The Effectiveness of Convex Programming in the Information and Physical Sciences

Emmanuel Candès

Simons Institute Open Lecture, UC Berkeley, October 2013

Three stories

Three stories

Today I want to tell you three stories from my life.
That's it. No big deal. Just three stories

Steve Jobs

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Often have missing information:

(1) Missing phase (phase retrieval)
(2) Missing and/or corrupted entries in data matrix (robust PCA)
(3) Missing high-frequency spectrum (super-resolution)

Makes signal/data recovery difficult

This lecture

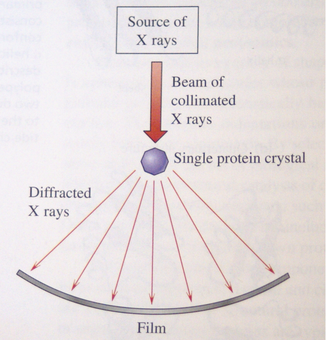
Convex programming usually (but not always) returns the right answer!

Story # 1: Phase Retrieval

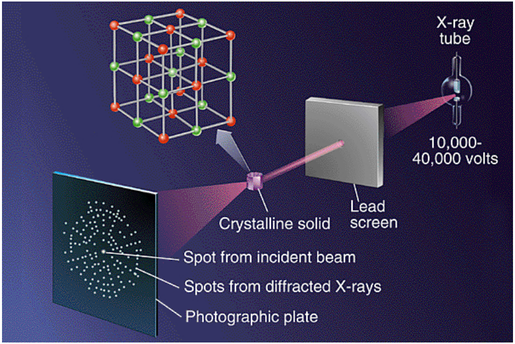
Collaborators: Y. Eldar, X. Li, T. Strohmer, V. Voroninski

X-ray crystallography

Method for determining atomic structure within a crystal



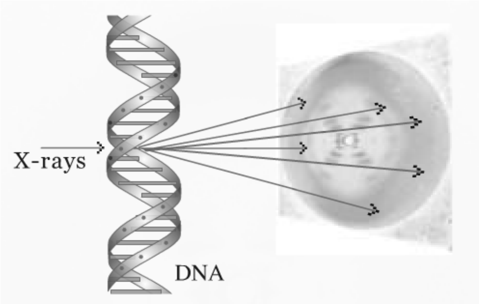
principle



typical setup

10 Nobel Prizes in X-ray crystallography, and counting...

Importance



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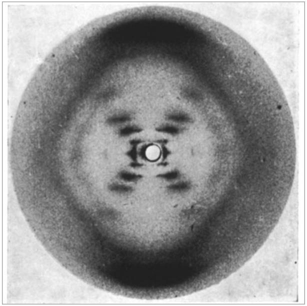
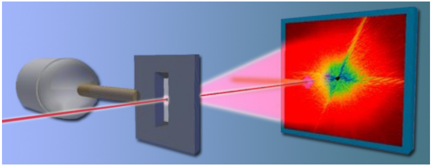


Franklin's photograph

Missing phase problem

Detectors only record intensities of diffracted rays

$\rightarrow$ magnitude measurements only!



Fraunhofer diffraction $\longrightarrow$ intensity of electrical field

$$|\hat{x}(f_1, f_2)|^2 = \left| \int x(t_1, t_2) e^{-i2\pi(f_1 t_1 + f_2 t_2)} \, dt_1 dt_2 \right|^2$$

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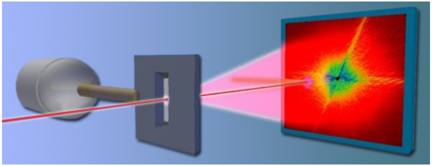

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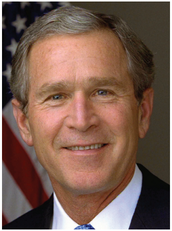
Phase retrieval problem (inversion)

How can we recover the phase (or equivalently signal $x(t_1, t_2)$) from $|\hat{x}(f_1, f_2)|$?

About the importance of phase...



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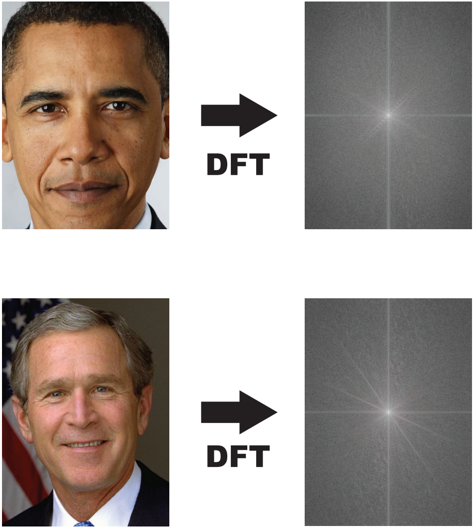
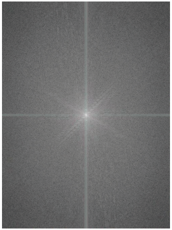
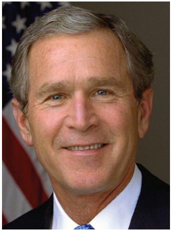


About the importance of phase...

DFT



keep magnitude
swap phase



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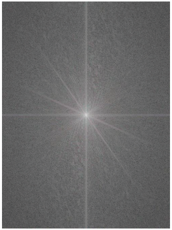

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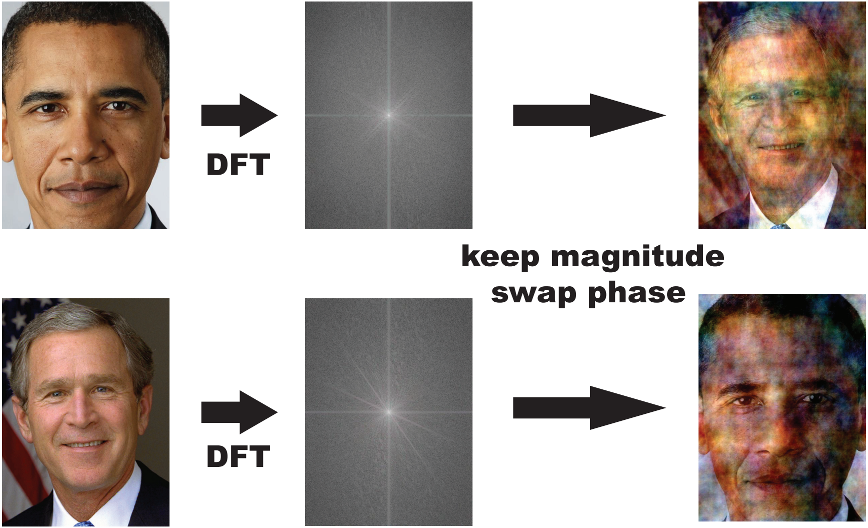
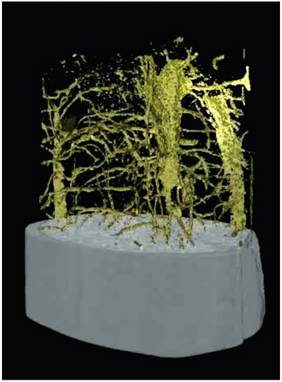


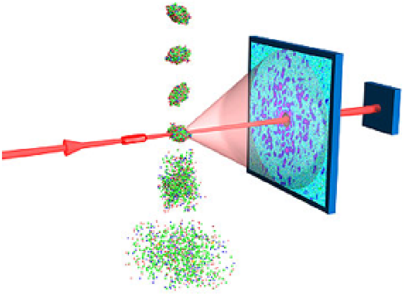
X-ray imaging: now and then

Röntgen (1895)



Dierolf (2010)

Ultrashort X-ray pulses



Imaging single large protein complexes

Discrete mathematical model

- Phaseless measurements about $x_0 \in \mathbb{C}^n$

$$b_k = |\langle a_k, x_0 \rangle|^2 \qquad k \in \{1, \ldots, m\} = [m]$$

- Phase retrieval is feasibility problem

$$\begin{array}{ll} \text{find} & x \\ \text{subject to} & |\langle a_k, x \rangle|^2 = b_k \quad k \in [m] \end{array}$$

