A Large Deviation Principle for Block Models

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  - Some small, dense graphs? "localization"

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- Setting $(A_{ij})_{i,j=1}^{n}$ to be the adjacency matrix,

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Today's focus: Large deviations in dense graphs

The Erdős-Rényi case

Key idea: represent an Erdős-Rényi random graph as a *graphon*[CV'11, LZ'15]

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- The region $[0,1]^2$ is divided into $n \times n$ cells.
- If $(i,j) \in E$, then the (i,j) cell takes value 1.
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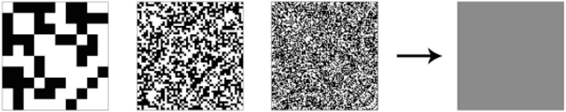

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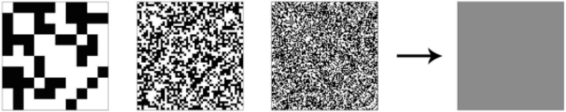

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Describe large deviations through the language of graphons!

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Theorem (Lovász & Szegedy (2007))

$(\widetilde{\mathcal{W}}, \delta_\square)$ *is a compact metric space.*

Homomorphism densities

Definition (Homomorphism density)

Fix a subgraph H. For $f \in \mathcal{W}$, define

$$t(H, f) = \int_{[0,1]^{|V(H)|}} \prod_{(i,j) \in E(H)} f(x_i, x_j) \prod_{i=1}^{|V(H)|} dx_i.$$

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Theorem (LS'07,BCLSV'08)

For any fixed graph H, $\tilde{f} \mapsto t(H, \tilde{f})$ is continuous under the cut topology.

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Definition (Relative entropy)

Define $I_{W_0} : \mathcal{W} \to \mathbb{R} \cup \{\infty\}$ as

$$I_{W_0}(f) = \frac{1}{2} \int_{[0,1]^2} h_p\left(f(x,y)\right) dxdy,$$

where $h_p(u)$ is the usual relative entropy,

$$h_p(u) = u \log \frac{u}{p} + (1-u) \log \frac{1-u}{1-p}.$$

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If $\tilde{F}^*$ is a singleton, the conditional distribution is concentrated at a single point!

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- If minimizer non-constant - what happens? (symmetry-breaking)

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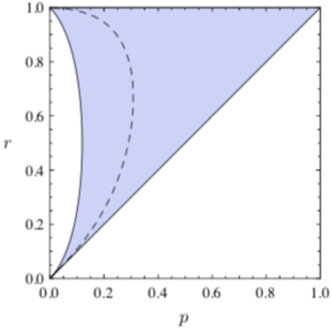

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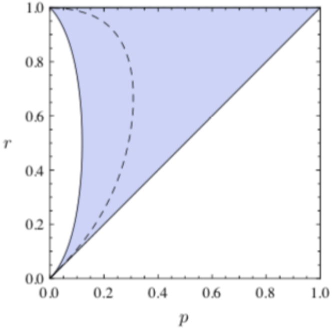

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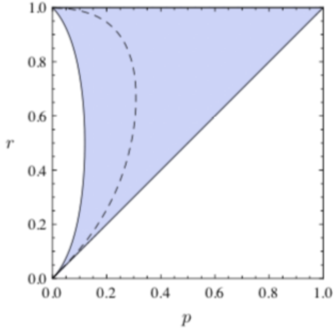

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- Blue region: *symmetric regime* $\rightarrow$ mimic $G(n, r)$
- White region: *non-symmetric regime* $\rightarrow$ distribution does not match $G(n, r)$

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Our focus: Large deviations beyond the Erdős-Rényi case

Beyond Erdos-Renyi graphs

- Random graphs with inhomogeneities or constraints are common.
  (a) The $G(n, m)$ model. [Dembo-Lubetzky (2018)]
  (b) Random regular graphs.
  (c) Block models.

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- Large deviations in this context is of natural interest!

