x = (\kappa - 1)^{2s}$, PI-Holant$_{\Omega_{2s}}$ is independent of s and has degree at most n with integer coefficients.

Using our oracle for PI-Holant(AD_κ), we can evaluate this polynomial at $n + 1$ distinct points $x = (\kappa - 1)^{2s}$ for $0 \leq s \leq n$. Then via polynomial interpolation, we can recover the coefficients of this polynomial efficiently.

Evaluating this polynomial at $x = 1 + \kappa$ gives the value of PI-Holant$_\Omega$.

We try to prove for every $f = \langle a, b, c \rangle$, Holant$(G; f)$ is either tractable or #P-hard.

Overview of Proof of Dichotomy Holant($G; f$)

We try to prove for every $f = \langle a, b, c \rangle$, Holant($G; f$) is either tractable or #P-hard.

1. We construct a special unary signature from the given $\langle a, b, c \rangle$.
2. We interpolate all binary succinct signatures of a certain type, assuming that we have special unary signature.
3. We utilize all these binary signatures to obtain a signature that we show is #P-hard.

Overview of Proof of Dichotomy Holant$(G; f)$

We try to prove for every $f = \langle a, b, c \rangle$, Holant$(G; f)$ is either tractable or #P-hard.

1. We construct a special unary signature from the given $\langle a, b, c \rangle$.
2. We interpolate all binary succinct signatures of a certain type, assuming that we have special unary signature.
3. We utilize all these binary signatures to obtain a signature that we show is #P-hard.

Along the way, we may find certain $\langle a, b, c \rangle$ does not allow us to achieve these steps.

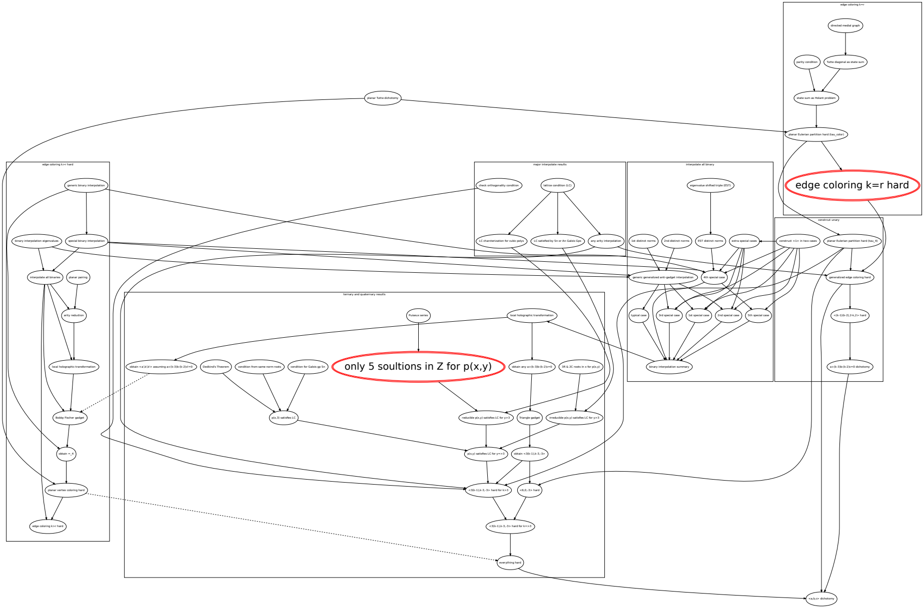
Overview of Proof of Dichotomy Holant$(G; f)$

We try to prove for every $f = \langle a, b, c \rangle$, Holant$(G; f)$ is either tractable or #P-hard.

1. We construct a special unary signature from the given $\langle a, b, c \rangle$.
2. We interpolate all binary succinct signatures of a certain type, assuming that we have special unary signature.
3. We utilize all these binary signatures to obtain a signature that we show is #P-hard.

Along the way, we may find certain $\langle a, b, c \rangle$ does not allow us to achieve these steps.

Instead, in those cases, we can directly prove that these problems are either in P or #P-hard (without the help of additional signtuares).



