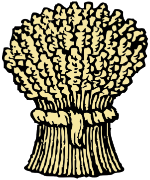
# A complexity theory of constructible sheaves

## Simon's Theory of Computing Institute, Berkeley

Saugata Basu

Department of Mathematics
Purdue University

October 22, 2014



# Outline

# Outline

1. Motivation
2. Qualitative/Background
3. Quantitative/Effective
4. Complexity-theoretic

# Outline

1. Motivation
2. Qualitative/Background
3. Quantitative/Effective
4. Complexity-theoretic

# Outline

1. Motivation
2. Qualitative/Background
3. Quantitative/Effective
4. Complexity-theoretic

# Why constructible sheaves ?

- Provides a more natural geometric language, more expressiveness than (first-order) logic.
- It provides a (topological) generalization of quantifier elimination (Tarski-Seidenberg). It is interesting to study quantitative/algorithmic questions in this more general setting.
- Applications in other areas ($D$-module theory, computational geometry ...).
- Interesting extensions of Blum-Shub-Smale complexity classes leading to $\mathbf{P}$ vs $\mathbf{NP}$ type questions which (paradoxically) might be *easier* to resolve than the classical (B-S-S) ones.
- Quantitative study of sheaf cohomology might be interesting on its own.

# Why constructible sheaves ?

- Provides a more natural geometric language, more expressiveness than (first-order) logic.
- It provides a (topological) generalization of quantifier elimination (Tarski-Seidenberg). It is interesting to study quantitative/algorithmic questions in this more general setting.
- Applications in other areas ($D$-module theory, computational geometry ...).
- Interesting extensions of Blum-Shub-Smale complexity classes leading to $\mathbf{P}$ vs $\mathbf{NP}$ type questions which (paradoxically) might be *easier* to resolve than the classical (B-S-S) ones.
- Quantitative study of sheaf cohomology might be interesting on its own.

# Why constructible sheaves ?

- Provides a more natural geometric language, more expressiveness than (first-order) logic.
- It provides a (topological) generalization of quantifier elimination (Tarski-Seidenberg). It is interesting to study quantitative/algorithmic questions in this more general setting.
- Applications in other areas ($D$-module theory, computational geometry ...).
- Interesting extensions of Blum-Shub-Smale complexity classes leading to $\mathbf{P}$ vs $\mathbf{NP}$ type questions which (paradoxically) might be *easier* to resolve than the classical (B-S-S) ones.
- Quantitative study of sheaf cohomology might be interesting on its own.

# Why constructible sheaves ?

- Provides a more natural geometric language, more expressiveness than (first-order) logic.
- It provides a (topological) generalization of quantifier elimination (Tarski-Seidenberg). It is interesting to study quantitative/algorithmic questions in this more general setting.
- Applications in other areas ($D$-module theory, computational geometry ...).
- Interesting extensions of Blum-Shub-Smale complexity classes leading to $\mathbf{P}$ vs $\mathbf{NP}$ type questions which (paradoxically) might be *easier* to resolve than the classical (B-S-S) ones.
- Quantitative study of sheaf cohomology might be interesting on its own.

# Why constructible sheaves ?

- Provides a more natural geometric language, more expressiveness than (first-order) logic.
- It provides a (topological) generalization of quantifier elimination (Tarski-Seidenberg). It is interesting to study quantitative/algorithmic questions in this more general setting.
- Applications in other areas ($D$-module theory, computational geometry ...).
- Interesting extensions of Blum-Shub-Smale complexity classes leading to $\mathbf{P}$ vs $\mathbf{NP}$ type questions which (paradoxically) might be *easier* to resolve than the classical (B-S-S) ones.
- Quantitative study of sheaf cohomology might be interesting on its own.

# Semi-algebraic sets and maps

- Semi-algebraic sets are subsets of $\mathbb{R}^n$ defined by Boolean formulas whose atoms are polynomial equalities and inequalities (i.e. $P = 0$, $P > 0$ for $P \in \mathbb{R}[X_1, \ldots, X_n]$).
- A semi-algebraic map is a map $X \xrightarrow{f} Y$ between semi-algebraic sets $X$ and $Y$, is a map whose graph is a semi-algebraic set.

# Semi-algebraic sets and maps

- Semi-algebraic sets are subsets of $\mathbb{R}^n$ defined by Boolean formulas whose atoms are polynomial equalities and inequalities (i.e. $P = 0$, $P > 0$ for $P \in \mathbb{R}[X_1, \ldots, X_n]$).
- A semi-algebraic map is a map $X \xrightarrow{f} Y$ between semi-algebraic sets $X$ and $Y$, is a map whose graph is a semi-algebraic set.

# Closure of semi-algebraic sets under different operations

Easy facts (i.e. follows more-or-less from the definitions) ...

Semi-algebraic sets are closed under:

- Finite unions and intersections, as well as taking complements.
- Cartesian products (or more generally fibered products over polynomial maps).
- Taking pull-backs (inverse images) under polynomial maps.

# Closure of semi-algebraic sets under different operations

Easy facts (i.e. follows more-or-less from the definitions) ...
Semi-algebraic sets are closed under:

- Finite unions and intersections, as well as taking complements.
- Cartesian products (or more generally fibered products over polynomial maps).
- Taking pull-backs (inverse images) under polynomial maps.

# Closure of semi-algebraic sets under different operations

Easy facts (i.e. follows more-or-less from the definitions) ...
Semi-algebraic sets are closed under:

- Finite unions and intersections, as well as taking complements.
- Cartesian products (or more generally fibered products over polynomial maps).
- Taking pull-backs (inverse images) under polynomial maps.

# Closure of semi-algebraic sets under different operations

Easy facts (i.e. follows more-or-less from the definitions) ...
Semi-algebraic sets are closed under:

- Finite unions and intersections, as well as taking complements.
- Cartesian products (or more generally fibered products over polynomial maps).
- Taking pull-backs (inverse images) under polynomial maps.

# Quantifier Elimination/ Tarski-Seidenberg

Harder fact (Tarski-Seidenberg Theorem (Tarski, 1951)) ...

- Images of a semi-algebraic sets under polynomial maps are also semi-algebraic.
- Equivalently, the first order theory of the reals admits quantifier elimination.

# Quantifier Elimination/ Tarski-Seidenberg

Harder fact (Tarski-Seidenberg Theorem (Tarski, 1951)) ...

- Images of a semi-algebraic sets under polynomial maps are also semi-algebraic.
- Equivalently, the first order theory of the reals admits quantifier elimination.

# Quantifier Elimination/ Tarski-Seidenberg

Harder fact (Tarski-Seidenberg Theorem (Tarski, 1951)) ...

- Images of a semi-algebraic sets under polynomial maps are also semi-algebraic.
- Equivalently, the first order theory of the reals admits quantifier elimination.

# In the language of maps instead of quantifiers

- Let $\mathbf{X} \xrightarrow{f} \mathbf{Y}$ be a map (between sets).
- Then there are induced maps:

$$2^{\mathbf{X}} \;\begin{matrix} \xrightarrow{f_\exists} \\ \xleftarrow{f^*} \\ \xrightarrow{f_\forall} \end{matrix}\; 2^{\mathbf{Y}} \qquad \left| \qquad \begin{matrix} f_\exists(A) := f(A) \\ f^*(B) := f^{-1}(B) \\ f_\forall(A) := Y - f(X - A) \end{matrix} \right.$$

- The pairs $(f_\exists, f^*)$ and $(f^*, f_\forall)$ are not quite pairs of inverses. But ... they do satisfy adjointness relations (namely):

$$f_\exists \dashv f^* \dashv f_\forall$$

as functors between the poset categories $2^{\mathbf{X}}, 2^{\mathbf{Y}}$ (the objects are subsets and arrows correspond to inclusions).
This is just a *chic* way of saying that for $A \in 2^{\mathbf{X}}$, $B \in 2^{\mathbf{Y}}$, $f_\exists(A) \subset B \Leftrightarrow A \subset f^*(B)$, and $A \subset f^*(B) \Leftrightarrow f_\forall(A) \subset B$.

# In the language of maps instead of quantifiers

- Let $\mathbf{X} \xrightarrow{f} \mathbf{Y}$ be a map (between sets).
- Then there are induced maps:

$$2^{\mathbf{X}} \; \begin{matrix} \xrightarrow{f_\exists} \\ \xleftarrow{f^*} \\ \xrightarrow{f_\forall} \end{matrix} \; 2^{\mathbf{Y}} \quad \left| \quad \begin{matrix} f_\exists(A) := f(A) \\ f^*(B) := f^{-1}(B) \\ f_\forall(A) := Y - f(X - A) \end{matrix} \right.$$

- The pairs $(f_\exists, f^*)$ and $(f^*, f_\forall)$ are not quite pairs of inverses. But ... they do satisfy adjointness relations (namely):

$$f_\exists \dashv f^* \dashv f_\forall$$

as functors between the poset categories $2^{\mathbf{X}}, 2^{\mathbf{Y}}$ (the objects are subsets and arrows correspond to inclusions).
This is just a *chic* way of saying that for $A \in 2^{\mathbf{X}}, B \in 2^{\mathbf{Y}}$, $f_\exists(A) \subset B \Leftrightarrow A \subset f^*(B)$, and $A \subset f^*(B) \Leftrightarrow f_\forall(A) \subset B$.