- Expect new phenomena . . .

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- In other words, if
  - Vertex i is in block a
  - Vertex j is in block b,

  then connect i and j with probability p_{ab}

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- Add edges independently with probability $p_{\lceil i/n\rceil\lceil j/n\rceil}$.
- In other words, if
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- Note: repeated p_{ab} are allowed; we can accommodate rational-length blocks.

Block Models

- Construct a graphon with k blocks of equal length.
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Sampled graph $\leftrightarrow$ Empirical graphon

Distribution over graphs $\leftrightarrow$ $\tilde{\mathbb{P}}_{n,W_0}$, the induced law on $(\widetilde{\mathcal{W}}, \delta_\square)$

Derive LDP for graphs in terms of $\tilde{\mathbb{P}}_{n,W_0}$!

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$\widetilde{\mathcal{W}}_\Omega$ closed, $\tilde{\mathbb{P}}_{n,W_0}$ supported on $\widetilde{\mathcal{W}}_\Omega$.

The rate function

Recall the rate function from the Erdős-Rényi setting:

$$I_{W_0}(f) = \frac{1}{2} \int_{[0,1]^2} h_p\left(f(x,y)\right) dxdy,$$

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- Markering '22 showed that the same rate function applies when $\log(W_0), \log(\mathbf{1} - W_0) \in L^1([0,1]^2)$.

Application to homomorphism densities

Theorem (BCGPS'20+)

Fix H. *Set* $t_{\max} = \max_{\tilde{f} \in \widetilde{\mathcal{W}}} t(H, \tilde{f})$. *For* $t < t_{\max}$ *define*

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- *If* $\tilde{F}^*$ *is a singleton, the conditional distribution is concentrated at a single point.*

A new notion of symmetry

Question

Do the minimizers of $\min\{J_{W_0}(\tilde{f}) : \tilde{f} \in \widetilde{\mathcal{W}}, t(H, \tilde{f}) \geq t\}$ have the same block structure as W_0?

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- Know the specific symmetry/symmetry-breaking boundary for Erdős-Rényi bipartite graphs.

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    - Compare base graphon to empirical graphon according to alignment of types

Proof Ideas: Symmetric Regime

Definition

Let $p \in (0,1)$ and $d \geq 2$. We define $\psi_p : [0,1] \to \mathbb{R}$ as

$$\psi_p(x) = h_p(x^{1/d}),$$

and let $\hat{\psi}_p(x)$ denote the convex minorant of $\psi_p(x)$.

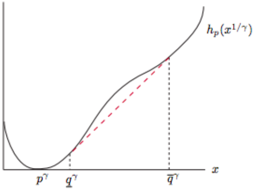

Figure 4: Illustration of the function $x \mapsto h_p(x^{1/\gamma})$ and its convex minorant (Lubetzky–Zhao 2015))

Proof Ideas: Symmetric Regime

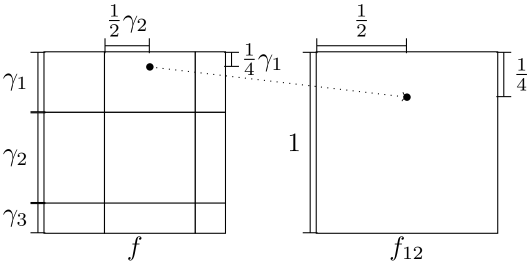

Figure 5: A graphon $f = (f_{ij})_{i,j\in[m]}$

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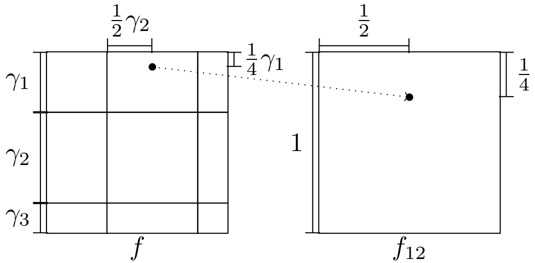

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Let $\|g\|_d = \left(\int_{[0,1]^2} g(x,y)^d dxdy \right)^{\frac{1}{d}}$.