planar Tutte dichotomy
edge coloring k=r
directed medial graph
parity condition
Tutte diagonal as state sum
state sum as Holant problem
planar Eulerian partition hard (tau_color)
edge coloring k=r hard
edge coloring k>r hard
generic binary interpolation
binary interpolation eigenvalues
special binary interpolation
interpolate all binaries
planar pairing
arity reduction
local holographic transformation
Bobby Fischer gadget
obtain =_4
planar vertex coloring hard
edge coloring k>r hard
major interpolate results
check orthogonality condition
lattice condition (LC)
LC charcterization for cubic polys
LC satisfied by Sn or An Galois Gps
any arity interpolation
interpolate all binary
eigenvalue shifted triple (EST)
1st distinct norms
2nd distinct norms
EST distinct norms
extra special cases
generic generalized anti-gadget interpolation
4th special case
typical case
3rd special case
1st special case
2nd special case
5th special case
binary interpolation summary
constrcut unary
construct <1> in two cases
planar Eulerian partition hard (tau_4)
generalized edge coloring hard
<(k-1)(k-2),2-k,2> hard
a+(k-3)b-(k-2)c=0 dichotomy
ternary and quaternary resutls
Puiseux series
local holographic transformation
obtain <a',b',b'> assuming a+(k-3)b-(k-2)c!=0
Dedkind's Theorem
condition from same norm roots
condition for Galois gp Sn
only 5 soultions in Z for p(x,y)
obtain any a+(k-3)b-(k-2)c=0
3R & 2C roots in x for p(x,y)
p(x,3) satisfies LC
reducible p(x,y) satisfies LC for y>3
Triangle gadget
irreducible p(x,y) satisfies LC for y>3
p(x,y) satisfies LC for y=>3
obtain <3(k-1),k-3,-3>
<3(k-1),k-3,-3> hard for k>3
<6,0,-3> hard
<3(k-1),k-3,-3> hard for k=>3
everything hard
<a,b,c> dichotomy

Definition

We say that $\lambda_1, \lambda_2, \ldots, \lambda_\ell \in \mathbb{C} - \{0\}$ satisfy the lattice condition if for all $x \in \mathbb{Z}^\ell - \{\mathbf{0}\}$ with $\sum_{i=1}^{\ell} x_i = 0$, we have

$$\prod_{i=1}^{\ell} \lambda_i^{x_i} \neq 1.$$

Lattice Condition

Definition

We say that $\lambda_1, \lambda_2, \ldots, \lambda_\ell \in \mathbb{C} - \{0\}$ satisfy the lattice condition if for all $x \in \mathbb{Z}^\ell - \{\mathbf{0}\}$ with $\sum_{i=1}^{\ell} x_i = 0$, we have

$$\prod_{i=1}^{\ell} \lambda_i^{x_i} \neq 1.$$

Taking the logarithms, this is really a condition about linear independence $\{\log \lambda_i\}$ over $\mathbb{Q}$.

Theorem

If there exists an infinite sequence of planar $\mathcal{F}$-gates defined by an initial signature $s \in \mathbb{C}^{n\times 1}$ and a recurrence matrix $M \in \mathbb{C}^{n\times n}$ satisfying the following conditions,

1. *M is diagonalizable with n linearly independent eigenvectors;*
2. *s is not orthogonal to exactly ℓ of these linearly independent row eigenvectors of M with eigenvalues $\lambda_1, \ldots, \lambda_\ell$;*
3. *$\lambda_1, \ldots, \lambda_\ell$ satisfy the lattice condition;*

then

$$\text{Pl-Holant}(\mathcal{F} \cup \{f\}) \leq_T \text{Pl-Holant}(\mathcal{F})$$

for any signature f that is orthogonal to the $n - \ell$ of these linearly independent eigenvectors of M to which s is also orthogonal.

Theorem

If there exists an infinite sequence of planar $\mathcal{F}$-gates defined by an initial signature $s \in \mathbb{C}^{n \times 1}$ and a recurrence matrix $M \in \mathbb{C}^{n \times n}$ satisfying the following conditions,

1. *M is diagonalizable with n linearly independent eigenvectors;*
2. *s is not orthogonal to exactly ℓ of these linearly independent row eigenvectors of M with eigenvalues $\lambda_1, \ldots, \lambda_\ell$;*
3. *$\lambda_1, \ldots, \lambda_\ell$ satisfy the lattice condition;*

then

$$\text{Pl-Holant}(\mathcal{F} \cup \{f\}) \leq_T \text{Pl-Holant}(\mathcal{F})$$

for any signature f that is orthogonal to the $n - \ell$ of these linearly independent eigenvectors of M to which s is also orthogonal.

To prove our dichotomy we use a combinatorial construction with $n = 9$ and $\ell = 5$.