# In the language of maps instead of quantifiers

- Let $\mathbf{X} \xrightarrow{f} \mathbf{Y}$ be a map (between sets).
- Then there are induced maps:

$$2^{\mathbf{X}} \underset{\xrightarrow{f_\forall}}{\overset{\xrightarrow{f_\exists}}{\xleftarrow{f^*}}} 2^{\mathbf{Y}} \quad \left| \quad \begin{array}{c} f_\exists(A) := f(A) \\ f^*(B) := f^{-1}(B) \\ f_\forall(A) := Y - f(X - A) \end{array} \right.$$

- The pairs $(f_\exists, f^*)$ and $(f^*, f_\forall)$ are not quite pairs of inverses. But ... they do satisfy adjointness relations (namely):

$$f_\exists \dashv f^* \dashv f_\forall$$

as functors between the poset categories $2^{\mathbf{X}}, 2^{\mathbf{Y}}$ (the objects are subsets and arrows correspond to inclusions).
This is just a *chic* way of saying that for $A \in 2^{\mathbf{X}}, B \in 2^{\mathbf{Y}}$, $f_\exists(A) \subset B \Leftrightarrow A \subset f^*(B)$, and $A \subset f^*(B) \Leftrightarrow f_\forall(A) \subset B$.

# In the language of maps instead of quantifiers

- Let $\mathbf{X} \xrightarrow{f} \mathbf{Y}$ be a map (between sets).
- Then there are induced maps:

$$2^{\mathbf{X}} \begin{array}{c} \xrightarrow{f_\exists} \\ \xleftarrow{f^*} \\ \xrightarrow{f_\forall} \end{array} 2^{\mathbf{Y}} \quad \Bigg| \quad \begin{array}{c} f_\exists(A) := f(A) \\ f^*(B) := f^{-1}(B) \\ f_\forall(A) := Y - f(X - A) \end{array}$$

- The pairs $(f_\exists, f^*)$ and $(f^*, f_\forall)$ are not quite pairs of inverses. But ... they do satisfy adjointness relations (namely):

$$f_\exists \dashv f^* \dashv f_\forall$$

as functors between the poset categories $2^{\mathbf{X}}, 2^{\mathbf{Y}}$ (the objects are subsets and arrows correspond to inclusions).
This is just a *chic* way of saying that for $A \in 2^{\mathbf{X}}, B \in 2^{\mathbf{Y}}$, $f_\exists(A) \subset B \Leftrightarrow A \subset f^*(B)$, and $A \subset f^*(B) \Leftrightarrow f_\forall(A) \subset B$.

# In the language of maps instead of quantifiers

- Let $\mathbf{X} \xrightarrow{f} \mathbf{Y}$ be a map (between sets).
- Then there are induced maps:

$$2^{\mathbf{X}} \begin{array}{c} \xrightarrow{f_\exists} \\ \xleftarrow{f^*} \\ \xrightarrow{f_\forall} \end{array} 2^{\mathbf{Y}} \quad \left| \quad \begin{array}{c} f_\exists(A) := f(A) \\ f^*(B) := f^{-1}(B) \\ f_\forall(A) := Y - f(X - A) \end{array} \right.$$

- The pairs $(f_\exists, f^*)$ and $(f^*, f_\forall)$ are not quite pairs of inverses. But ... they do satisfy adjointness relations (namely):

$$f_\exists \dashv f^* \dashv f_\forall$$

as functors between the poset categories $2^{\mathbf{X}}, 2^{\mathbf{Y}}$ (the objects are subsets and arrows correspond to inclusions).

This is just a *chic* way of saying that for $A \in 2^{\mathbf{X}}, B \in 2^{\mathbf{Y}}$, $f_\exists(A) \subset B \Leftrightarrow A \subset f^*(B)$, and $A \subset f^*(B) \Leftrightarrow f_\forall(A) \subset B$.

# In the language of maps instead of quantifiers

- Let $\mathbf{X} \xrightarrow{f} \mathbf{Y}$ be a map (between sets).
- Then there are induced maps:

$$2^{\mathbf{X}} \overset{\xrightarrow{f_\exists}}{\underset{\xrightarrow{f_\forall}}{\xleftarrow{f^*}}} 2^{\mathbf{Y}} \quad \left| \quad \begin{array}{c} f_\exists(A) := f(A) \\ f^*(B) := f^{-1}(B) \\ f_\forall(A) := Y - f(X - A) \end{array} \right.$$

- The pairs $(f_\exists, f^*)$ and $(f^*, f_\forall)$ are not quite pairs of inverses. But ... they do satisfy adjointness relations (namely):

$$f_\exists \dashv f^* \dashv f_\forall$$

as functors between the poset categories $2^{\mathbf{X}}, 2^{\mathbf{Y}}$ (the objects are subsets and arrows correspond to inclusions).
This is just a *chic* way of saying that for $A \in 2^{\mathbf{X}}, B \in 2^{\mathbf{Y}}$,
$f_\exists(A) \subset B \Leftrightarrow A \subset f^*(B)$, and $A \subset f^*(B) \Leftrightarrow f_\forall(A) \subset B$.

# Tarski-Seidenberg arrow-theoretically ....

- For any semi-algebraic set $\mathbf{X}$, let $\mathcal{S}(\mathbf{X})$ denote the set of semi-algebraic subsets of $\mathbf{X}$.
- Let $\mathbf{X}, \mathbf{Y}$ be semi-algebraic sets, and $\mathbf{X} \xrightarrow{f} \mathbf{Y}$ a polynomial map.
- (Tarski-Seidenberg restated) The restrictions of the maps $f^{\exists}, f^{*}, f^{\forall}$ give functors (maps)

$$\mathcal{S}(\mathbf{X}) \begin{array}{c} \xrightarrow{f_{\exists}} \\ \xleftarrow{f^{*}} \\ \xrightarrow{f_{\forall}} \end{array} \mathcal{S}(\mathbf{Y})$$

(i.e. they carry semi-algebraic subsets to semi-algebraic subsets).

# Tarski-Seidenberg arrow-theoretically ....

- For any semi-algebraic set $\mathbf{X}$, let $\mathcal{S}(\mathbf{X})$ denote the set of semi-algebraic subsets of $\mathbf{X}$.
- Let $\mathbf{X}, \mathbf{Y}$ be semi-algebraic sets, and $\mathbf{X} \xrightarrow{f} \mathbf{Y}$ a polynomial map.
- (Tarski-Seidenberg restated) The restrictions of the maps $f^{\exists}, f^{*}, f^{\forall}$ give functors (maps)

$$\mathcal{S}(\mathbf{X}) \begin{array}{c} \xrightarrow{f_{\exists}} \\ \xleftarrow{f^{*}} \\ \xrightarrow{f_{\forall}} \end{array} \mathcal{S}(\mathbf{Y})$$

(i.e. they carry semi-algebraic subsets to semi-algebraic subsets).

# Tarski-Seidenberg arrow-theoretically ....

- For any semi-algebraic set $\mathbf{X}$, let $\mathcal{S}(\mathbf{X})$ denote the set of semi-algebraic subsets of $\mathbf{X}$.
- Let $\mathbf{X}, \mathbf{Y}$ be semi-algebraic sets, and $\mathbf{X} \xrightarrow{f} \mathbf{Y}$ a polynomial map.
- (Tarski-Seidenberg restated) The restrictions of the maps $f^{\exists}, f^{*}, f^{\forall}$ give functors (maps)

$$\mathcal{S}(\mathbf{X}) \begin{array}{c} \xrightarrow{f_{\exists}} \\ \xleftarrow{f^{*}} \\ \xrightarrow{f_{\forall}} \end{array} \mathcal{S}(\mathbf{Y})$$

(i.e. they carry semi-algebraic subsets to semi-algebraic subsets).

# Triviality of semi-algebraic maps

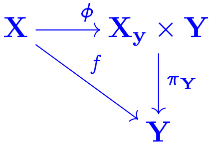
Yet harder. More than just Tarski-Seidenberg is true...

We say that a semi-algebraic map $\mathbf{X} \xrightarrow{f} \mathbf{Y}$ is semi-algebraically trivial, if there exists $\mathbf{y} \in \mathbf{Y}$, and a semi-algebraic homemorphism $\phi : \mathbf{X} \to \mathbf{X_y} \times \mathbf{Y}$ (denoting $\mathbf{X_y} = f^{-1}(\mathbf{y})$) such that the following diagram is commutative.

# Triviality of semi-algebraic maps

Yet harder. More than just Tarski-Seidenberg is true...

We say that a semi-algebraic map $\mathbf{X} \xrightarrow{f} \mathbf{Y}$ is semi-algebraically trivial, if there exists $\mathbf{y} \in \mathbf{Y}$, and a semi-algebraic homemorphism $\phi : \mathbf{X} \to \mathbf{X_y} \times \mathbf{Y}$ (denoting $\mathbf{X_y} = f^{-1}(\mathbf{y})$) such that the following diagram is commutative.