- Solving quadratic equations is NP hard in general

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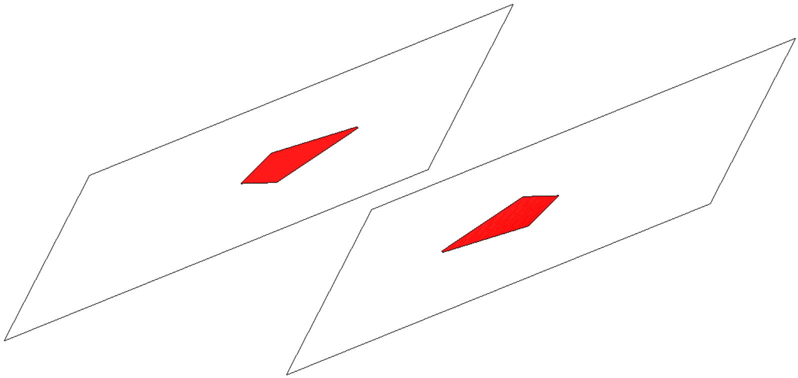
Standard approach: Gerchberg Saxton (or Fienup) iterative algorithm

- Sometimes works well
- Sometimes does not

Quadratic equations: geometric view I



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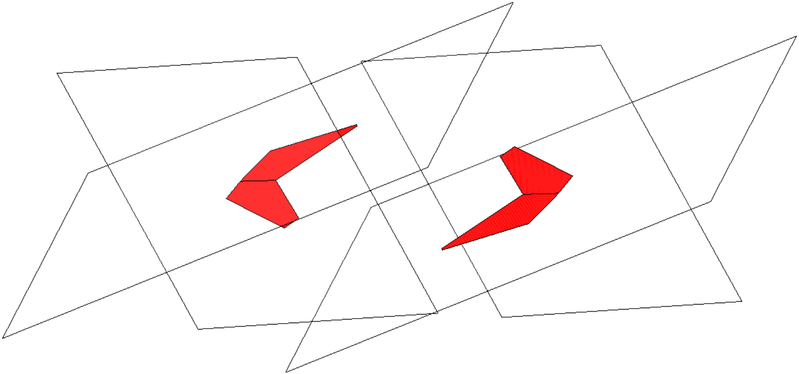

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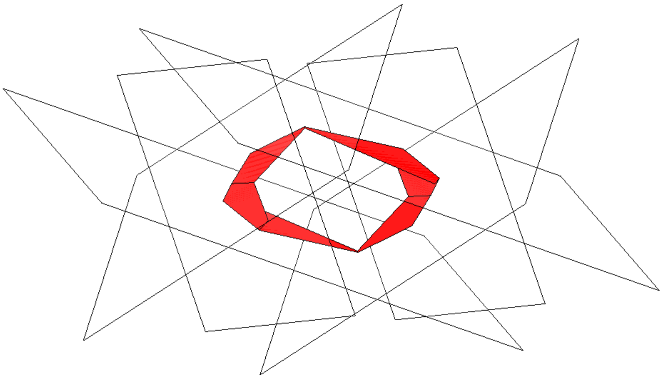

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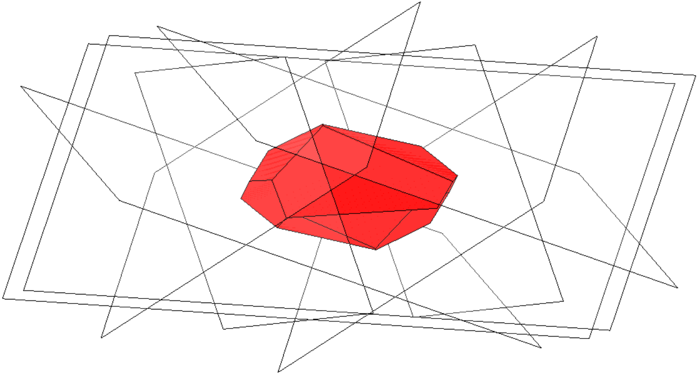

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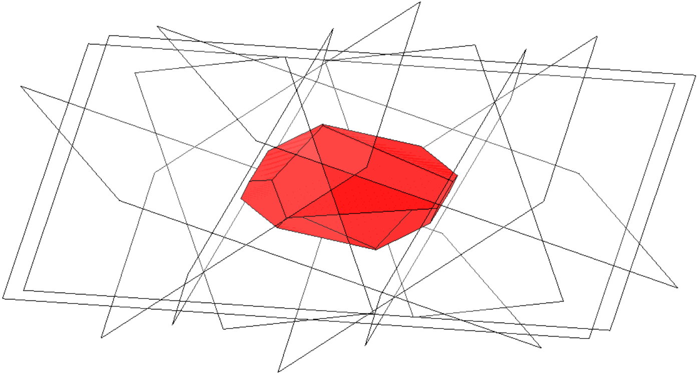

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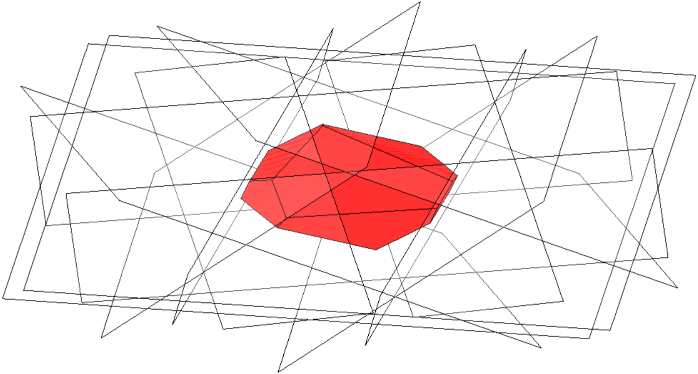
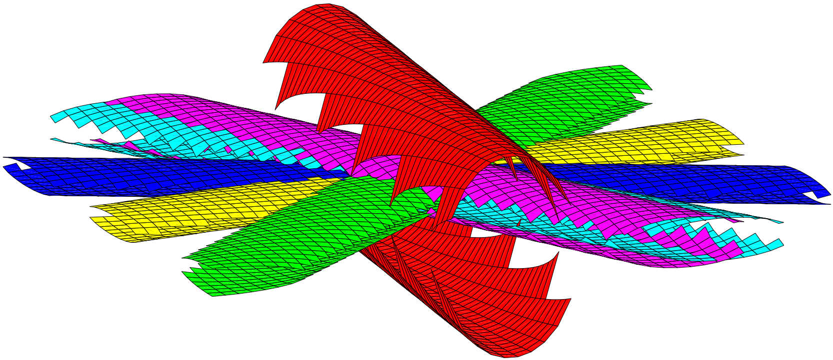


Quadratic equations: geometric view II



PhaseLift

$$|\langle a_k, x\rangle|^2 = b_k \qquad k \in [m]$$

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Lifting: $X = xx^*$

$$|\langle a_k, x\rangle|^2 = \mathsf{Tr}(x^* a_k a_k^* x) = \mathsf{Tr}(a_k a_k^* x x^*) := \mathsf{Tr}(A_k X) \qquad a_k a_k^* = A_k$$

Turns quadratic measurements into linear measurements about xx^*

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Phase retrieval: equivalent formulation

$$\begin{array}{lll} \text{find} & X & \\ \text{s. t.} & \mathrm{Tr}(A_k X) = b_k \quad k \in [m] & \\ & X \succeq 0,\ \mathrm{rank}(X) = 1 & \end{array} \iff \begin{array}{ll} \min & \mathrm{rank}(X) \\ \text{s. t.} & \mathrm{Tr}(A_k X) = b_k \quad k \in [m] \\ & X \succeq 0 \end{array}$$

Combinatorially hard

PhaseLift

$$|\langle a_k, x\rangle|^2 = b_k \qquad k \in [m]$$

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Turns quadratic measurements into linear measurements about xx^*