$$
\begin{bmatrix}
(\kappa-1)\left(\kappa^2+9\kappa-9\right) & 12(\kappa-3)(\kappa-1)^2 & (\kappa-3)^2(\kappa-1) & 2(\kappa-3)^2(\kappa-2)(\kappa-1) & (\kappa-3)^2(\kappa-1) & 2(\kappa-3)^2(\kappa-2)(\kappa-1) & (\kappa-1)(2\kappa-3)(4\kappa-3) & 6(\kappa-3)(\kappa-2)(\kappa-1)^2 & (\kappa-3)^3(\kappa-2)(\kappa-1) \\
3(\kappa-3)(\kappa-1) & 3\kappa^3-28\kappa^2+60\kappa-36 & -(\kappa-3)(2\kappa-3) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & -(\kappa-3)(2\kappa-3) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)(\kappa-1)^2 & (\kappa-2)\left(\kappa^3-14\kappa^2+30\kappa-18\right) & -(\kappa-3)^2(\kappa-2)(2\kappa-3) \\
(2\kappa-3)(4\kappa-3) & 12(\kappa-3)(\kappa-1)^2 & (\kappa-3)^2(\kappa-1) & 2(\kappa-3)^2(\kappa-2)(\kappa-1) & (\kappa-3)^2(\kappa-1) & 2(\kappa-3)^2(\kappa-2)(\kappa-1) & 9\kappa^3-26\kappa^2+27\kappa-9 & 6(\kappa-3)(\kappa-2)(\kappa-1)^2 & (\kappa-3)^3(\kappa-2)(\kappa-1) \\
3(\kappa-3)(\kappa-1) & 2\left(\kappa^3-14\kappa^2+30\kappa-18\right) & -(\kappa-3)(2\kappa-3) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & -(\kappa-3)(2\kappa-3) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)(\kappa-1)^2 & (\kappa-3)\left(\kappa^3-12\kappa^2+22\kappa-12\right) & -(\kappa-3)^2(\kappa-2)(2\kappa-3) \\
(\kappa-3)^2 & -4(\kappa-3)(2\kappa-3) & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & \kappa^3+3\kappa-9 & 6(\kappa-3)(\kappa-2) & (\kappa-3)^2(\kappa-1) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)^2(\kappa-2) \\
(\kappa-3)^2 & -4(\kappa-3)(2\kappa-3) & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & 3(\kappa-3) & \kappa^3+6\kappa^2-30\kappa+36 & (\kappa-3)^2(\kappa-1) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)^2(\kappa-2) \\
(\kappa-3)^2 & -4(\kappa-3)(2\kappa-3) & \kappa^3+3\kappa-9 & 6(\kappa-3)(\kappa-2) & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & (\kappa-3)^2(\kappa-1) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)^2(\kappa-2) \\
(\kappa-3)^2 & -4(\kappa-3)(2\kappa-3) & 3(\kappa-3) & \kappa^3+6\kappa^2-30\kappa+36 & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & (\kappa-3)^2(\kappa-1) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)^2(\kappa-2) \\
(\kappa-3)^2 & -4(\kappa-3)(2\kappa-3) & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & (\kappa-3)^2(\kappa-1) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & (2\kappa-3)\left(2\kappa^2-9\kappa+18\right)
\end{bmatrix}
$$

$$\begin{bmatrix}
(\kappa-1)(\kappa^2+9\kappa-9) & 12(\kappa-3)(\kappa-1)^2 & (\kappa-3)^2(\kappa-1) & 2(\kappa-3)^2(\kappa-2)(\kappa-1) & (\kappa-3)^2(\kappa-1) & 2(\kappa-3)^2(\kappa-2)(\kappa-1) & (\kappa-1)(2\kappa-3)(4\kappa-3) & 6(\kappa-3)(\kappa-2)(\kappa-1)^2 & (\kappa-3)^3(\kappa-2)(\kappa-1) \\
3(\kappa-3)(\kappa-1) & 3\kappa^3-28\kappa^2+60\kappa-36 & -(\kappa-3)(2\kappa-3) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & -(\kappa-3)(2\kappa-3) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)(\kappa-1)^2 & (\kappa-2)(\kappa^3-14\kappa^2+30\kappa-18) & -(\kappa-3)^2(\kappa-2)(2\kappa-3) \\
(2\kappa-3)(4\kappa-3) & 12(\kappa-3)(\kappa-1)^2 & (\kappa-3)^2(\kappa-1) & 2(\kappa-3)^2(\kappa-2)(\kappa-1) & (\kappa-3)^2(\kappa-1) & 2(\kappa-3)^2(\kappa-2)(\kappa-1) & 9\kappa^3-26\kappa^2+27\kappa-9 & 6(\kappa-3)(\kappa-2)(\kappa-1)^2 & (\kappa-3)^3(\kappa-2)(\kappa-1) \\
3(\kappa-3)(\kappa-1) & 2(\kappa^3-14\kappa^2+30\kappa-18) & -(\kappa-3)(2\kappa-3) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & -(\kappa-3)(2\kappa-3) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)(\kappa-1)^2 & (\kappa-3)(\kappa^3-12\kappa^2+22\kappa-12) & -(\kappa-3)^2(\kappa-2)(2\kappa-3) \\
(\kappa-3)^2 & -4(\kappa-3)(2\kappa-3) & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & \kappa^3+3\kappa-9 & 6(\kappa-3)(\kappa-2) & (\kappa-3)^2(\kappa-1) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)^2(\kappa-2) \\
(\kappa-3)^2 & -4(\kappa-3)(2\kappa-3) & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & 3(\kappa-3) & \kappa^3+6\kappa^2-30\kappa+36 & (\kappa-3)^2(\kappa-1) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)^2(\kappa-2) \\
(\kappa-3)^2 & -4(\kappa-3)(2\kappa-3) & \kappa^3+3\kappa-9 & 6(\kappa-3)(\kappa-2) & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & (\kappa-3)^2(\kappa-1) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)^2(\kappa-2) \\
(\kappa-3)^2 & -4(\kappa-3)(2\kappa-3) & 3(\kappa-3) & \kappa^3+6\kappa^2-30\kappa+36 & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & (\kappa-3)^2(\kappa-1) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)^2(\kappa-2) \\
(\kappa-3)^2 & -4(\kappa-3)(2\kappa-3) & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & (\kappa-3)^2(\kappa-1) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & (2\kappa-3)(2\kappa^2-9\kappa+18)
\end{bmatrix}$$