# Local triviality of semi-algebraic maps

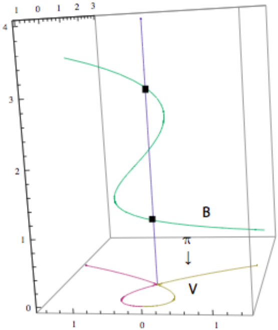
**Theorem (Hardt (1980))**

*Let $\mathbf{X} \xrightarrow{f} \mathbf{Y}$ be a semi-algebraic map. Then, there is a finite partition $\{\mathbf{Y}_i\}_{i \in I}$ of $\mathbf{Y}$ into locally closed semi-algebraic subsets $\mathbf{Y}_i$, such that for each $i \in I$, $f|_{f^{-1}(\mathbf{Y}_i)} : f^{-1}(\mathbf{Y}_i) \to \mathbf{Y}_i$ is semi-algebraically trivial.*

# Local triviality of semi-algebraic maps

**Theorem (Hardt (1980))**

*Let $\mathbf{X} \xrightarrow{f} \mathbf{Y}$ be a semi-algebraic map. Then, there is a finite partition $\{\mathbf{Y}_i\}_{i \in I}$ of $\mathbf{Y}$ into locally closed semi-algebraic subsets $\mathbf{Y}_i$, such that for each $i \in I$, $f|_{f^{-1}(\mathbf{Y}_i)} : f^{-1}(\mathbf{Y}_i) \to \mathbf{Y}_i$ is semi-algebraically trivial.*



# Local triviality of semi-algebraic maps

Theorem (Hardt (1980))

*Let* $\mathbf{X} \xrightarrow{f} \mathbf{Y}$ *be a semi-algebraic map. Then, there is a finite partition* $\{\mathbf{Y}_i\}_{i \in I}$ *of* $\mathbf{Y}$ *into locally closed semi-algebraic subsets* $\mathbf{Y}_i$*, such that for each* $i \in I$*,* $f|_{f^{-1}(\mathbf{Y}_i)} : f^{-1}(\mathbf{Y}_i) \to \mathbf{Y}_i$ *is semi-algebraically trivial.*

Generalization of Tarski-Seidenberg, since the image $f(\mathbf{X})$ is a (disjoint) union of a sub-collection of the $\mathbf{Y}_i$'s (and so in particular semi-algebraic). But some drawbacks ...

# Local triviality of semi-algebraic maps

**Theorem (Hardt (1980))**

*Let $\mathbf{X} \xrightarrow{f} \mathbf{Y}$ be a semi-algebraic map. Then, there is a finite partition $\{\mathbf{Y}_i\}_{i \in I}$ of $\mathbf{Y}$ into locally closed semi-algebraic subsets $\mathbf{Y}_i$, such that for each $i \in I$, $f|_{f^{-1}(\mathbf{Y}_i)} : f^{-1}(\mathbf{Y}_i) \to \mathbf{Y}_i$ is semi-algebraically trivial.*

Generalization of Tarski-Seidenberg, since the image $f(\mathbf{X})$ is a (disjoint) union of a sub-collection of the $\mathbf{Y}_i$'s (and so in particular semi-algebraic).

But some drawbacks ...

Homeomorphism type is difficult to quantify, undecidable to check, and the best complexity upper bound known for the induced partition is doubly exponential.

# Local triviality of semi-algebraic maps

Theorem (Hardt (1980))

*Let* $\mathbf{X} \xrightarrow{f} \mathbf{Y}$ *be a semi-algebraic map. Then, there is a finite partition* $\{\mathbf{Y}_i\}_{i \in I}$ *of* $\mathbf{Y}$ *into locally closed semi-algebraic subsets* $\mathbf{Y}_i$*, such that for each* $i \in I$*,* $f|_{f^{-1}(\mathbf{Y}_i)} : f^{-1}(\mathbf{Y}_i) \to \mathbf{Y}_i$ *is semi-algebraically trivial.*

Generalization of Tarski-Seidenberg, since the image $f(\mathbf{X})$ is a (disjoint) union of a sub-collection of the $\mathbf{Y}_i$'s (and so in particular semi-algebraic).
But some drawbacks ...
Homeomorphism type is difficult to quantify, undecidable to check, and the best complexity upper bound known for the induced partition is doubly exponential.
The formalism of "constructible sheaves" seems to be just the right compromise.

# Little detour – Pre-sheaves of $A$-modules

Let $A$ be a fixed commutative ring. For simplicity we will soon take $A = \mathbb{Q}$.

**Definition (Pre-sheaf of $A$-modules)**

A *pre-sheaf* $\mathcal{F}$ of $A$-modules over a topological space $\mathbf{X}$ associates to each open subset $U \subset \mathbf{X}$ an $A$-module $\mathcal{F}(\mathbf{U})$, such that that for all pairs of open subsets $\mathbf{U}, \mathbf{V}$ of $\mathbf{X}$, with $\mathbf{V} \subset \mathbf{U}$, there exists a *restriction* homomorphism $r_{\mathbf{U},\mathbf{V}} : \mathcal{F}(\mathbf{U}) \to \mathcal{F}(\mathbf{V})$ satisfying:

1. $r_{\mathbf{U},\mathbf{U}} = \mathrm{Id}_{\mathcal{F}(\mathbf{U})}$;
2. for $\mathbf{U}, \mathbf{V}, \mathbf{W}$ open subsets of $\mathbf{X}$, with $\mathbf{W} \subset \mathbf{V} \subset \mathbf{U}$,

$$r_{\mathbf{U},\mathbf{W}} = r_{\mathbf{V},\mathbf{W}} \circ r_{\mathbf{U},\mathbf{V}}.$$

(For open subsets $\mathbf{U}, \mathbf{V} \subset \mathbf{X}$, $\mathbf{V} \subset \mathbf{U}$, and $s \in \mathcal{F}(\mathbf{U})$, we will sometimes denote the element $r_{\mathbf{U},\mathbf{V}}(s) \in \mathcal{F}(\mathbf{V})$ simply by $s|_{\mathbf{V}}$.)

# Little detour – Pre-sheaves of $A$-modules

Let $A$ be a fixed commutative ring. For simplicity we will soon take $A = \mathbb{Q}$.

**Definition (Pre-sheaf of $A$-modules)**

A *pre-sheaf* $\mathcal{F}$ of $A$-modules over a topological space $\mathbf{X}$ associates to each open subset $U \subset \mathbf{X}$ an $A$-module $\mathcal{F}(\mathbf{U})$, such that that for all pairs of open subsets $\mathbf{U}, \mathbf{V}$ of $\mathbf{X}$, with $\mathbf{V} \subset \mathbf{U}$, there exists a *restriction* homomorphism $r_{\mathbf{U},\mathbf{V}} : \mathcal{F}(\mathbf{U}) \to \mathcal{F}(\mathbf{V})$ satisfying:

1. $r_{\mathbf{U},\mathbf{U}} = \mathrm{Id}_{\mathcal{F}(\mathbf{U})}$,
2. for $\mathbf{U}, \mathbf{V}, \mathbf{W}$ open subsets of $\mathbf{X}$, with $\mathbf{W} \subset \mathbf{V} \subset \mathbf{U}$,

$$r_{\mathbf{U},\mathbf{W}} = r_{\mathbf{V},\mathbf{W}} \circ r_{\mathbf{U},\mathbf{V}}.$$

(For open subsets $\mathbf{U}, \mathbf{V} \subset \mathbf{X}$, $\mathbf{V} \subset \mathbf{U}$, and $s \in \mathcal{F}(\mathbf{U})$, we will sometimes denote the element $r_{\mathbf{U},\mathbf{V}}(s) \in \mathcal{F}(\mathbf{V})$ simply by $s|_{\mathbf{V}}$.)

# Little detour – Pre-sheaves of $A$-modules

Let $A$ be a fixed commutative ring. For simplicity we will soon take $A = \mathbb{Q}$.

Definition (Pre-sheaf of $A$-modules)

A *pre-sheaf* $\mathcal{F}$ *of* $A$-*modules* over a topological space $\mathbf{X}$ associates to each open subset $U \subset \mathbf{X}$ an $A$-module $\mathcal{F}(\mathbf{U})$, such that that for all pairs of open subsets $\mathbf{U}, \mathbf{V}$ of $\mathbf{X}$, with $\mathbf{V} \subset \mathbf{U}$, there exists a *restriction* homomorphism $r_{\mathbf{U},\mathbf{V}} : \mathcal{F}(\mathbf{U}) \to \mathcal{F}(\mathbf{V})$ satisfying:

1. $r_{\mathbf{U},\mathbf{U}} = \mathrm{Id}_{\mathcal{F}(\mathbf{U})}$,
2. for $\mathbf{U}, \mathbf{V}, \mathbf{W}$ open subsets of $\mathbf{X}$, with $\mathbf{W} \subset \mathbf{V} \subset \mathbf{U}$,

$$r_{\mathbf{U},\mathbf{W}} = r_{\mathbf{V},\mathbf{W}} \circ r_{\mathbf{U},\mathbf{V}}.$$

(For open subsets $\mathbf{U}, \mathbf{V} \subset \mathbf{X}$, $\mathbf{V} \subset \mathbf{U}$, and $s \in \mathcal{F}(\mathbf{U})$, we will sometimes denote the element $r_{\mathbf{U},\mathbf{V}}(s) \in \mathcal{F}(\mathbf{V})$ simply by $s|_{\mathbf{V}}$.)

# Little detour – Pre-sheaves of $A$-modules

Let $A$ be a fixed commutative ring. For simplicity we will soon take $A = \mathbb{Q}$.

Definition (Pre-sheaf of $A$-modules)

A *pre-sheaf $\mathcal{F}$ of $A$-modules* over a topological space $\mathbf{X}$ associates to each open subset $U \subset \mathbf{X}$ an $A$-module $\mathcal{F}(\mathbf{U})$, such that that for all pairs of open subsets $\mathbf{U}, \mathbf{V}$ of $\mathbf{X}$, with $\mathbf{V} \subset \mathbf{U}$, there exists a *restriction* homomorphism $r_{\mathbf{U},\mathbf{V}} : \mathcal{F}(\mathbf{U}) \to \mathcal{F}(\mathbf{V})$ satisfying:

1. $r_{\mathbf{U},\mathbf{U}} = \mathrm{Id}_{\mathcal{F}(\mathbf{U})}$,
2. for $\mathbf{U}, \mathbf{V}, \mathbf{W}$ open subsets of $\mathbf{X}$, with $\mathbf{W} \subset \mathbf{V} \subset \mathbf{U}$,

$$r_{\mathbf{U},\mathbf{W}} = r_{\mathbf{V},\mathbf{W}} \circ r_{\mathbf{U},\mathbf{V}}.$$

(For open subsets $\mathbf{U}, \mathbf{V} \subset \mathbf{X}$, $\mathbf{V} \subset \mathbf{U}$, and $s \in \mathcal{F}(\mathbf{U})$, we will sometimes denote the element $r_{\mathbf{U},\mathbf{V}}(s) \in \mathcal{F}(\mathbf{V})$ simply by $s|_{\mathbf{V}}$.)