PhaseLift: tractable semidefinite relaxation

$$\begin{array}{ll} \text{minimize} & \mathsf{Tr}(X) \\ \text{subject to} & \mathsf{Tr}(A_k X) = b_k \quad k \in [m] \\ & X \succeq 0 \end{array}$$

- This is a semidefinite program (SDP)
- Trace is convex proxy for rank

Semidefinite programming (SDP)

- Special class of convex optimization problems
- Relatively natural extension of linear programming (LP)
- 'Efficient' numerical solvers (interior point methods)

LP (std. form): $x \in \mathbb{R}^n$

$$\begin{array}{ll} \text{minimize} & \langle c, x \rangle \\ \text{subject to} & a_k^T x = b_k \ \ k = 1, \ldots \\ & x \geq 0 \end{array}$$

SDP (std. form): $X \in \mathbb{R}^{n \times n}$

$$\begin{array}{ll} \text{minimize} & \langle C, X \rangle \\ \text{subject to} & \langle A_k, X \rangle = b_k \ \ k = 1, \ldots \\ & X \succeq 0 \end{array}$$

Standard inner product: $\langle C, X \rangle = \mathrm{Tr}(C^* X)$

From overdetermined to highly underdetermined

Quadratic equations

$$b_k = |\langle a_k, x\rangle|^2 \quad k \in [m] \qquad b = \mathcal{A}(xx^*)$$

Lift

$$\begin{array}{ll} \text{minimize} & \mathrm{Tr}(X) \\ \text{subject to} & \mathcal{A}(X) = b \\ & X \succeq 0 \end{array}$$

Have we made things worse?

overdetermined $(m > n)$ $\rightarrow$ highly underdetermined $(m \ll n^2)$

This is not really new...

Relaxation of quadratically constrained QP's

- Shor (87) [Lower bounds on nonconvex quadratic optimization problems]
- Goemans and Williamson (95) [MAX-CUT]
- Ben-Tal and Nemirovskii (01) [Monograph]
- ...

Similar approach for array imaging: Chai, Moscoso, Papanicolaou (11)

Exact phase retrieval via SDP

Quadratic equations

$$b_k = |\langle a_k, x\rangle|^2 \quad k \in [m] \qquad b = \mathcal{A}(xx^*)$$

Simplest model: a_k independently and uniformly sampled on unit sphere

- of $\mathbb{C}^n$ if $x \in \mathbb{C}^n$ (complex-valued problem)
- of $\mathbb{R}^n$ if $x \in \mathbb{R}^n$ (real-valued problem)

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Theorem (C. and Li ('12); C., Strohmer and Voroninski ('11))

Assume $m \gtrsim n$. With prob. $1 - O(e^{-\gamma m})$, for all $x \in \mathbb{C}^n$, only point in feasible set

$$\{X : \mathcal{A}(X) = b \quad \textit{and} \quad X \succeq 0\} \qquad \textit{is } xx^*$$

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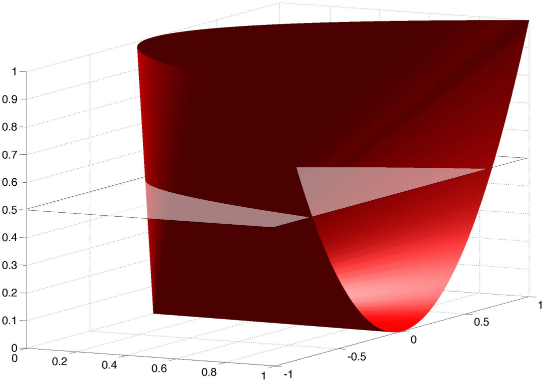
Injectivity if $m \geq 4n - 2$ (Balan, Bodmann, Casazza, Edidin '09)

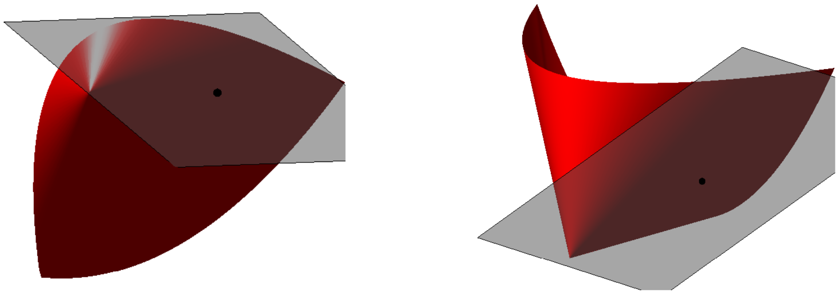
How is this possible?

How can feasible set $\{X \succeq 0\} \cap \{\mathcal{A}(X) = b\}$ have a unique point?



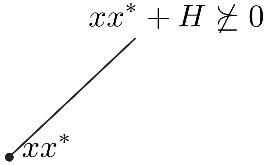
Intersection of $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \succeq 0$ with affine space

Correct representation



Rank-1 matrices are on the boundary (extreme rays) of PSD cone

My mental representation



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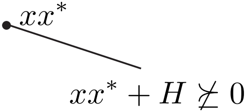

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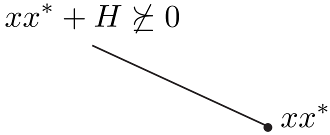

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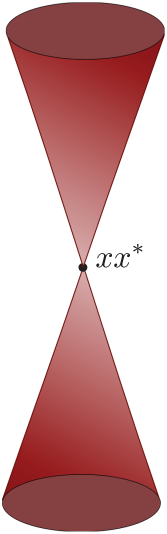

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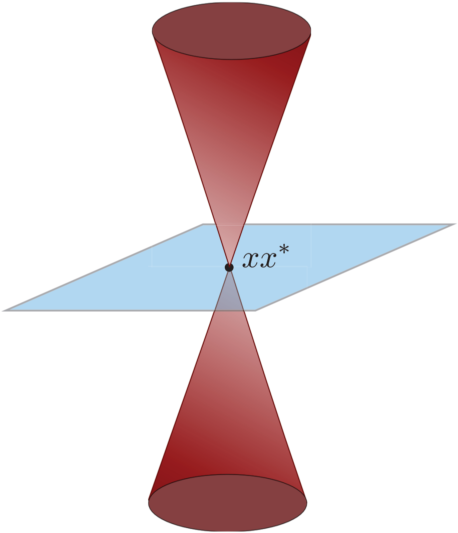
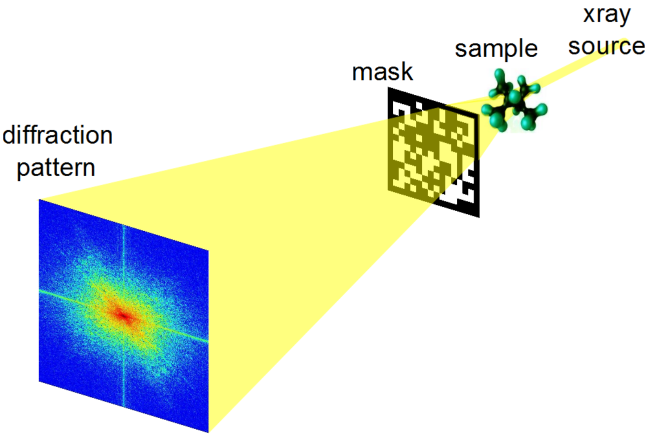


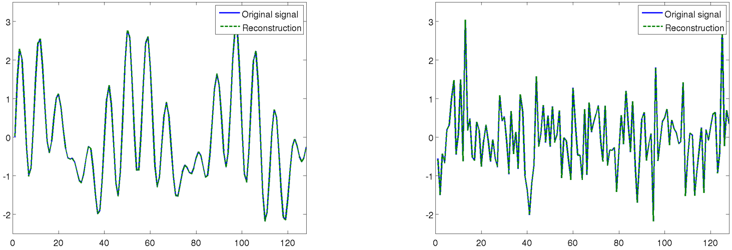
Extensions to physical setups



Random masking + diffraction

Similar theory: C. , Li and Soltanolkotabi ('13)

Numerical results: noiseless recovery



(a) Smooth signal (real part) (b) Random signal (real part)