The characteristic polynomial of M is $\lambda_M(x,\kappa) = (x-\kappa^3)^4 f(x,\kappa)$, where $f(x,\kappa) = x^5 - \kappa^6(2\kappa-1)x^3 - \kappa^9(\kappa^2-2\kappa+3)x^2 + (\kappa-2)(\kappa-1)\kappa^{12}x + (\kappa-1)^3\kappa^{15}$.

$$\begin{bmatrix}
(\kappa-1)\left(\kappa^2+9\kappa-9\right) & 12(\kappa-3)(\kappa-1)^2 & (\kappa-3)^2(\kappa-1) & 2(\kappa-3)^2(\kappa-2)(\kappa-1) & (\kappa-3)^2(\kappa-1) & 2(\kappa-3)^2(\kappa-2)(\kappa-1) & (\kappa-1)(2\kappa-3)(4\kappa-3) & 6(\kappa-3)(\kappa-2)(\kappa-1)^2 & (\kappa-3)^3(\kappa-2)(\kappa-1) \\
3(\kappa-3)(\kappa-1) & 3\kappa^3-28\kappa^2+60\kappa-36 & -(\kappa-3)(2\kappa-3) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & -(\kappa-3)(2\kappa-3) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)(\kappa-1)^2 & (\kappa-2)\left(\kappa^3-14\kappa^2+30\kappa-18\right) & -(\kappa-3)^2(\kappa-2)(2\kappa-3) \\
(2\kappa-3)(4\kappa-3) & 12(\kappa-3)(\kappa-1)^2 & (\kappa-3)^2(\kappa-1) & 2(\kappa-3)^2(\kappa-2)(\kappa-1) & (\kappa-3)^2(\kappa-1) & 2(\kappa-3)^2(\kappa-2)(\kappa-1) & 9\kappa^3-26\kappa^2+27\kappa-9 & 6(\kappa-3)(\kappa-2)(\kappa-1)^2 & (\kappa-3)^3(\kappa-2)(\kappa-1) \\
3(\kappa-3)(\kappa-1) & 2\left(\kappa^3-14\kappa^2+30\kappa-18\right) & -(\kappa-3)(2\kappa-3) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & -(\kappa-3)(2\kappa-3) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)(\kappa-1)^2 & (\kappa-3)\left(\kappa^3-12\kappa^2+22\kappa-12\right) & -(\kappa-3)^2(\kappa-2)(2\kappa-3) \\
(\kappa-3)^2 & -4(\kappa-3)(2\kappa-3) & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & \kappa^3+3\kappa-9 & 6(\kappa-3)(\kappa-2) & (\kappa-3)^2(\kappa-1) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)^2(\kappa-2) \\
(\kappa-3)^2 & -4(\kappa-3)(2\kappa-3) & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & 3(\kappa-3) & \kappa^3+6\kappa^2-30\kappa+36 & (\kappa-3)^2(\kappa-1) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)^2(\kappa-2) \\
(\kappa-3)^2 & -4(\kappa-3)(2\kappa-3) & \kappa^3+3\kappa-9 & 6(\kappa-3)(\kappa-2) & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & (\kappa-3)^2(\kappa-1) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)^2(\kappa-2) \\
(\kappa-3)^2 & -4(\kappa-3)(2\kappa-3) & 3(\kappa-3) & \kappa^3+6\kappa^2-30\kappa+36 & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & (\kappa-3)^2(\kappa-1) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)^2(\kappa-2) \\
(\kappa-3)^2 & -4(\kappa-3)(2\kappa-3) & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & (\kappa-3)^2(\kappa-1) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & (2\kappa-3)\left(2\kappa^2-9\kappa+18\right)
\end{bmatrix}$$

The characteristic polynomial of M is $\lambda_M(x,\kappa) = (x-\kappa^3)^4 f(x,\kappa)$, where $f(x,\kappa) = x^5 - \kappa^6(2\kappa-1)x^3 - \kappa^9(\kappa^2-2\kappa+3)x^2 + (\kappa-2)(\kappa-1)\kappa^{12}x + (\kappa-1)^3\kappa^{15}$.