# Little detour – Pre-sheaves of $A$-modules

Let $A$ be a fixed commutative ring. For simplicity we will soon take $A = \mathbb{Q}$.

## Definition (Pre-sheaf of $A$-modules)

A *pre-sheaf* $\mathcal{F}$ of $A$-modules over a topological space $\mathbf{X}$ associates to each open subset $U \subset \mathbf{X}$ an $A$-module $\mathcal{F}(\mathbf{U})$, such that that for all pairs of open subsets $\mathbf{U}, \mathbf{V}$ of $\mathbf{X}$, with $\mathbf{V} \subset \mathbf{U}$, there exists a *restriction* homomorphism $r_{\mathbf{U},\mathbf{V}} : \mathcal{F}(\mathbf{U}) \to \mathcal{F}(\mathbf{V})$ satisfying:

1. $r_{\mathbf{U},\mathbf{U}} = \mathrm{Id}_{\mathcal{F}(\mathbf{U})}$,
2. for $\mathbf{U}, \mathbf{V}, \mathbf{W}$ open subsets of $\mathbf{X}$, with $\mathbf{W} \subset \mathbf{V} \subset \mathbf{U}$,

$$r_{\mathbf{U},\mathbf{W}} = r_{\mathbf{V},\mathbf{W}} \circ r_{\mathbf{U},\mathbf{V}}.$$

(For open subsets $\mathbf{U}, \mathbf{V} \subset \mathbf{X}$, $\mathbf{V} \subset \mathbf{U}$, and $s \in \mathcal{F}(\mathbf{U})$, we will sometimes denote the element $r_{\mathbf{U},\mathbf{V}}(s) \in \mathcal{F}(\mathbf{V})$ simply by $s|_{\mathbf{V}}$.)

# Little detour – Pre-sheaves of $A$-modules

Let $A$ be a fixed commutative ring. For simplicity we will soon take $A = \mathbb{Q}$.

Definition (Pre-sheaf of $A$-modules)

A *pre-sheaf* $\mathcal{F}$ *of* $A$*-modules* over a topological space $\mathbf{X}$ associates to each open subset $U \subset \mathbf{X}$ an $A$-module $\mathcal{F}(\mathbf{U})$, such that that for all pairs of open subsets $\mathbf{U}, \mathbf{V}$ of $\mathbf{X}$, with $\mathbf{V} \subset \mathbf{U}$, there exists a *restriction* homomorphism $r_{\mathbf{U},\mathbf{V}} : \mathcal{F}(\mathbf{U}) \to \mathcal{F}(\mathbf{V})$ satisfying:

1. $r_{\mathbf{U},\mathbf{U}} = \mathrm{Id}_{\mathcal{F}(\mathbf{U})}$,
2. for $\mathbf{U}, \mathbf{V}, \mathbf{W}$ open subsets of $\mathbf{X}$, with $\mathbf{W} \subset \mathbf{V} \subset \mathbf{U}$,

$$r_{\mathbf{U},\mathbf{W}} = r_{\mathbf{V},\mathbf{W}} \circ r_{\mathbf{U},\mathbf{V}}.$$

(For open subsets $\mathbf{U}, \mathbf{V} \subset \mathbf{X}$, $\mathbf{V} \subset \mathbf{U}$, and $s \in \mathcal{F}(\mathbf{U})$, we will sometimes denote the element $r_{\mathbf{U},\mathbf{V}}(s) \in \mathcal{F}(\mathbf{V})$ simply by $s|_{\mathbf{V}}$.)

# Little detour – Pre-sheaves of $A$-modules

Let $A$ be a fixed commutative ring. For simplicity we will soon take $A = \mathbb{Q}$.

Definition (Pre-sheaf of $A$-modules)

A *pre-sheaf* $\mathcal{F}$ *of* $A$*-modules* over a topological space $\mathbf{X}$ associates to each open subset $U \subset \mathbf{X}$ an $A$-module $\mathcal{F}(\mathbf{U})$, such that that for all pairs of open subsets $\mathbf{U}, \mathbf{V}$ of $\mathbf{X}$, with $\mathbf{V} \subset \mathbf{U}$, there exists a *restriction* homomorphism $r_{\mathbf{U},\mathbf{V}} : \mathcal{F}(\mathbf{U}) \to \mathcal{F}(\mathbf{V})$ satisfying:

1. $r_{\mathbf{U},\mathbf{U}} = \mathrm{Id}_{\mathcal{F}(\mathbf{U})}$,
2. for $\mathbf{U}, \mathbf{V}, \mathbf{W}$ open subsets of $\mathbf{X}$, with $\mathbf{W} \subset \mathbf{V} \subset \mathbf{U}$,

$$r_{\mathbf{U},\mathbf{W}} = r_{\mathbf{V},\mathbf{W}} \circ r_{\mathbf{U},\mathbf{V}}.$$

(For open subsets $\mathbf{U}, \mathbf{V} \subset \mathbf{X}$, $\mathbf{V} \subset \mathbf{U}$, and $s \in \mathcal{F}(\mathbf{U})$, we will sometimes denote the element $r_{\mathbf{U},\mathbf{V}}(s) \in \mathcal{F}(\mathbf{V})$ simply by $s|_{\mathbf{V}}$.)

# Sheaves with constant coefficients

## Definition (Sheaf of $A$-modules)

A pre-sheaf $\mathcal{F}$ of $A$-modules on $\mathbf{X}$ is said to be a *sheaf* if it satisfies the following two axioms. For any collection of open subsets $\{\mathbf{U}_i\}_{i \in I}$ of $\mathbf{X}$ with $\mathbf{U} = \bigcup_{i \in I} \mathbf{U}_i$;

1. if $s \in \mathcal{F}(\mathbf{U})$ and $s|_{\mathbf{U}_i} = 0$ for all $i \in I$, then $s = 0$;
2. if for all $i \in I$ there exists $s_i \in \mathcal{F}(\mathbf{U}_i)$ such that

$$s_i|_{\mathbf{U}_i \cap \mathbf{U}_j} = s_j|_{\mathbf{U}_i \cap \mathbf{U}_j}$$

for all $i, j \in I$, then there exists $s \in \mathcal{F}(\mathbf{U})$ such that $s|_{\mathbf{U}_i} = s_i$ for each $i \in I$.

# Sheaves with constant coefficients

Definition (Sheaf of $A$-modules)

A pre-sheaf $\mathcal{F}$ of $A$-modules on $\mathbf{X}$ is said to be a *sheaf* if it satisfies the following two axioms. For any collection of open subsets $\{\mathbf{U}_i\}_{i\in I}$ of $\mathbf{X}$ with $\mathbf{U} = \bigcup_{i\in I} \mathbf{U}_i$;

1. if $s \in \mathcal{F}(\mathbf{U})$ and $s|_{\mathbf{U}_i} = 0$ for all $i \in I$, then $s = 0$;
2. if for all $i \in I$ there exists $s_i \in \mathcal{F}(\mathbf{U}_i)$ such that

$$s_i|_{\mathbf{U}_i\cap\mathbf{U}_j} = s_j|_{\mathbf{U}_i\cap\mathbf{U}_j}$$

for all $i, j \in I$, then there exists $s \in \mathcal{F}(\mathbf{U})$ such that $s|_{\mathbf{U}_i} = s_i$ for each $i \in I$.

# Sheaves with constant coefficients

### Definition (Sheaf of $A$-modules)

A pre-sheaf $\mathcal{F}$ of $A$-modules on $\mathbf{X}$ is said to be a *sheaf* if it satisfies the following two axioms. For any collection of open subsets $\{\mathbf{U}_i\}_{i \in I}$ of $\mathbf{X}$ with $\mathbf{U} = \bigcup_{i \in I} \mathbf{U}_i$;

1. if $s \in \mathcal{F}(\mathbf{U})$ and $s|_{\mathbf{U}_i} = 0$ for all $i \in I$, then $s = 0$;
2. if for all $i \in I$ there exists $s_i \in \mathcal{F}(\mathbf{U}_i)$ such that

$$s_i|_{\mathbf{U}_i \cap \mathbf{U}_j} = s_j|_{\mathbf{U}_i \cap \mathbf{U}_j}$$

   for all $i, j \in I$, then there exists $s \in \mathcal{F}(\mathbf{U})$ such that $s|_{\mathbf{U}_i} = s_i$ for each $i \in I$.