Figure: Recovery (with reweighting) of n-dimensional complex signal ($2n$ unknowns) from $4n$ quadratic measurements (random binary masks)

With noise

$$b_k \approx |\langle x, a_k \rangle|^2 \quad k \in [m]$$

Noise aware recovery (SDP)

$$\begin{array}{ll} \text{minimize} & \|\mathcal{A}(X) - b\|_1 = \sum_k |\mathrm{Tr}(a_k a_k^* X) - b_k| \\ \text{subject to} & X \succeq 0 \end{array}$$

Signal $\hat{x}$ obtained by extracting first eigenvector (PC) of solution matrix

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Signal $\hat{x}$ obtained by extracting first eigenvector (PC) of solution matrix

In same setup as before and for realistic noise models, no method whatsoever can possibly yield a fundamentally smaller recovery error [C. and Li (2012)]

Numerical results: noisy recovery

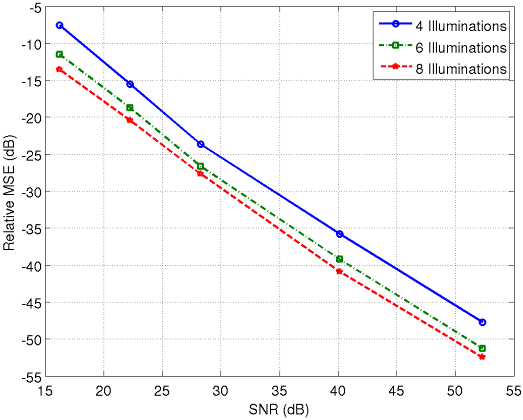
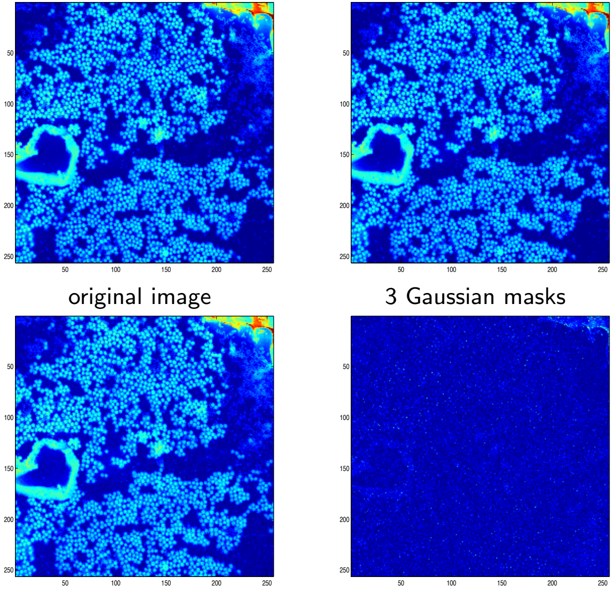


Figure: SNR versus relative MSE on a dB-scale for different numbers of illuminations with binary masks

Numerical results: noiseless 2D images



original image

3 Gaussian masks

8 binary masks

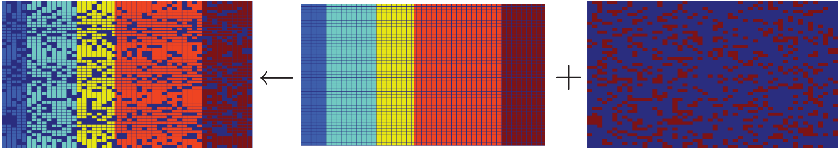
error with 8 binary masks

Courtesy
S. Marchesini (LBL)

Story #2: Robust Principal Component Analysis

Collaborators: X. Li, Y. Ma, J. Wright

The separation problem (Chandrasekahran et al.)



$$M = L + S$$

- M: data matrix (observed)
- L: low-rank (unobserved)
- S: sparse (unobserved)

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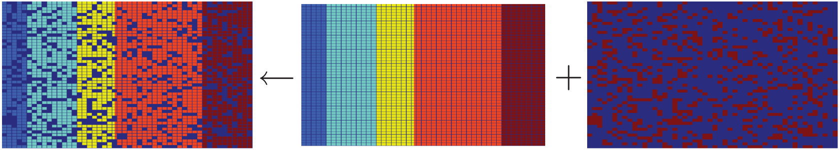

$$M = L + S$$

- M: data matrix (observed)
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- S: sparse (unobserved)

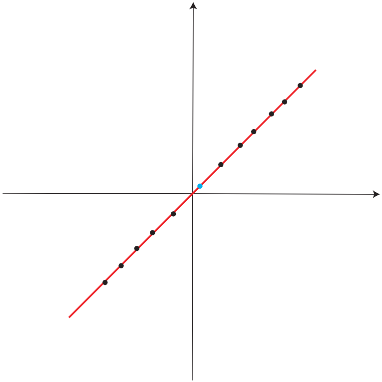
Problem: can we recover L and S accurately?

Again, missing information

Motivation: robust principal component analysis (RPCA)

PCA sensitive to outliers: breaks down with one (badly) corrupted data point

$$\begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{d1} & x_{d2} & \dots & x_{dn} \end{bmatrix}$$



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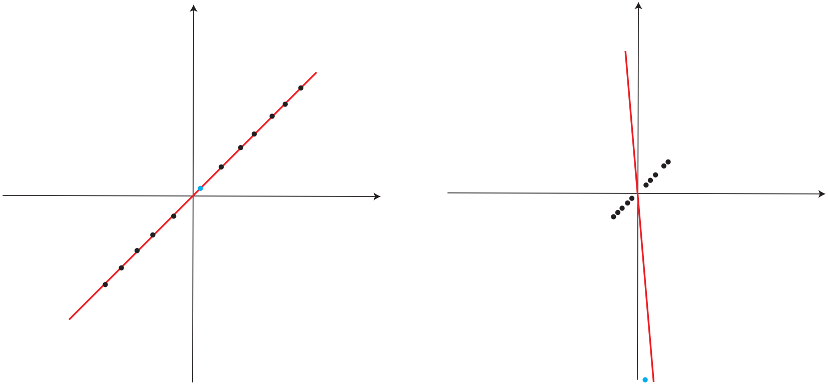
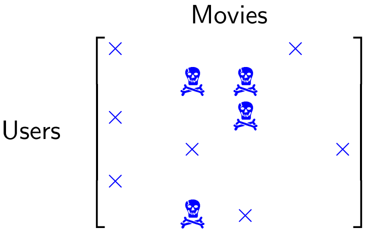


Robust PCA

- Data increasingly high dimensional
- Gross errors frequently occur in many applications
  - Image processing
  - Web data analysis
  - Bioinformatics
  - ...
  - Occlusions
  - Malicious tampering
  - Sensor failures
  - ...

Important to make PCA robust

Gross errors



Observe corrupted entries

$$Y_{ij} = L_{ij} + S_{ij} \qquad (i,j) \in \Omega_{\text{obs}}$$

- L low-rank matrix
- S entries that have been tampered with (impulsive noise)

Problem

Recover L from missing and corrupted samples

The L+S model

(Partial) information $y = \mathcal{A}(M)$ about

$$\underbrace{M}_{\text{object}} = \underbrace{L}_{\text{low rank}} + \underbrace{S}_{\text{sparse}}$$

- RPCA

  data = low-dimensional structure + corruption

- Dynamic MR

  video seq. = static background + sparse innovation

- Graphical modeling with hidden variables: Chandrasekaran, Sanghavi, Parrilo, Willsky ('09, '11)

  marginal inverse covariance of observed variables = low-rank + sparse

When does separation make sense?

$$M = L + S$$

Low-rank component cannot be sparse: $L = \begin{bmatrix} * & * & * & * & \cdots & * & * \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$ ✗

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Sparse component cannot be low rank: $S = \begin{bmatrix} * & 0 & 0 & 0 & \cdots & 0 & 0 \\ * & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ * & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$ ✗

Low-rank component cannot be sparse

$$L = \begin{bmatrix} * & * & * & * & \cdots & * & * \\ * & * & * & * & \cdots & * & * \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Incoherent condition [C. and Recht ('08)]: column and row spaces not aligned with coordinate axes (singular vectors are not sparse)