After setting

$$\tilde{f}(x,\kappa) = \frac{1}{\kappa^{15}} f(\kappa^3 x,\kappa) = x^5 - (2\kappa-1)x^3 - (\kappa^2-2\kappa+3)x^2 + (\kappa-2)(\kappa-1)x + (\kappa-1)^3$$

$$\begin{bmatrix}
(\kappa-1)\left(\kappa^2+9\kappa-9\right) & 12(\kappa-3)(\kappa-1)^2 & (\kappa-3)^2(\kappa-1) & 2(\kappa-3)^2(\kappa-2)(\kappa-1) & (\kappa-3)^2(\kappa-1) & 2(\kappa-3)^2(\kappa-2)(\kappa-1) & (\kappa-1)(2\kappa-3)(4\kappa-3) & 6(\kappa-3)(\kappa-2)(\kappa-1)^2 & (\kappa-3)^3(\kappa-2)(\kappa-1) \\
3(\kappa-3)(\kappa-1) & 3\kappa^3-28\kappa^2+60\kappa-36 & -(\kappa-3)(2\kappa-3) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & -(\kappa-3)(2\kappa-3) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)(\kappa-1)^2 & (\kappa-2)\left(\kappa^3-14\kappa^2+30\kappa-18\right) & -(\kappa-3)^2(\kappa-2)(2\kappa-3) \\
(2\kappa-3)(4\kappa-3) & 12(\kappa-3)(\kappa-1)^2 & (\kappa-3)^2(\kappa-1) & 2(\kappa-3)^2(\kappa-2)(\kappa-1) & (\kappa-3)^2(\kappa-1) & 2(\kappa-3)^2(\kappa-2)(\kappa-1) & 9\kappa^3-26\kappa^2+27\kappa-9 & 6(\kappa-3)(\kappa-2)(\kappa-1)^2 & (\kappa-3)^3(\kappa-2)(\kappa-1) \\
3(\kappa-3)(\kappa-1) & 2\left(\kappa^3-14\kappa^2+30\kappa-18\right) & -(\kappa-3)(2\kappa-3) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & -(\kappa-3)(2\kappa-3) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)(\kappa-1)^2 & (\kappa-3)\left(\kappa^3-12\kappa^2+22\kappa-12\right) & -(\kappa-3)^2(\kappa-2)(2\kappa-3) \\
(\kappa-3)^2 & -4(\kappa-3)(2\kappa-3) & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & \kappa^3+3\kappa-9 & 6(\kappa-3)(\kappa-2) & (\kappa-3)^2(\kappa-1) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)^2(\kappa-2) \\
(\kappa-3)^2 & -4(\kappa-3)(2\kappa-3) & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & 3(\kappa-3) & \kappa^3+6\kappa^2-30\kappa+36 & (\kappa-3)^2(\kappa-1) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)^2(\kappa-2) \\
(\kappa-3)^2 & -4(\kappa-3)(2\kappa-3) & \kappa^3+3\kappa-9 & 6(\kappa-3)(\kappa-2) & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & (\kappa-3)^2(\kappa-1) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)^2(\kappa-2) \\
(\kappa-3)^2 & -4(\kappa-3)(2\kappa-3) & 3(\kappa-3) & \kappa^3+6\kappa^2-30\kappa+36 & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & (\kappa-3)^2(\kappa-1) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)^2(\kappa-2) \\
(\kappa-3)^2 & -4(\kappa-3)(2\kappa-3) & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & (\kappa-3)^2(\kappa-1) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & (2\kappa-3)\left(2\kappa^2-9\kappa+18\right)
\end{bmatrix}$$

The characteristic polynomial of M is $\lambda_M(x,\kappa) = (x-\kappa^3)^4 f(x,\kappa)$, where $f(x,\kappa) = x^5 - \kappa^6(2\kappa-1)x^3 - \kappa^9(\kappa^2-2\kappa+3)x^2 + (\kappa-2)(\kappa-1)\kappa^{12}x + (\kappa-1)^3\kappa^{15}$.