# Stalks of a sheaf

**Definition (Stalk of a sheaf at a point)**

Let $\mathcal{F}$ be a (pre)-sheaf of $A$-modules on $\mathbf{X}$ and $\mathbf{x} \in X$. The *stalk* $\mathcal{F}_{\mathbf{x}}$ of $\mathcal{F}$ at $\mathbf{x}$ is defined as the inductive limit

$$\mathcal{F}_{\mathbf{x}} = \varinjlim_{\mathbf{U} \ni \mathbf{x}} \mathcal{F}(\mathbf{U}).$$

# Derived category of sheaves on $X$

- One first considers the category whose objects are *complexes of sheaves* on $X$, and whose morphisms are *homotopy classes* of morphisms of complexes of sheaves.
- One then localizes with respect to a class of arrows so that complexes homotopic to 0 become isomorphic, to obtain the derived category $\mathrm{D}(X)$ (resp. $\mathrm{D}^b(X)$).
- This is no longer an abelian category but a *triangulated category*. Exact sequences replaced by distinguished triangles and so on...
- For our purposes it is "ok" to think of an object in $\mathrm{D}(X)$ as a "complex of sheaves".
- If $X = \{\mathrm{pt}\}$, then an object in $\mathrm{D}^b(X)$ is represented by a bounded complex $\mathrm{C}^\bullet$ of $A$-modules, and $\mathrm{C}^\bullet$ is isomorphic in the derived category to the complex $\mathrm{H}^*(\mathrm{C}^\bullet)$ (with all differentials $= 0$).
- In other words, $\mathrm{C}^\bullet \cong \oplus_{n\in\mathbb{Z}}\mathrm{H}^n(\mathrm{C}^\bullet)[-n]$. But this is *not true* in general (i.e. if $X$ is not a point).

# Derived category of sheaves on $X$

- One first considers the category whose objects are *complexes of sheaves* on $X$, and whose morphisms are *homotopy classes* of morphisms of complexes of sheaves.
- One then localizes with respect to a class of arrows so that complexes homotopic to 0 become isomorphic, to obtain the derived category $\mathrm{D}(X)$ (resp. $\mathrm{D}^b(X)$).
- This is no longer an abelian category but a *triangulated category*. Exact sequences replaced by distinguished triangles and so on...
- For our purposes it is "ok" to think of an object in $\mathrm{D}(X)$ as a "complex of sheaves".
- If $X = \{\mathrm{pt}\}$, then an object in $\mathrm{D}^b(X)$ is represented by a bounded complex $\mathrm{C}^\bullet$ of $A$-modules, and $\mathrm{C}^\bullet$ is isomorphic in the derived category to the complex $\mathrm{H}^*(\mathrm{C}^\bullet)$ (with all differentials $= 0$).
- In other words, $\mathrm{C}^\bullet \cong \oplus_{n\in\mathbb{Z}}\mathrm{H}^n(\mathrm{C}^\bullet)[-n]$. But this is *not true* in general (i.e. if $X$ is not a point).

# Derived category of sheaves on $X$

- One first considers the category whose objects are *complexes of sheaves* on $X$, and whose morphisms are *homotopy classes* of morphisms of complexes of sheaves.
- One then localizes with respect to a class of arrows so that complexes homotopic to 0 become isomorphic, to obtain the derived category $\mathrm{D}(X)$ (resp. $\mathrm{D}^b(X)$).
- This is no longer an abelian category but a *triangulated category*. Exact sequences replaced by distinguished triangles and so on...
- For our purposes it is "ok" to think of an object in $\mathrm{D}(X)$ as a "complex of sheaves".
- If $X = \{\mathrm{pt}\}$, then an object in $\mathrm{D}^b(X)$ is represented by a bounded complex $\mathrm{C}^\bullet$ of $A$-modules, and $\mathrm{C}^\bullet$ is isomorphic in the derived category to the complex $\mathrm{H}^*(\mathrm{C}^\bullet)$ (with all differentials $= 0$).
- In other words, $\mathrm{C}^\bullet \cong \oplus_{n\in\mathbb{Z}}\mathrm{H}^n(\mathrm{C}^\bullet)[-n]$. But this is *not true* in general (i.e. if $X$ is not a point).

# Derived category of sheaves on $X$

- One first considers the category whose objects are *complexes of sheaves* on $X$, and whose morphisms are *homotopy classes* of morphisms of complexes of sheaves.
- One then localizes with respect to a class of arrows so that complexes homotopic to 0 become isomorphic, to obtain the derived category $\mathrm{D}(X)$ (resp. $\mathrm{D}^b(X)$).
- This is no longer an abelian category but a *triangulated category*. Exact sequences replaced by distinguished triangles and so on...
- For our purposes it is "ok" to think of an object in $\mathrm{D}(X)$ as a "complex of sheaves".
- If $X = \{\mathrm{pt}\}$, then an object in $\mathrm{D}^b(X)$ is represented by a bounded complex $\mathrm{C}^\bullet$ of $A$-modules, and $\mathrm{C}^\bullet$ is isomorphic in the derived category to the complex $\mathrm{H}^*(\mathrm{C}^\bullet)$ (with all differentials $= 0$).
- In other words, $\mathrm{C}^\bullet \cong \oplus_{n\in\mathbb{Z}}\mathrm{H}^n(\mathrm{C}^\bullet)[-n]$. But this is *not true* in general (i.e. if $X$ is not a point).

# Derived category of sheaves on $X$

- One first considers the category whose objects are *complexes of sheaves* on $X$, and whose morphisms are *homotopy classes* of morphisms of complexes of sheaves.
- One then localizes with respect to a class of arrows so that complexes homotopic to 0 become isomorphic, to obtain the derived category $\mathrm{D}(X)$ (resp. $\mathrm{D}^b(X)$).
- This is no longer an abelian category but a *triangulated category*. Exact sequences replaced by distinguished triangles and so on...
- For our purposes it is "ok" to think of an object in $\mathrm{D}(X)$ as a "complex of sheaves".
- If $X = \{\mathrm{pt}\}$, then an object in $\mathrm{D}^b(X)$ is represented by a bounded complex $\mathrm{C}^\bullet$ of $A$-modules, and $\mathrm{C}^\bullet$ is isomorphic in the derived category to the complex $\mathrm{H}^*(\mathrm{C}^\bullet)$ (with all differentials $= 0$).
- In other words, $\mathrm{C}^\bullet \cong \oplus_{n\in\mathbb{Z}} \mathrm{H}^n(\mathrm{C}^\bullet)[-n]$. But this is *not true* in general (i.e. if $X$ is not a point).

# Derived category of sheaves on $X$

- One first considers the category whose objects are *complexes of sheaves* on $X$, and whose morphisms are *homotopy classes* of morphisms of complexes of sheaves.
- One then localizes with respect to a class of arrows so that complexes homotopic to 0 become isomorphic, to obtain the derived category $\mathrm{D}(X)$ (resp. $\mathrm{D}^b(X)$).
- This is no longer an abelian category but a *triangulated category*. Exact sequences replaced by distinguished triangles and so on...
- For our purposes it is "ok" to think of an object in $\mathrm{D}(X)$ as a "complex of sheaves".
- If $X = \{\mathrm{pt}\}$, then an object in $\mathrm{D}^b(X)$ is represented by a bounded complex $\mathrm{C}^\bullet$ of $A$-modules, and $\mathrm{C}^\bullet$ is isomorphic in the derived category to the complex $\mathrm{H}^*(\mathrm{C}^\bullet)$ (with all differentials $= 0$).
- In other words, $\mathrm{C}^\bullet \cong \oplus_{n\in\mathbb{Z}}\mathrm{H}^n(\mathrm{C}^\bullet)[-n]$. But this is *not true* in general (i.e. if $X$ is not a point).

# Operations on sheaves, derived images

Let $\mathcal{F}$ be a sheaf on $\mathbf{X}$, and $\mathcal{G}$ a sheaf on $\mathbf{Y}$, and $f : \mathbf{X} \to \mathbf{Y}$ a continuous map. Then, there exists naturally defined sheaves:

- $f^{-1}(\mathcal{G})$ – a sheaf on $\mathbf{X}$ (pull back). ($f^{-1}$ is an exact functor.)
- The derived direct image denoted $Rf_*(\mathcal{F})$ is an object in $\mathrm{D}(Y)$ (and thus should be thought of as a complex of sheaves on $Y$).
- We denote for $i \in \mathbb{Z}$, $R^i f_*(\mathcal{F})$ the sheaf $\mathcal{H}^i(Rf_*(\mathcal{F}))$ – but these separately don't determine $Rf_*(\mathcal{F})$.
- In the special case when $\mathcal{F} = A_{\mathbf{X}}$ (the constant sheaf on $\mathbf{X}$), $Rf_*(\mathcal{F})$ is obtained by associating to each open $\mathbf{U} \subset \mathbf{Y}$, a complex of $A$-modules obtained by taking sections of a flabby resolution of the sheaf $A_{f^{-1}(U)}$.
- In this case, for $\mathbf{y} \in \mathbf{Y}$, the stalk $Rf_*(\mathcal{F})_{\mathbf{y}}$ is an object of the *derived category of A-modules* and is isomorphic (in the derived category) to $\oplus_n \mathrm{H}^*(f^{-1}(\mathbf{y}), A)[-n]$.

# Operations on sheaves, derived images

Let $\mathcal{F}$ be a sheaf on $\mathbf{X}$, and $\mathcal{G}$ a sheaf on $\mathbf{Y}$, and $f : \mathbf{X} \to \mathbf{Y}$ a continuous map. Then, there exists naturally defined sheaves:

- $f^{-1}(\mathcal{G})$ – a sheaf on $\mathbf{X}$ (pull back). ($f^{-1}$ is an exact functor.)
- The derived direct image denoted $Rf_*(\mathcal{F})$ is an object in $\mathrm{D}(Y)$ (and thus should be thought of as a complex of sheaves on $Y$).
- We denote for $i \in \mathbb{Z}$, $R^i f_*(\mathcal{F})$ the sheaf $\mathcal{H}^i(Rf_*(\mathcal{F}))$ – but these separately don't determine $Rf_*(\mathcal{F})$.
- In the special case when $\mathcal{F} = A_{\mathbf{X}}$ (the constant sheaf on $\mathbf{X}$), $Rf_*(\mathcal{F})$ is obtained by associating to each open $\mathbf{U} \subset \mathbf{Y}$, a complex of $A$-modules obtained by taking sections of a flabby resolution of the sheaf $A_{f^{-1}(U)}$.
- In this case, for $\mathbf{y} \in \mathbf{Y}$, the stalk $Rf_*(\mathcal{F})_{\mathbf{y}}$ is an object of the *derived category of A-modules* and is isomorphic (in the derived category) to $\oplus_n \mathrm{H}^*(f^{-1}(\mathbf{y}), A)[-n]$.