Low-rank component cannot be sparse

$$M = \begin{bmatrix} * & * & \skull & * & \cdots & * & * \\ * & * & \skull & * & \cdots & * & \skull \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Incoherent condition [C. and Recht ('08)]: column and row spaces not aligned with coordinate axes (singular vectors are not sparse)

Sparse component cannot be low-rank

$$L = \underbrace{\begin{bmatrix} x_1 & x_2 & \cdots & x_{n-1} & x_n \\ x_1 & x_2 & \cdots & x_{n-1} & x_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1 & x_2 & \cdots & x_{n-1} & x_n \end{bmatrix}}_{1x^*} \Rightarrow L + S = \begin{bmatrix} \skull & x_2 & \cdots & x_{n-1} & x_n \\ \skull & x_2 & \cdots & x_{n-1} & x_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \skull & x_2 & \cdots & x_{n-1} & x_n \end{bmatrix}$$

Sparsity pattern will be assumed (uniform) random

Demixing by convex programming

$$M = L + S$$

- L unknown (rank unknown)
- S unknown (# of entries $\neq 0$, locations, magnitudes all unknown)

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Recovery via SDP

$$\begin{array}{ll} \text{minimize} & \|\hat{L}\|_* + \lambda \|\hat{S}\|_1 \\ \text{subject to} & \hat{L} + \hat{S} = M \end{array}$$

See also Chandrasekaran, Sanghavi, Parrilo, Willsky ('09)

- nuclear norm: $\|L\|_* = \sum_i \sigma_i(L)$ (sum of sing. values)
- ℓ_1 norm: $\|S\|_1 = \sum_{ij} |S_{ij}|$ (sum of abs. values)

Exact recovery via SDP

$$\min \|\hat{L}\|_* + \lambda\|\hat{S}\|_1 \quad \text{s. t.} \quad \hat{L} + \hat{S} = M$$

Exact recovery via SDP

$$\text{min } \|\hat{L}\|_* + \lambda\|\hat{S}\|_1 \quad \text{s. t.} \quad \hat{L} + \hat{S} = M$$

Theorem

- *L is $n \times n$ of rank$(L) \le \rho_r n \, (\log n)^{-2}$ and incoherent*
- *S is $n \times n$, random sparsity pattern of cardinality at most $\rho_s n^2$*

Then with probability $1 - O(n^{-10})$, SDP with $\lambda = 1/\sqrt{n}$ is exact:

$$\hat{L} = L, \quad \hat{S} = S$$

Same conclusion for rectangular matrices with $\lambda = 1/\sqrt{\text{max dim}}$

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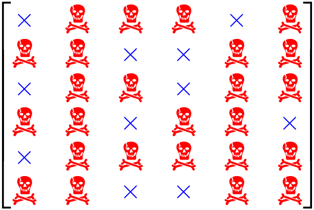
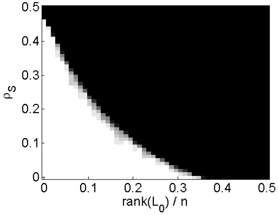
Same conclusion for rectangular matrices with $\lambda = 1/\sqrt{\text{max dim}}$

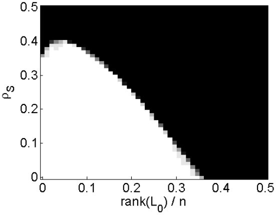
- No tuning parameter!
- Whatever the magnitudes of L and S



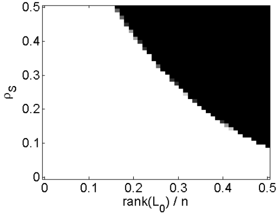
Phase transitions in probability of success



(a) PCP, Random Signs



(b) PCP, Coherent Signs



(c) Matrix Completion

$L = XY^T$ is a product of independent $n \times r$ i.i.d. $\mathcal{N}(0, 1/n)$ matrices

Missing *and* corrupted

RPCA

$$\begin{array}{ll} \text{min} & \|\hat{L}\|_* + \lambda \|\hat{S}\|_1 \\ \text{s. t.} & \hat{L}_{ij} + \hat{S}_{ij} = L_{ij} + S_{ij} \;\; (i,j) \in \Omega_{\text{obs}} \end{array}$$

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Theorem

- *L as before*
- *Ω_{obs} random set of size $0.1n^2$ (missing frac. is arbitrary)*
- *Each observed entry corrupted with prob. $\tau \leq \tau_0$*

Then with prob. $1 - O(n^{-10})$, PCP with $\lambda = 1/\sqrt{0.1n}$ is exact:

$$\hat{L} = L$$

Same conclusion for rectangular matrices with $\lambda = 1/\sqrt{0.1 \mathit{max\ dim}}$

Background subtraction

With noise

With Li, Ma, Wright & Zhou ('10)

Z stochastic or deterministic perturbation

$$Y_{ij} = L_{ij} + S_{ij} + Z_{ij} \quad (i,j) \in \Omega$$

$$\begin{array}{ll} \text{minimize} & \|\hat{L}\|_* + \lambda\|\hat{S}\|_1 \\ \text{subject to} & \sum_{(i,j)\in\Omega}(M_{ij} - \hat{L}_{ij} - \hat{S}_{ij})^2 \le \delta^2 \end{array}$$

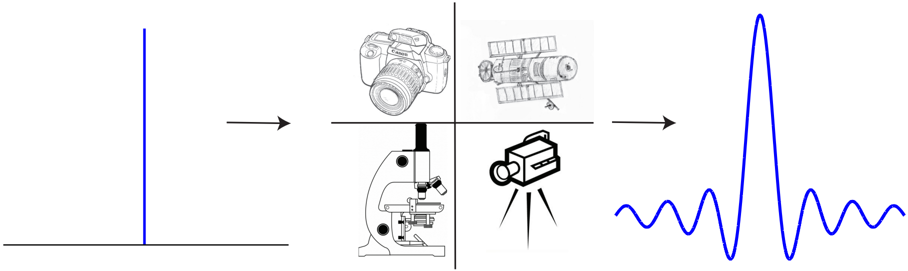
When perfect (noiseless) separation occurs $\Longrightarrow$ noisy variant is stable

Story #3: Super-resolution

Collaborator: C. Fernandez-Granda

Limits of resolution

In any optical imaging system, diffraction imposes fundamental limit on resolution



$\delta(t-\tau)$ optical system $h(t-\tau)$

The physical phenomenon called diffraction *is of the utmost importance in the theory of optical imaging systems* (Joseph Goodman)

Bandlimited imaging systems (Fourier optics)

$$f_{\text{obs}}(t) = (h * f)(t)$$
$$\hat{f}_{\text{obs}}(\omega) = \hat{h}(\omega)\hat{f}(\omega)$$

h : point spread function (PSF)
$\hat{h}$: transfer function (TF)

Bandlimited system

$$|\omega| > \Omega \quad \Rightarrow \quad |\hat{h}(\omega)| = 0$$

$\hat{f}_{\text{obs}}(\omega) = \hat{h}(\omega)\,\hat{f}(\omega) \rightarrow$ suppresses *all* high-frequency components

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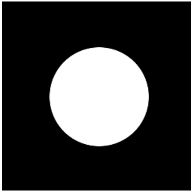
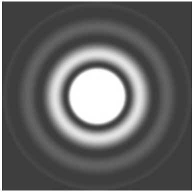
$\hat{f}_{\text{obs}}(\omega) = \hat{h}(\omega)\,\hat{f}(\omega) \rightarrow$ suppresses *all* high-frequency components

Example: coherent imaging

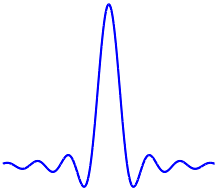
$\hat{h}(\omega) = 1_P(\omega)$ indicator of pupil element