After setting

$$\tilde{f}(x,\kappa) = \frac{1}{\kappa^{15}} f(\kappa^3 x,\kappa) = x^5-(2\kappa-1)x^3-(\kappa^2-2\kappa+3)x^2+(\kappa-2)(\kappa-1)x+(\kappa-1)^3$$

and replacing κ by $y+1$ we get

$$p(x,y) = x^5 - (2y+1)x^3 - (y^2+2)x^2 + (y-1)yx + y^3.$$

$$
\begin{bmatrix}
(\kappa-1)\left(\kappa^2+9\kappa-9\right) & 12(\kappa-3)(\kappa-1)^2 & (\kappa-3)^2(\kappa-1) & 2(\kappa-3)^2(\kappa-2)(\kappa-1) & (\kappa-3)^2(\kappa-1) & 2(\kappa-3)^2(\kappa-2)(\kappa-1) & (\kappa-1)(2\kappa-3)(4\kappa-3) & 6(\kappa-3)(\kappa-2)(\kappa-1)^2 & (\kappa-3)^3(\kappa-2)(\kappa-1) \\
3(\kappa-3)(\kappa-1) & 3\kappa^3-28\kappa^2+60\kappa-36 & -(\kappa-3)(2\kappa-3) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & -(\kappa-3)(2\kappa-3) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)(\kappa-1)^2 & (\kappa-2)\left(\kappa^3-14\kappa^2+30\kappa-18\right) & -(\kappa-3)^2(\kappa-2)(2\kappa-3) \\
(2\kappa-3)(4\kappa-3) & 12(\kappa-3)(\kappa-1)^2 & (\kappa-3)^2(\kappa-1) & 2(\kappa-3)^2(\kappa-2)(\kappa-1) & (\kappa-3)^2(\kappa-1) & 2(\kappa-3)^2(\kappa-2)(\kappa-1) & 9\kappa^3-26\kappa^2+27\kappa-9 & 6(\kappa-3)(\kappa-2)(\kappa-1)^2 & (\kappa-3)^3(\kappa-2)(\kappa-1) \\
3(\kappa-3)(\kappa-1) & 2\left(\kappa^3-14\kappa^2+30\kappa-18\right) & -(\kappa-3)(2\kappa-3) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & -(\kappa-3)(2\kappa-3) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)(\kappa-1)^2 & (\kappa-3)\left(\kappa^3-12\kappa^2+22\kappa-12\right) & -(\kappa-3)^2(\kappa-2)(2\kappa-3) \\
(\kappa-3)^2 & -4(\kappa-3)(2\kappa-3) & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & \kappa^3+3\kappa-9 & 6(\kappa-3)(\kappa-2) & (\kappa-3)^2(\kappa-1) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)^2(\kappa-2) \\
(\kappa-3)^2 & -4(\kappa-3)(2\kappa-3) & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & 3(\kappa-3) & \kappa^3+6\kappa^2-30\kappa+36 & (\kappa-3)^2(\kappa-1) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)^2(\kappa-2) \\
(\kappa-3)^2 & -4(\kappa-3)(2\kappa-3) & \kappa^3+3\kappa-9 & 6(\kappa-3)(\kappa-2) & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & (\kappa-3)^2(\kappa-1) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)^2(\kappa-2) \\
(\kappa-3)^2 & -4(\kappa-3)(2\kappa-3) & 3(\kappa-3) & \kappa^3+6\kappa^2-30\kappa+36 & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & (\kappa-3)^2(\kappa-1) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & 3(\kappa-3)^2(\kappa-2) \\
(\kappa-3)^2 & -4(\kappa-3)(2\kappa-3) & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & 3(\kappa-3) & 6(\kappa-3)(\kappa-2) & (\kappa-3)^2(\kappa-1) & -2(\kappa-3)(\kappa-2)(2\kappa-3) & (2\kappa-3)\left(2\kappa^2-9\kappa+18\right)
\end{bmatrix}
$$

The characteristic polynomial of M is $\lambda_M(x,\kappa) = (x-\kappa^3)^4 f(x,\kappa)$, where $f(x,\kappa) = x^5 - \kappa^6(2\kappa-1)x^3 - \kappa^9(\kappa^2-2\kappa+3)x^2 + (\kappa-2)(\kappa-1)\kappa^{12}x + (\kappa-1)^3\kappa^{15}$.

After setting

$$\tilde{f}(x,\kappa) = \frac{1}{\kappa^{15}} f(\kappa^3 x,\kappa) = x^5-(2\kappa-1)x^3-(\kappa^2-2\kappa+3)x^2+(\kappa-2)(\kappa-1)x+(\kappa-1)^3$$

and replacing κ by $y+1$ we get

$$p(x,y) = x^5 - (2y+1)x^3 - (y^2+2)x^2 + (y-1)yx + y^3.$$

We want to prove that for all integer $y \geq 4$, the roots of $p(x,y)$ satisfy the lattice condition.

Irreducible over $\mathbb{Q}[x]$?

We suspect that for any integer $y \geq 4$, $p(x, y)$ is in fact irreducible in $\mathbb{Q}[x]$.

Irreducible over $\mathbb{Q}[x]$?

We suspect that for any integer $y \geq 4$, $p(x, y)$ is in fact irreducible in $\mathbb{Q}[x]$.

Can't prove that.

Irreducible over $\mathbb{Q}[x]$?