# Operations on sheaves, derived images

Let $\mathcal{F}$ be a sheaf on $\mathbf{X}$, and $\mathcal{G}$ a sheaf on $\mathbf{Y}$, and $f : \mathbf{X} \to \mathbf{Y}$ a continuous map. Then, there exists naturally defined sheaves:

- $f^{-1}(\mathcal{G})$ – a sheaf on $\mathbf{X}$ (pull back). ($f^{-1}$ is an exact functor.)
- The derived direct image denoted $Rf_*(\mathcal{F})$ is an object in $\mathrm{D}(Y)$ (and thus should be thought of as a complex of sheaves on $Y$).
- We denote for $i \in \mathbb{Z}$, $R^i f_*(\mathcal{F})$ the sheaf $\mathcal{H}^i(Rf_*(\mathcal{F}))$ – but these separately don't determine $Rf_*(\mathcal{F})$.
- In the special case when $\mathcal{F} = A_{\mathbf{X}}$ (the constant sheaf on $\mathbf{X}$), $Rf_*(\mathcal{F})$ is obtained by associating to each open $\mathbf{U} \subset \mathbf{Y}$, a complex of $A$-modules obtained by taking sections of a flabby resolution of the sheaf $A_{f^{-1}(U)}$.
- In this case, for $\mathbf{y} \in \mathbf{Y}$, the stalk $Rf_*(\mathcal{F})_{\mathbf{y}}$ is an object of the *derived category of A-modules* and is isomorphic (in the derived category) to $\oplus_n \mathrm{H}^*(f^{-1}(\mathbf{y}), A)[-n]$.

# Operations on sheaves, derived images

Let $\mathcal{F}$ be a sheaf on $\mathbf{X}$, and $\mathcal{G}$ a sheaf on $\mathbf{Y}$, and $f : \mathbf{X} \to \mathbf{Y}$ a continuous map. Then, there exists naturally defined sheaves:

- $f^{-1}(\mathcal{G})$ – a sheaf on $\mathbf{X}$ (pull back). ($f^{-1}$ is an exact functor.)
- The derived direct image denoted $Rf_*(\mathcal{F})$ is an object in $\mathrm{D}(Y)$ (and thus should be thought of as a complex of sheaves on $Y$).
- We denote for $i \in \mathbb{Z}$, $R^i f_*(\mathcal{F})$ the sheaf $\mathcal{H}^i(Rf_*(\mathcal{F}))$ – but these separately don't determine $Rf_*(\mathcal{F})$.
- In the special case when $\mathcal{F} = A_{\mathbf{X}}$ (the constant sheaf on $\mathbf{X}$), $Rf_*(\mathcal{F})$ is obtained by associating to each open $\mathbf{U} \subset \mathbf{Y}$, a complex of $A$-modules obtained by taking sections of a flabby resolution of the sheaf $A_{f^{-1}(U)}$.
- In this case, for $\mathbf{y} \in \mathbf{Y}$, the stalk $Rf_*(\mathcal{F})_{\mathbf{y}}$ is an object of the *derived category of A-modules* and is isomorphic (in the derived category) to $\oplus_n \mathrm{H}^*(f^{-1}(\mathbf{y}), A)[-n]$.

# Operations on sheaves, derived images

Let $\mathcal{F}$ be a sheaf on $\mathbf{X}$, and $\mathcal{G}$ a sheaf on $\mathbf{Y}$, and $f : \mathbf{X} \rightarrow \mathbf{Y}$ a continuous map. Then, there exists naturally defined sheaves:

- $f^{-1}(\mathcal{G})$ – a sheaf on $\mathbf{X}$ (pull back). ($f^{-1}$ is an exact functor.)
- The derived direct image denoted $Rf_*(\mathcal{F})$ is an object in $\mathrm{D}(Y)$ (and thus should be thought of as a complex of sheaves on $Y$).
- We denote for $i \in \mathbb{Z}$, $R^i f_*(\mathcal{F})$ the sheaf $\mathcal{H}^i(Rf_*(\mathcal{F}))$ – but these separately don't determine $Rf_*(\mathcal{F})$.
- In the special case when $\mathcal{F} = A_{\mathbf{X}}$ (the constant sheaf on $\mathbf{X}$), $Rf_*(\mathcal{F})$ is obtained by associating to each open $\mathbf{U} \subset \mathbf{Y}$, a complex of $A$-modules obtained by taking sections of a flabby resolution of the sheaf $A_{f^{-1}(U)}$.
- In this case, for $\mathbf{y} \in \mathbf{Y}$, the stalk $Rf_*(\mathcal{F})_{\mathbf{y}}$ is an object of the *derived category of A-modules* and is isomorphic (in the derived category) to $\oplus_n \mathrm{H}^*(f^{-1}(\mathbf{y}), A)[-n]$.

# High school example – discriminant of a real quadratic

**Logical formulation**

$$(\exists X) X^2 + 2BX + C = 0$$
$$\Updownarrow$$
$$B^2 - C \geq 0$$

# High school example – discriminant of a real quadratic

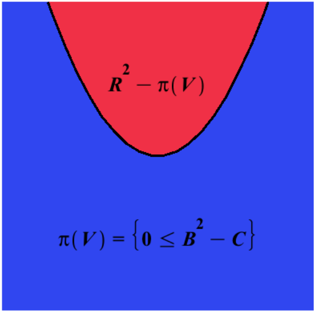
**Geometric formulation**

Defining $V \subset \mathbb{R}^3$ (with coordinates $X, B, C$) defined by $X^2 + 2BX + C = 0$ and $\pi : \mathbb{R}^3 \to \mathbb{R}^2, (x, b, c) \mapsto (b, c)$,

# High school example – discriminant of a real quadratic

## Geometric formulation

Defining $V \subset \mathbb{R}^3$ (with coordinates $X, B, C$) defined by $X^2 + 2BX + C = 0$ and $\pi : \mathbb{R}^3 \to \mathbb{R}^2, (x, b, c) \mapsto (b, c)$,



# High school example – discriminant of a real quadratic

### Sheaf theoretic formulation

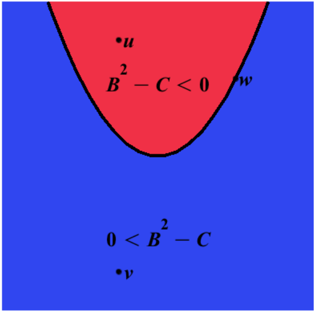
Denoting $j : V \hookrightarrow \mathbb{R}^3$, consider the sheaf $j_*(\mathbb{Q}_V) \cong \mathbb{Q}_{\mathbb{R}^3}|_V$, and its (derived) direct image $R\pi_*(j_*(\mathbb{Q}_V))$.

# High school example – discriminant of a real quadratic

## Sheaf theoretic formulation

Denoting $j : V \hookrightarrow \mathbb{R}^3$, consider the sheaf $j_*(\mathbb{Q}_V) \cong \mathbb{Q}_{\mathbb{R}^3}|_V$, and its (derived) direct image $R\pi_*(j_*(\mathbb{Q}_V))$.

The stalks of $R\pi_*(j_*(\mathbb{Q}_V))$ induce a finer partition:

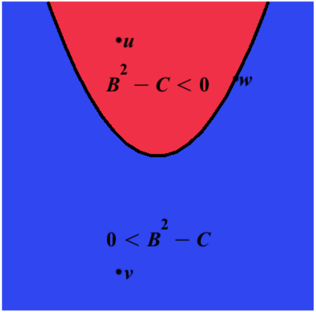


# High school example – discriminant of a real quadratic

## Sheaf theoretic formulation

Denoting $j : V \hookrightarrow \mathbb{R}^3$, consider the sheaf $j_*(\mathbb{Q}_V) \cong \mathbb{Q}_{\mathbb{R}^3}|_V$, and its (derived) direct image $R\pi_*(j_*(\mathbb{Q}_V))$.

The stalks of $R\pi_*(j_*(\mathbb{Q}_V))$ induce a finer partition:



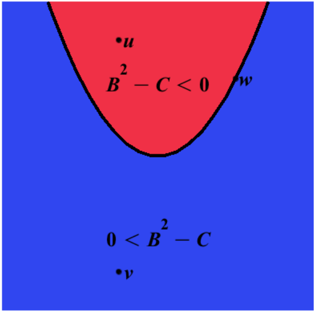

$(R\pi_*(j_*\mathbb{Q}_V))_u \cong 0$,

# High school example – discriminant of a real quadratic

## Sheaf theoretic formulation

Denoting $j : V \hookrightarrow \mathbb{R}^3$, consider the sheaf $j_*(\mathbb{Q}_V) \cong \mathbb{Q}_{\mathbb{R}^3}|_V$, and its (derived) direct image $R\pi_*(j_*(\mathbb{Q}_V))$.