TF
Pupil

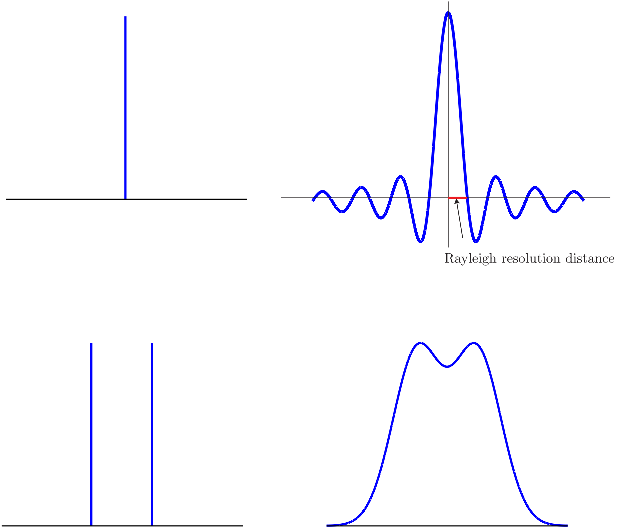


PSF
Airy disk



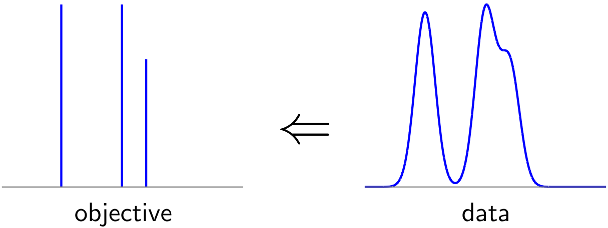
cross-section (PSF)

Rayleigh resolution limit

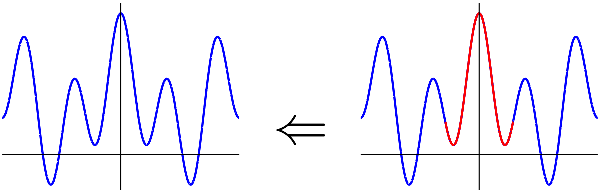


Lord Rayleigh

The super-resolution problem

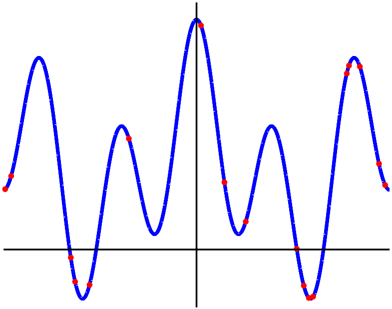


Retrieve fine scale information from low-pass data

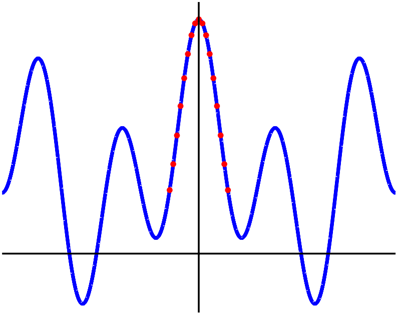


Equivalent description: extrapolate spectrum (ill posed)

Random vs. low-frequency sampling



Random sampling (CS)



Low-frequency sampling (SR)

Compressive sensing: spectrum interpolation
Super-resolution: spectrum extrapolation

Super-resolving point sources

Signal of interest is superposition of point sources

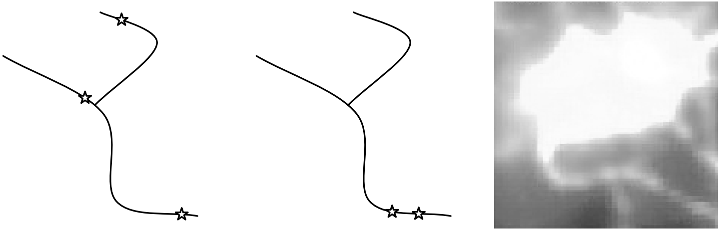
- Celestial bodies in astronomy
- Line spectra in speech analysis
- Fluorescent molecules in single-molecule microscopy

Many applications

- Radar
- Spectroscopy
- Medical imaging
- Astronomy
- Geophysics
- ...

Single molecule imaging (with WE Moerner's Lab)

Microscope receives light from fluorescent molecules



Problem

Resolution is much coarser than size of individual molecules (low-pass data)

Can we 'beat' the diffraction limit and super-resolve those molecules?

Higher molecule density $\longrightarrow$ faster imaging

Mathematical model

- Signal

$$x = \sum_j a_j \delta_{\tau_j}$$
$$a_j \in \mathbb{C},\ \tau_j \in T \subset [0,1]$$

- Data $y = \mathcal{F}_n x$: $n = 2f_{\text{lo}} + 1$ low-frequency coefficients (Nyquist sampling)

$$y(k) = \int_0^1 e^{-i2\pi kt} x(\mathrm{d}t) = \sum_j a_j e^{-i2\pi k\tau_j} \quad k \in \mathbb{Z},\ |k| \leq f_{\text{lo}}$$

- Resolution limit: ($\lambda_{\text{lo}}/2$ is Rayleigh distance)

$$1/f_{\text{lo}} = \lambda_{\text{lo}}$$

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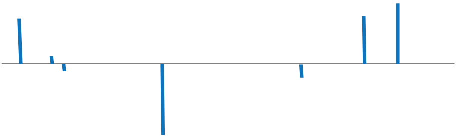

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- Resolution limit: ($\lambda_{\text{lo}}/2$ is Rayleigh distance)

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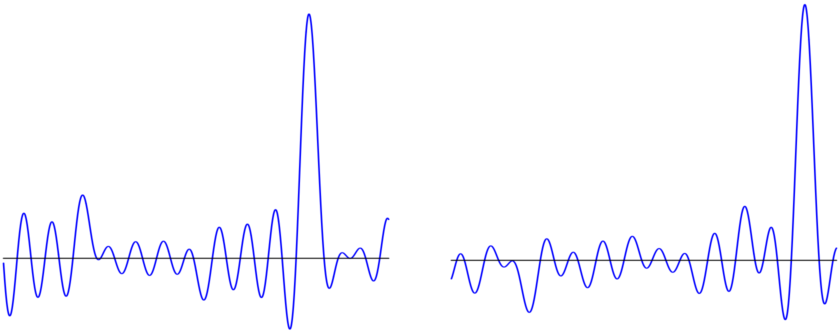
Question

Can we resolve the signal beyond this limit?

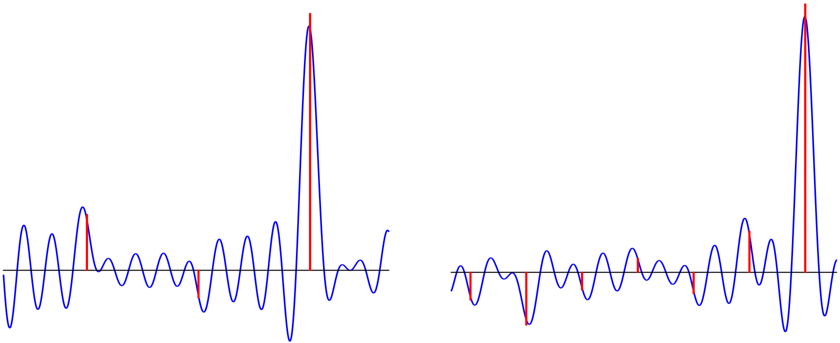
Swap time and frequency $\longrightarrow$ spectral estimation

Can you find the spikes?



Low-frequency data about spike train

Can you find the spikes?



Low-frequency data about spike train

Recovery by minimum total-variation

Recovery by cvx prog.

$$\min \ \|\tilde{x}\|_{\mathsf{TV}} \quad \text{subject to} \quad \mathcal{F}_n\, \tilde{x} = y$$

$\|x\|_{\mathsf{TV}} = \int |x(\mathrm{d}t)|$ is continuous analog of ℓ_1 norm

$$x = \sum_j a_j \delta_{\tau_j} \quad \Longrightarrow \quad \|x\|_{\mathsf{TV}} = \sum_j |a_j|$$

With noise

$$\min \ \frac{1}{2}\|y - \mathcal{F}_n\, \tilde{x}\|_{\ell_2}^2 + \lambda \|\tilde{x}\|_{\mathsf{TV}}$$

Recovery by convex programming

$$y(k) = \int_0^1 e^{-i2\pi kt} x(\mathsf{d}t) \qquad |k| \leq f_{\mathsf{lo}}$$

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Theorem (C. and Fernandez Granda (2012))

If spikes are separated by at least

$$2\,/\,f_{\mathsf{lo}} := 2\,\lambda_{\mathsf{lo}}$$

then min TV solution is exact! For real-valued x, a min dist. of $1.87\lambda_{\mathsf{lo}}$ suffices

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- Infinite precision!