We suspect that for any integer $y \geq 4$, $p(x, y)$ is in fact irreducible in $\mathbb{Q}[x]$.

Can't prove that.

We know five integer solutions $(x, y) \in \mathbb{Z}^2$, so for these five values of $y \in \mathbb{Z}$, $p(x, y)$ is reducible as a polynomial in x:

$$p(x,y) = \begin{cases} (x-1)(x^4+x^3+2x^2-x+1) & y=-1 \\ x^2(x^3-x-2) & y=0 \\ (x+1)(x^4-x^3-2x^2-x+1) & y=1 \\ (x-1)(x^2-x-4)(x^2+2x+2) & y=2 \\ (x-3)(x^4+3x^3+2x^2-5x-9) & y=3. \end{cases}$$

Effective Siegel's Theorem?

We will show there are no other integer solutions $(x, y) \in \mathbb{Z}^2$.

Effective Siegel's Theorem?

We will show there are no other integer solutions $(x, y) \in \mathbb{Z}^2$.

This means, for all integer $y \geq 4$, $p(x, y)$ is either irreducible or is a product of two irreducible polynomials of degree 2 and 3 respectively.

Effective Siegel's Theorem?

We will show there are no other integer solutions $(x, y) \in \mathbb{Z}^2$.

This means, for all integer $y \geq 4$, $p(x, y)$ is either irreducible or is a product of two irreducible polynomials of degree 2 and 3 respectively.

Note that, by Gauss Lemma, for any integer y, the monic polynomial $p(x, y)$ in x is irreducible over $\mathbb{Z}$ iff it is irreducible over $\mathbb{Q}$.

We will show there are no other integer solutions $(x, y) \in \mathbb{Z}^2$.

This means, for all integer $y \geq 4$, $p(x, y)$ is either irreducible or is a product of two irreducible polynomials of degree 2 and 3 respectively.

Note that, by Gauss Lemma, for any integer y, the monic polynomial $p(x, y)$ in x is irreducible over $\mathbb{Z}$ iff it is irreducible over $\mathbb{Q}$.

Lemma

Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree $n \geq 2$. If the Galois group of f over $\mathbb{Q}$ is S_n or A_n and the roots of f do not all have the same complex norm, then the roots of f satisfy the lattice condition.

Irreducible Quintic

Lemma

For any integer $y \geq 1$, the polynomial $p(x, y)$ has three distinct real roots and two nonreal complex conjugate roots in x.

Proof by discriminant.

Irreducible Quintic

Lemma

For any integer $y \geq 1$, the polynomial $p(x, y)$ has three distinct real roots and two nonreal complex conjugate roots in x.

Proof by discriminant.

Lemma

For any integer $y \geq 4$, if $p(x, y)$ is irreducible in $\mathbb{Q}[x]$, then the roots of $p(x, y)$ satisfy the lattice condition.

Proof.

Three distinct real roots do not have the same complex norm.
An irreducible polynomial of prime degree n with exactly two nonreal roots has S_n as its Galois group over $\mathbb{Q}$.
Hence they satisfy the lattice condition. □

What if it is Reducible?

Some more Galois Theory is needed if it is a product of two irreducible polynomials of degree 2 and 3.

What if it is Reducible?

Some more Galois Theory is needed if it is a product of two irreducible polynomials of degree 2 and 3.

They still satisfy the lattice condition.

But Why No Other Integer Solutions?

Lemma

The only integer solutions to $p(x, y) = 0$ *are*

$$(1, -1), (0, 0), (-1, 1), (1, 2), \textit{ and } (3, 3).$$

But Why No Other Integer Solutions?

Lemma

The only integer solutions to $p(x, y) = 0$ *are*

$$(1, -1), (0, 0), (-1, 1), (1, 2), \textit{and } (3, 3).$$

Let $p(a, b) = 0$ with $a \neq 0$.

But Why No Other Integer Solutions?

Lemma

The only integer solutions to $p(x, y) = 0$ *are*

$$(1, -1), (0, 0), (-1, 1), (1, 2), \textit{and } (3, 3).$$

Let $p(a, b) = 0$ with $a \neq 0$.

$$p(x, y) = x^5 - (2y + 1)x^3 - (y^2 + 2)x^2 + (y - 1)yx + y^3.$$

But Why No Other Integer Solutions?

Lemma

The only integer solutions to $p(x, y) = 0$ *are*

$$(1, -1), (0, 0), (-1, 1), (1, 2), \textit{ and } (3, 3).$$

Let $p(a, b) = 0$ with $a \neq 0$.

$$p(x, y) = x^5 - (2y + 1)x^3 - (y^2 + 2)x^2 + (y - 1)yx + y^3.$$

One can show that $a | b^2$.

But Why No Other Integer Solutions?