The stalks of $R\pi_*(j_*(\mathbb{Q}_V))$ induce a finer partition:



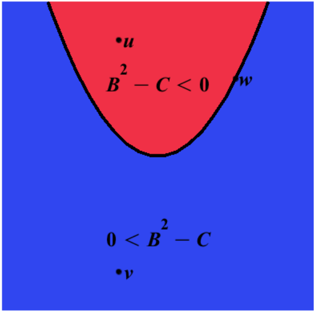

$(R\pi_*(j_*\mathbb{Q}_V))_u \cong 0, \quad (R\pi_*(j_*\mathbb{Q}_V))_v \cong \mathbb{Q} \oplus \mathbb{Q},$

# High school example – discriminant of a real quadratic

## Sheaf theoretic formulation

Denoting $j : V \hookrightarrow \mathbb{R}^3$, consider the sheaf $j_*(\mathbb{Q}_V) \cong \mathbb{Q}_{\mathbb{R}^3}|_V$, and its (derived) direct image $R\pi_*(j_*(\mathbb{Q}_V))$.

The stalks of $R\pi_*(j_*(\mathbb{Q}_V))$ induce a finer partition:




$(R\pi_*(j_*\mathbb{Q}_V))_u \cong 0, \quad (R\pi_*(j_*\mathbb{Q}_V))_v \cong \mathbb{Q} \oplus \mathbb{Q}, \quad (R\pi_*(j_*\mathbb{Q}_V))_w \cong \mathbb{Q}.$

# Example: Hopf vs trivial

Suppose that:

$$\mathbf{X} = \mathbb{S}^3 := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\},$$

$$\mathbf{X}' = \mathbb{S}^1 \times \mathbb{S}^2,$$

$$\mathbf{Y} = \mathbb{P}^1_{\mathbb{C}} \cong \mathbb{S}^2,$$

$$f : \mathbf{X} \to \mathbf{Y}, (z_1, z_2) \mapsto (z_1 : z_2) \text{ (Hopf fibration)},$$

$$g : \mathbf{X}' \to \mathbf{Y}, \text{ projection to the second factor.}$$

# Example: Hopf vs trivial

Suppose that:

$$\mathbf{X} = \mathbb{S}^3 := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\},$$

$$\mathbf{X}' = \mathbb{S}^1 \times \mathbb{S}^2,$$

$$\mathbf{Y} = \mathbb{P}^1_{\mathbb{C}} \cong \mathbb{S}^2,$$

$$f : \mathbf{X} \to \mathbf{Y}, (z_1, z_2) \mapsto (z_1 : z_2) \text{ (Hopf fibration)},$$

$$g : \mathbf{X}' \to \mathbf{Y}, \text{ projection to the second factor.}$$

# Example: Hopf vs trivial

Suppose that:

$$\mathbf{X} = \mathbb{S}^3 := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\},$$

$$\mathbf{X}' = \mathbb{S}^1 \times \mathbb{S}^2,$$

$$\mathbf{Y} = \mathbb{P}^1_{\mathbb{C}} \cong \mathbb{S}^2,$$

$$f : \mathbf{X} \to \mathbf{Y}, (z_1, z_2) \mapsto (z_1 : z_2) \text{ (Hopf fibration)},$$

$$g : \mathbf{X}' \to \mathbf{Y}, \text{ projection to the second factor.}$$

# Example: Hopf vs trivial

Suppose that:

$$\mathbf{X} = \mathbb{S}^3 := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\},$$

$$\mathbf{X}' = \mathbb{S}^1 \times \mathbb{S}^2,$$

$$\mathbf{Y} = \mathbb{P}^1_{\mathbb{C}} \cong \mathbb{S}^2,$$

$$f : \mathbf{X} \to \mathbf{Y}, (z_1, z_2) \mapsto (z_1 : z_2) \text{ (Hopf fibration)},$$

$$g : \mathbf{X}' \to \mathbf{Y}, \text{ projection to the second factor.}$$

# Example: Hopf vs trivial

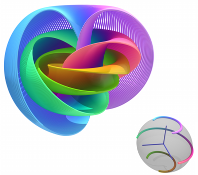
Suppose that:

$$\mathbf{X} = \mathbb{S}^3 := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\},$$

$$\mathbf{X}' = \mathbb{S}^1 \times \mathbb{S}^2,$$

$$\mathbf{Y} = \mathbb{P}^1_{\mathbb{C}} \cong \mathbb{S}^2,$$

$$f : \mathbf{X} \to \mathbf{Y}, (z_1, z_2) \mapsto (z_1 : z_2) \text{ (Hopf fibration)},$$



# Example: Hopf vs trivial

Suppose that:

$$\mathbf{X} = \mathbb{S}^3 := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\},$$

$$\mathbf{X}' = \mathbb{S}^1 \times \mathbb{S}^2,$$

$$\mathbf{Y} = \mathbb{P}^1_{\mathbb{C}} \cong \mathbb{S}^2,$$

$$f : \mathbf{X} \to \mathbf{Y}, (z_1, z_2) \mapsto (z_1 : z_2) \text{ (Hopf fibration)},$$

$$g : \mathbf{X}' \to \mathbf{Y}, \text{ projection to the second factor.}$$

# The higher derived images of the sheaves $\mathbb{Q}_{\mathbf{X}}$ and $\mathbb{Q}_{\mathbf{X}'}$ under $f$ and $g$

They are isomorphic !

$$R^0 f_*(\mathbb{Q}_{\mathbf{X}}) \cong \mathbb{Q}_{\mathbf{Y}} \cong R^0 g_*(\mathbb{Q}_{\mathbf{X}'}),$$

$$R^1 f_*(\mathbb{Q}_{\mathbf{X}}) \cong \mathbb{Q}_{\mathbf{Y}} \cong R^1 g_*(\mathbb{Q}_{\mathbf{X}'}).$$

# The higher derived images of the sheaves $\mathbb{Q}_{\mathbf{X}}$ and $\mathbb{Q}_{\mathbf{X}'}$ under $f$ and $g$

They are isomorphic !

$$R^0 f_*(\mathbb{Q}_{\mathbf{X}}) \cong \mathbb{Q}_{\mathbf{Y}} \cong R^0 g_*(\mathbb{Q}_{\mathbf{X}'}),$$

$$R^1 f_*(\mathbb{Q}_{\mathbf{X}}) \cong \mathbb{Q}_{\mathbf{Y}} \cong R^1 g_*(\mathbb{Q}_{\mathbf{X}'}).$$

# The higher derived images of the sheaves $\mathbb{Q}_{\mathbf{X}}$ and $\mathbb{Q}_{\mathbf{X}'}$ under $f$ and $g$

They are isomorphic !

$$R^0 f_*(\mathbb{Q}_{\mathbf{X}}) \cong \mathbb{Q}_{\mathbf{Y}} \cong R^0 g_*(\mathbb{Q}_{\mathbf{X}'}),$$

$$R^1 f_*(\mathbb{Q}_{\mathbf{X}}) \cong \mathbb{Q}_{\mathbf{Y}} \cong R^1 g_*(\mathbb{Q}_{\mathbf{X}'}).$$

# But in the derived category ...

stalks are isomorphic – for all $\mathbf{y} \in \mathbf{Y}$,

$$Rf_*(\mathbb{Q}_\mathbf{X})_\mathbf{y} \cong Rg_*(\mathbb{Q}_{\mathbf{X}'})_\mathbf{y} \cong \mathbb{Q}[-1] \oplus \mathbb{Q},$$

but ...

$$Rf_*(\mathbb{Q}_\mathbf{X}) \not\cong \mathbb{Q}_\mathbf{Y}[-1] \oplus \mathbb{Q}_\mathbf{Y} \cong Rg_*(\mathbb{Q}_{\mathbf{X}'}),$$

and to see that they are not isomorphic one has to notice ...

$$\mathbb{H}^*(\mathbf{Y}, Rf_*(\mathbb{Q}_\mathbf{X})) \cong \mathrm{H}^*(\mathbb{S}^3, \mathbb{Q}),$$

but ...

$$\mathbb{H}^*(\mathbf{Y}, Rg_*(\mathbb{Q}_{\mathbf{X}'})) \cong \mathrm{H}^*(\mathbb{S}^1 \times \mathbb{S}^2, \mathbb{Q}).$$

# But in the derived category ...

stalks are isomorphic – for all $\mathbf{y} \in \mathbf{Y}$,

$$Rf_*(\mathbb{Q}_{\mathbf{X}})_{\mathbf{y}} \cong Rg_*(\mathbb{Q}_{\mathbf{X}'})_{\mathbf{y}} \cong \mathbb{Q}[-1] \oplus \mathbb{Q},$$

but ...

$$Rf_*(\mathbb{Q}_{\mathbf{X}}) \not\cong \mathbb{Q}_{\mathbf{Y}}[-1] \oplus \mathbb{Q}_{\mathbf{Y}} \cong Rg_*(\mathbb{Q}_{\mathbf{X}'}),$$

and to see that they are not isomorphic one has to notice ...

$$\mathbb{H}^*(\mathbf{Y}, Rf_*(\mathbb{Q}_{\mathbf{X}})) \cong \mathrm{H}^*(\mathbb{S}^3, \mathbb{Q}),$$

but ...

$$\mathbb{H}^*(\mathbf{Y}, Rg_*(\mathbb{Q}_{\mathbf{X}'})) \cong \mathrm{H}^*(\mathbb{S}^1 \times \mathbb{S}^2, \mathbb{Q}).$$

# But in the derived category ...

stalks are isomorphic – for all $\mathbf{y} \in \mathbf{Y}$,

$$Rf_*(\mathbb{Q}_{\mathbf{X}})_{\mathbf{y}} \cong Rg_*(\mathbb{Q}_{\mathbf{X}'})_{\mathbf{y}} \cong \mathbb{Q}[-1] \oplus \mathbb{Q},$$

but ...

$$Rf_*(\mathbb{Q}_{\mathbf{X}}) \not\cong \mathbb{Q}_{\mathbf{Y}}[-1] \oplus \mathbb{Q}_{\mathbf{Y}} \cong Rg_*(\mathbb{Q}_{\mathbf{X}'}),$$

and to see that they are not isomorphic one has to notice ...