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- Whatever the amplitudes
- Can recover $(2\lambda_{\mathsf{lo}})^{-1} = f_{\mathsf{lo}}/2 = n/4$ spikes from n low-freq. samples

Recovery by convex programming

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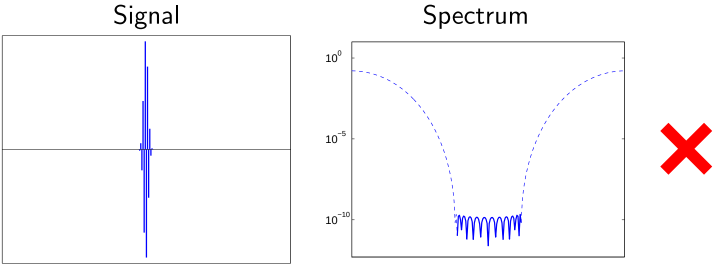
- Infinite precision!
- Whatever the amplitudes
- Can recover $(2\lambda_{\mathrm{lo}})^{-1} = f_{\mathrm{lo}}/2 = n/4$ spikes from n low-freq. samples
- Cannot go below λ_{lo}
- Essentially same result in higher dimensions

About separation: sparsity is not enough!

- CS: sparse signals are 'away' from null space of sampling operator
- Super-res: this is not the case



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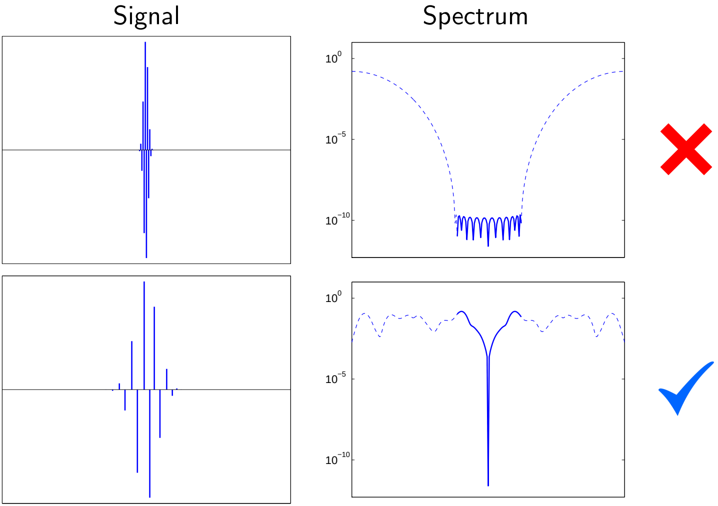

Analysis via prolate spheroidal functions

David Slepian

If distance between spikes less than $\lambda_{\text{lo}}/2$ (Rayleigh), problem hopelessly ill posed

Formulation as a finite-dimensional problem

Primal problem

$$\min \ \|x\|_{\text{TV}} \text{ s. t. } \mathcal{F}_n x = y$$

- Infinite-dimensional variable x
- Finitely many constraints

Dual problem

$$\max \ \text{Re}\langle y, c\rangle \text{ s. t. } \|\mathcal{F}_n^* c\|_\infty \leq 1$$

- Finite-dimensional variable c
- Infinitely many constraints

$$(\mathcal{F}_n^* c)(t) = \sum_{|k| \leq f_{\text{lo}}} c_k e^{i2\pi kt}$$

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Semidefinite representability

$|(\mathcal{F}_n^* \, c)(t)| \leq 1$ for all $t \in [0,1]$ equivalent to

(1) there is Q Hermitian s. t.

$$\begin{bmatrix} Q & c \\ c^* & 1 \end{bmatrix} \succeq 0$$

(2) $\text{Tr}(Q) = 1$

(3) sums along superdiagonals vanish: $\sum_{i=1}^{n-j} Q_{i,i+j} = 0$ for $1 \leq j \leq n-1$

SDP formulation

Dual as an SDP

$$\text{maximize} \quad \text{Re}\langle y, c\rangle \qquad \text{subject to} \quad \begin{bmatrix} Q & c \\ c^* & 1 \end{bmatrix} \succeq 0$$
$$\sum_{i=1}^{n-j} Q_{i,i+j} = \delta_j \quad 0 \leq j \leq n-1$$

Dual solution c: coeffs. of low-pass trig. polynomial $\sum_k c_k e^{i2\pi kt}$ interpolating the sign of the primal solution

SDP formulation

Dual as an SDP

maximize $\quad \mathrm{Re}\langle y, c\rangle \quad$ subject to

$$\begin{bmatrix} Q & c \\ c^* & 1 \end{bmatrix} \succeq 0$$

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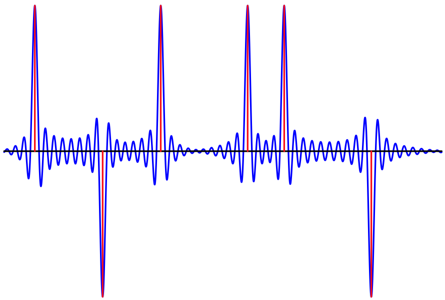


To recover spike locations

(1) Solve dual

(2) Check when polynomial takes on magnitude 1

With noise

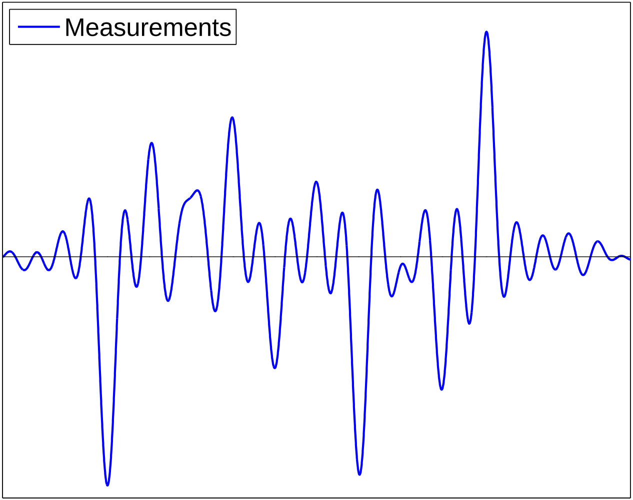
$$y = \mathcal{F}_n x + \text{noise}$$

$$\text{min} \quad \frac{1}{2}\|y - \mathcal{F}_n \tilde{x}\|_{\ell_2}^2 + \lambda \|\tilde{x}\|_{\text{TV}}$$

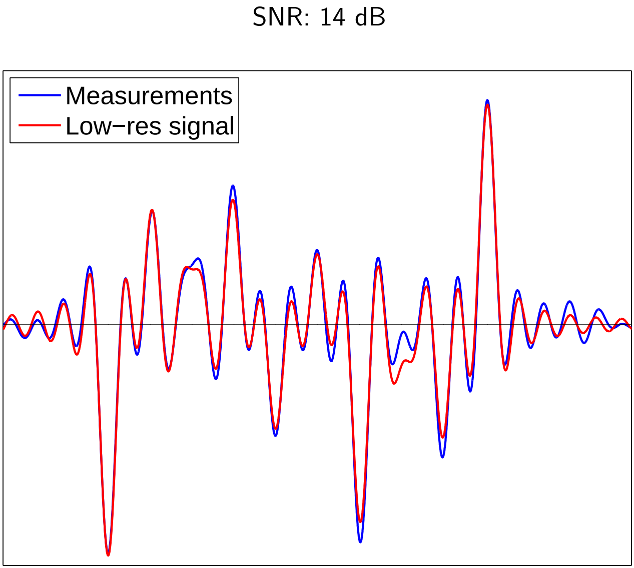
- Also an SDP
- Theory: C. and Fernandez Granda ('12)