Lemma

The only integer solutions to $p(x, y) = 0$ are

$$(1, -1), (0, 0), (-1, 1), (1, 2), \textit{ and } (3, 3).$$

Let $p(a, b) = 0$ with $a \neq 0$.

$$p(x, y) = x^5 - (2y + 1)x^3 - (y^2 + 2)x^2 + (y - 1)yx + y^3.$$

One can show that $a | b^2$.

Consider

$$g_1(x, y) = y - x^2 \quad \text{and} \quad g_2(x, y) = \frac{y^2}{x} + y - x^2 + 1.$$

(This particular choice is due to Aaron Levin.) Whenever $p(a, b) = 0$ with $a \neq 0$, $g_1(a, b)$ and $g_2(a, b)$ are integers. However, we show that if $a \leq -3$ or $a \geq 17$, then either $g_1(a, b)$ or $g_2(a, b)$ is not an integer.

Puiseux Series

The Puiseux series expansions for $p(x, y)$ are

$$y_1(x) = x^2 + 2x^{-1} + 2x^{-2} - 6x^{-4} - 18x^{-5} + O(x^{-6})$$
$$y_2(x) = x^{3/2} - \frac{1}{2}x + \frac{1}{8}x^{1/2} - \frac{65}{128}x^{-1/2} - x^{-1} - \frac{1471}{1024}x^{-3/2} - x^{-2} + O(x^{-5/2})$$
$$y_3(x) = -x^{3/2} - \frac{1}{2}x - \frac{1}{8}x^{1/2} + \frac{65}{128}x^{-1/2} - x^{-1} + \frac{1471}{1024}x^{-3/2} - x^{-2} + O(x^{-5/2})$$

Puiseux Series

The Puiseux series expansions for $p(x, y)$ are

$$y_1(x) = x^2 + 2x^{-1} + 2x^{-2} - 6x^{-4} - 18x^{-5} + O(x^{-6})$$
$$y_2(x) = x^{3/2} - \frac{1}{2}x + \frac{1}{8}x^{1/2} - \frac{65}{128}x^{-1/2} - x^{-1} - \frac{1471}{1024}x^{-3/2} - x^{-2} + O(x^{-5/2})$$
$$y_3(x) = -x^{3/2} - \frac{1}{2}x - \frac{1}{8}x^{1/2} + \frac{65}{128}x^{-1/2} - x^{-1} + \frac{1471}{1024}x^{-3/2} - x^{-2} + O(x^{-5/2})$$

If we substitute say $y_2(x)$ in $g_2(x, y_2(x))$, we get $O(x^{-1/2})$, where the multiplier in the O-notation is bounded both above and below by a nonzero constant in absolute value.

Puiseux Series

The Puiseux series expansions for $p(x, y)$ are

$$y_1(x) = x^2 + 2x^{-1} + 2x^{-2} - 6x^{-4} - 18x^{-5} + O(x^{-6})$$
$$y_2(x) = x^{3/2} - \frac{1}{2}x + \frac{1}{8}x^{1/2} - \frac{65}{128}x^{-1/2} - x^{-1} - \frac{1471}{1024}x^{-3/2} - x^{-2} + O(x^{-5/2})$$
$$y_3(x) = -x^{3/2} - \frac{1}{2}x - \frac{1}{8}x^{1/2} + \frac{65}{128}x^{-1/2} - x^{-1} + \frac{1471}{1024}x^{-3/2} - x^{-2} + O(x^{-5/2})$$

If we substitute say $y_2(x)$ in $g_2(x, y_2(x))$, we get $O(x^{-1/2})$, where the multiplier in the O-notation is bounded both above and below by a nonzero constant in absolute value.

So for large x, it is non-zero and non-integral.

Puiseux Series

The Puiseux series expansions for $p(x, y)$ are

$$y_1(x) = x^2 + 2x^{-1} + 2x^{-2} - 6x^{-4} - 18x^{-5} + O(x^{-6})$$
$$y_2(x) = x^{3/2} - \frac{1}{2}x + \frac{1}{8}x^{1/2} - \frac{65}{128}x^{-1/2} - x^{-1} - \frac{1471}{1024}x^{-3/2} - x^{-2} + O(x^{-5/2})$$
$$y_3(x) = -x^{3/2} - \frac{1}{2}x - \frac{1}{8}x^{1/2} + \frac{65}{128}x^{-1/2} - x^{-1} + \frac{1471}{1024}x^{-3/2} - x^{-2} + O(x^{-5/2})$$

If we substitute say $y_2(x)$ in $g_2(x, y_2(x))$, we get $O(x^{-1/2})$, where the multiplier in the O-notation is bounded both above and below by a nonzero constant in absolute value.

So for large x, it is non-zero and non-integral.

Hence there are no large integral solutions.

Some papers can be found on my web site
`http://www.cs.wisc.edu/~jyc`

THANK YOU!