The Convex Minorant Condition

Definition

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We say f satisfies the ε-neighborhood minorant condition if for all (i, j) such that $p_{ij} \in (0, 1)$

$$x \in \left(\|f_{ij}\|_d^d - \epsilon, \|f_{ij}\|_d^d + \epsilon\right) \cap [0, 1] \implies \psi_{p_{ij}}(x) = \hat{\psi}_{p_{ij}}(x).$$

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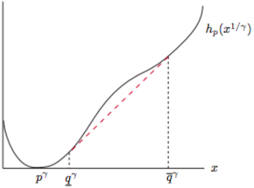

Figure 6: The function $x \mapsto h_p(x^{1/\gamma})$ and its convex minorant [LZ '15]

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Lemma

- *Let* $W_0 = (p_{ij})_{i,j\in[m]}$ *be the base graphon*
- *Suppose* $\tilde{f}$ *is a minimizer of the variational problem for* $\tau = t(H, \cdot)$
- *Suppose there exists a sequence of graphons* $f_n \in \mathcal{W}_\Omega$ *such that*
  - *Each* f_n *satisfies the* ε*-neighborhood minorant condition,*
  - $\delta_\square(f_n, \tilde{f}) \to 0$,
  - *and* $I_{W_0}(f_n) \to J_{W_0}(\tilde{f})$.

Then $\tilde{f}$ *matches the block structure of* W_0.

Proof Ideas: Non-Symmetric Regime

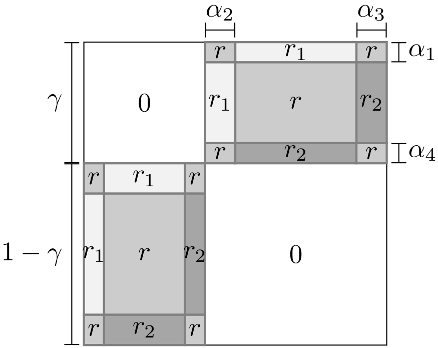
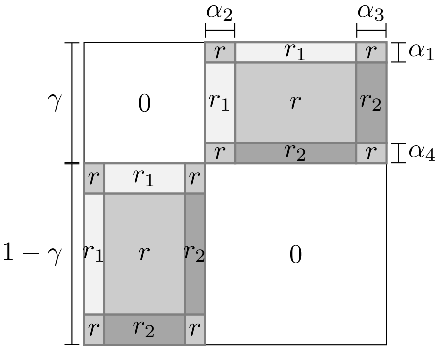


Figure 7: Construction of a non-symmetric graphon

Build a non-symmetric graphon such that

Proof Ideas: Non-Symmetric Regime



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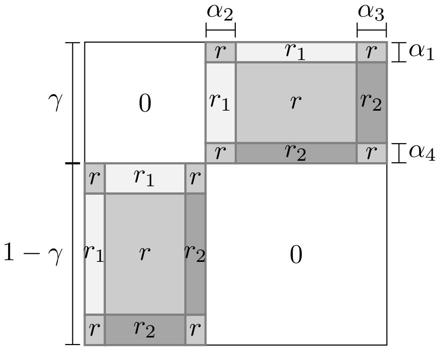

Figure 7: Construction of a non-symmetric graphon

Build a non-symmetric graphon such that

- The constraint (e.g. homomorphism density) is satisfied
- The relative entropy is strictly lower than what the symmetric solution attains.

Subsequent Developments

- Dupuis, Medvedev'20—inhomogeneous LDP (proof using weak convergence methods)
- Chakraborty, Hazra, den Hollander, Sfragara '20 (variational problem for spectral radius)
- Braunsteins, den Hollander, Mandjes'20 (sample path large deviations)
- Grebik, Pikhurko '21 (irrational block lengths)

Open Problems

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Thank you!