Noisy example

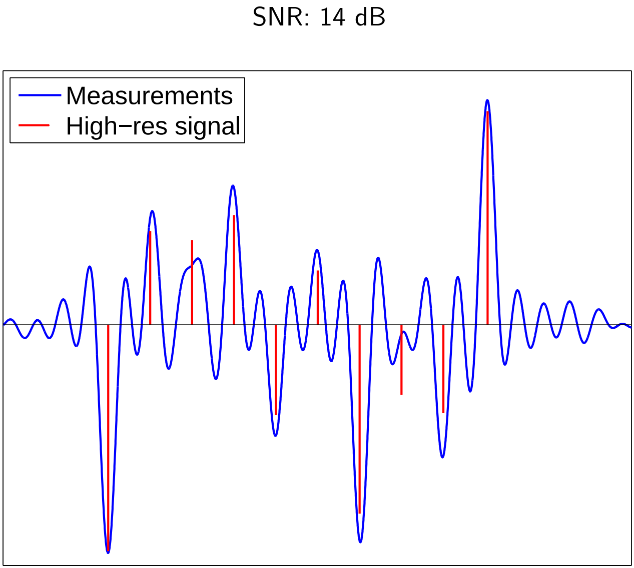
SNR: 14 dB



Noisy example

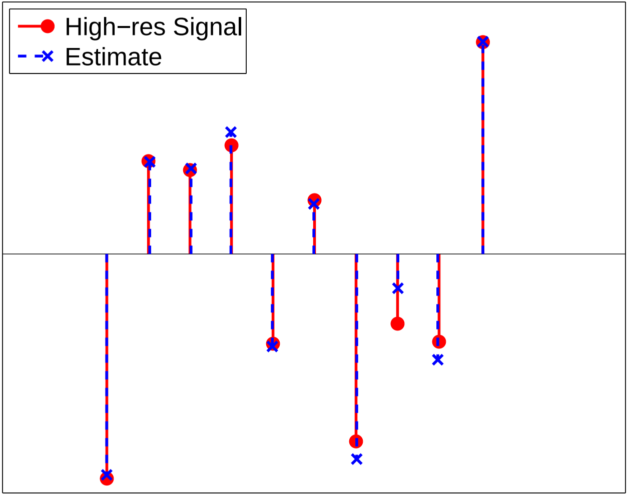


Noisy example



Noisy example

Average localization error: 6.54×10^{-4}



Summary

- Three important problems with missing data
  - Phase retrieval
  - Matrix completion/RPCA
  - Super-resolution
- Three simple and model-free recovery procedures via convex programming
- Three near-perfect solutions

Apologies: things I have not talked about

- Algorithms
- Applications
- Avalanche of related works

A small sample of papers I have greatly enjoyed

- Phase retrieval
  - Netrapalli, Jain, Sanghavi, *Phase retrieval using alternating minimization* ('13)
  - Waldspurger, d'Aspremont, Mallat, *Phase recovery, MaxCut and complex semidefinite programming* ('12)
- Robust PCA
  - Gross, *Recovering low-rank matrices from few coefficients in any basis* ('09)
  - Chandrasekaran, Parrilo and Willsky, *Latent variable graphical model selection via convex optimization* ('11)
  - Hsu, Kakade and Zhang, *Robust matrix decomposition with outliers* ('11)
- Super-resolution
  - Kahane, *Analyse et synthèse harmoniques* ('11)
  - Slepian, *Prolate spheroidal wave functions, Fourier analysis, and uncertainty. V - The discrete case* ('78)

General SDP formulation

Nuclear norm and spectral norms are dual: $\|X\|_* = \mathsf{val}(P)$

$$(P) \quad \begin{array}{ll} \text{maximize} & \langle U, X \rangle \\ \text{subject to} & \|U\| \le 1 \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \text{maximize} & \langle U, X \rangle \\ \text{subject to} & \begin{bmatrix} I & U \\ U^* & I \end{bmatrix} \succeq 0 \end{array}$$

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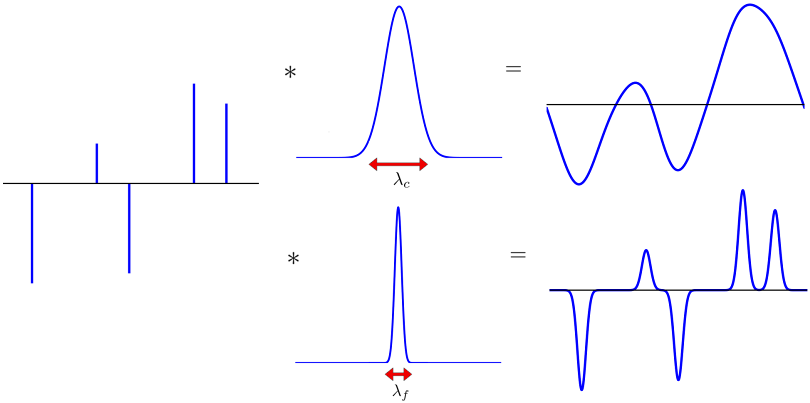
Duality: $\|X\|_* = \text{val}(D)$

$$(D) \quad \begin{array}{ll} \text{minimize} & .5(\text{Tr}(W_1) + \text{Tr}(W_2)) \\ \text{subject to} & \begin{bmatrix} W_1 & X \\ X^* & W_2 \end{bmatrix} \succeq 0 \end{array}$$

Optimization variables: $W_1 \in \mathbb{R}^{n_1 \times n_1}$, $W_2 \in \mathbb{R}^{n_2 \times n_2}$

Nuclear norm heuristics: Fazel (2002), Hindi, Boyd & Fazel (2001)

The super-resolution factor

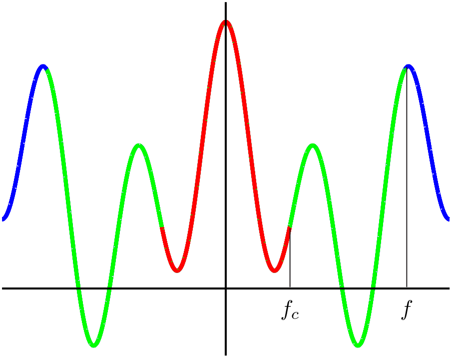


- Have data at resolution λ_{lo}
- Wish resolution λ_{hi}

Super-resolution factor

$$\text{SRF} = \frac{\lambda_{\text{lo}}}{\lambda_{\text{hi}}}$$

The super-resolution factor (SRF): frequency viewpoint



- Observe spectrum up to f_{lo}
- Wish to extrapolate up to f_{hi}

Super-resolution factor

$$\text{SRF} = \frac{f_{\text{hi}}}{f_{\text{lo}}}$$

With noise

$$y = \mathcal{F}_n x + \text{noise} \qquad \begin{array}{l} \mathcal{F}_n x = \int_0^1 e^{-i2\pi kt}\, x(\mathrm{d}t) \\ |k| \leq f_{\mathsf{lo}} \end{array}$$

At 'native' resolution

$$\|(\hat{x} - x) * \varphi_{\lambda_{\mathsf{lo}}}\|_{\mathsf{TV}} \lesssim \text{noise level}$$

With noise

$$y = \mathcal{F}_n x + \text{noise} \qquad \begin{array}{l} \mathcal{F}_n x = \int_0^1 e^{-i2\pi kt}\, x(\mathrm{d}t) \\ |k| \le f_{\mathsf{lo}} \end{array}$$

At 'finer' resolution $\lambda_{\mathsf{hi}} = \lambda_{\mathsf{lo}}/\mathsf{SRF}$, convex programming achieves

$$\|(\hat{x} - x) * \varphi_{\lambda_{\mathsf{hi}}}\|_{\mathsf{TV}} \lesssim \mathsf{SRF}^2 \times \text{noise level}$$

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Modulus of continuity studies for super-resolution: Donoho ('92)