$$\mathbb{H}^*(\mathbf{Y}, Rf_*(\mathbb{Q}_{\mathbf{X}})) \cong \mathrm{H}^*(\mathbb{S}^3, \mathbb{Q}),$$

but ...

$$\mathbb{H}^*(\mathbf{Y}, Rg_*(\mathbb{Q}_{\mathbf{X}'})) \cong \mathrm{H}^*(\mathbb{S}^1 \times \mathbb{S}^2, \mathbb{Q}).$$

# But in the derived category ...

stalks are isomorphic – for all $\mathbf{y} \in \mathbf{Y}$,

$$Rf_*(\mathbb{Q}_{\mathbf{X}})_{\mathbf{y}} \cong Rg_*(\mathbb{Q}_{\mathbf{X}'})_{\mathbf{y}} \cong \mathbb{Q}[-1] \oplus \mathbb{Q},$$

but ...

$$Rf_*(\mathbb{Q}_{\mathbf{X}}) \not\cong \mathbb{Q}_{\mathbf{Y}}[-1] \oplus \mathbb{Q}_{\mathbf{Y}} \cong Rg_*(\mathbb{Q}_{\mathbf{X}'}),$$

and to see that they are not isomorphic one has to notice ...

$$\mathbb{H}^*(\mathbf{Y}, Rf_*(\mathbb{Q}_{\mathbf{X}})) \cong \mathrm{H}^*(\mathbb{S}^3, \mathbb{Q}),$$

but ...

$$\mathbb{H}^*(\mathbf{Y}, Rg_*(\mathbb{Q}_{\mathbf{X}'})) \cong \mathrm{H}^*(\mathbb{S}^1 \times \mathbb{S}^2, \mathbb{Q}).$$

# Constructible sheaves

## Definition (Constructible Sheaves)

Let $\mathbf{X}$ be a locally closed semi-algebraic set. Following [Kashiwara-Schapira], an object $\mathcal{F} \in \mathrm{Ob}(\mathbf{D}^b(\mathbf{X}))$ is said to be *constructible* if it satisfies the following two conditions:

(a) There exists a finite partition $\mathbf{X} = \coprod_{i \in I} C_i$ of $\mathbf{X}$ by locally closed semi-algebraic subsets such that for $j \in \mathbb{Z}$ and $i \in I$, the $\mathrm{H}^j(\mathcal{F})|_{C_i}$ are locally constant. We will call such a partition *subordinate* to $\mathcal{F}$.

(b) For each $\mathrm{x} \in \mathbf{X}$, the stalk $\mathcal{F}_{\mathrm{x}}$ has the following properties:

 (i) for each $j \in \mathbb{Z}$, the cohomology groups $\mathrm{H}^j(\mathcal{F}_{\mathrm{x}})$ are finitely generated, and

 (ii) there exists $N$ such that $\mathrm{H}^j(\mathcal{F}_{\mathrm{x}}) = 0$ for all $\mathrm{x} \in \mathbf{X}$ and $|j| > N$.

We will denote the category of constructible sheaves on $\mathbf{X}$ by $\mathbf{D}^b_{\mathrm{sa}}(\mathbf{X})$, and denote by

$$\mathcal{CS}(X) := \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{X})).$$

# Constructible sheaves

## Definition (Constructible Sheaves)

Let $\mathbf{X}$ be a locally closed semi-algebraic set. Following [Kashiwara-Schapira], an object $\mathcal{F} \in \mathrm{Ob}(\mathbf{D}^b(\mathbf{X}))$ is said to be *constructible* if it satisfies the following two conditions:

(a) There exists a finite partition $\mathbf{X} = \coprod_{i \in I} C_i$ of $\mathbf{X}$ by locally closed semi-algebraic subsets such that for $j \in \mathbb{Z}$ and $i \in I$, the $\mathrm{H}^j(\mathcal{F})|_{C_i}$ are locally constant. We will call such a partition *subordinate* to $\mathcal{F}$.

(b) For each $\mathrm{x} \in \mathbf{X}$, the stalk $\mathcal{F}_{\mathrm{x}}$ has the following properties:

(i) for each $j \in \mathbb{Z}$, the cohomology groups $\mathrm{H}^j(\mathcal{F}_{\mathrm{x}})$ are finitely generated, and

(ii) there exists $N$ such that $\mathrm{H}^j(\mathcal{F}_{\mathrm{x}}) = 0$ for all $\mathrm{x} \in \mathbf{X}$ and $|j| > N$.

We will denote the category of constructible sheaves on $\mathbf{X}$ by $\mathbf{D}^b_{\mathrm{sa}}(\mathbf{X})$, and denote by

$$\mathcal{CS}(X) := \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{X})).$$

# Constructible sheaves

## Definition (Constructible Sheaves)

Let $\mathbf{X}$ be a locally closed semi-algebraic set. Following [Kashiwara-Schapira], an object $\mathcal{F} \in \mathrm{Ob}(\mathbf{D}^b(\mathbf{X}))$ is said to be *constructible* if it satisfies the following two conditions:

(a) There exists a finite partition $\mathbf{X} = \coprod_{i \in I} C_i$ of $\mathbf{X}$ by locally closed semi-algebraic subsets such that for $j \in \mathbb{Z}$ and $i \in I$, the $\mathrm{H}^j(\mathcal{F})|_{C_i}$ are locally constant. We will call such a partition *subordinate* to $\mathcal{F}$.

(b) For each $\mathbf{x} \in \mathbf{X}$, the stalk $\mathcal{F}_{\mathbf{x}}$ has the following properties:

(i) for each $j \in \mathbb{Z}$, the cohomology groups $\mathrm{H}^j(\mathcal{F}_{\mathbf{x}})$ are finitely generated, and

(ii) there exists $N$ such that $\mathrm{H}^j(\mathcal{F}_{\mathbf{x}}) = 0$ for all $\mathbf{x} \in \mathbf{X}$ and $|j| > N$.

We will denote the category of constructible sheaves on $\mathbf{X}$ by $\mathbf{D}^b_{\mathrm{sa}}(\mathbf{X})$, and denote by

$$\mathcal{CS}(X) := \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{X})).$$

# Constructible sheaves

## Definition (Constructible Sheaves)

Let $\mathbf{X}$ be a locally closed semi-algebraic set. Following [Kashiwara-Schapira], an object $\mathcal{F} \in \mathrm{Ob}(\mathbf{D}^b(\mathbf{X}))$ is said to be *constructible* if it satisfies the following two conditions:

(a) There exists a finite partition $\mathbf{X} = \coprod_{i \in I} C_i$ of $\mathbf{X}$ by locally closed semi-algebraic subsets such that for $j \in \mathbb{Z}$ and $i \in I$, the $\mathrm{H}^j(\mathcal{F})|_{C_i}$ are locally constant. We will call such a partition *subordinate* to $\mathcal{F}$.

(b) For each $\mathbf{x} \in \mathbf{X}$, the stalk $\mathcal{F}_\mathbf{x}$ has the following properties:
  - (i) for each $j \in \mathbb{Z}$, the cohomology groups $\mathrm{H}^j(\mathcal{F}_\mathbf{x})$ are finitely generated, and
  - (ii) there exists $N$ such that $\mathrm{H}^j(\mathcal{F}_\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbf{X}$ and $|j| > N$.

We will denote the category of constructible sheaves on $\mathbf{X}$ by $\mathbf{D}^b_{\mathrm{sa}}(\mathbf{X})$, and denote by

$$\mathcal{CS}(X) := \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{X})).$$

# Constructible sheaves

## Definition (Constructible Sheaves)

Let $\mathbf{X}$ be a locally closed semi-algebraic set. Following [Kashiwara-Schapira], an object $\mathcal{F} \in \mathrm{Ob}(\mathbf{D}^b(\mathbf{X}))$ is said to be *constructible* if it satisfies the following two conditions:

(a) There exists a finite partition $\mathbf{X} = \coprod_{i \in I} C_i$ of $\mathbf{X}$ by locally closed semi-algebraic subsets such that for $j \in \mathbb{Z}$ and $i \in I$, the $\mathrm{H}^j(\mathcal{F})|_{C_i}$ are locally constant. We will call such a partition *subordinate* to $\mathcal{F}$.

(b) For each $\mathbf{x} \in \mathbf{X}$, the stalk $\mathcal{F}_{\mathbf{x}}$ has the following properties:
   - (i) for each $j \in \mathbb{Z}$, the cohomology groups $\mathrm{H}^j(\mathcal{F}_{\mathbf{x}})$ are finitely generated, and
   - (ii) there exists $N$ such that $\mathrm{H}^j(\mathcal{F}_{\mathbf{x}}) = 0$ for all $\mathbf{x} \in \mathbf{X}$ and $|j| > N$.

We will denote the category of constructible sheaves on $\mathbf{X}$ by $\mathbf{D}^b_{\mathrm{sa}}(\mathbf{X})$, and denote by

$$\mathcal{CS}(X) := \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{X})).$$

# Constructible sheaves

## Definition (Constructible Sheaves)

Let $\mathbf{X}$ be a locally closed semi-algebraic set. Following [Kashiwara-Schapira], an object $\mathcal{F} \in \mathrm{Ob}(\mathbf{D}^b(\mathbf{X}))$ is said to be *constructible* if it satisfies the following two conditions:

(a) There exists a finite partition $\mathbf{X} = \coprod_{i \in I} C_i$ of $\mathbf{X}$ by locally closed semi-algebraic subsets such that for $j \in \mathbb{Z}$ and $i \in I$, the $\mathrm{H}^j(\mathcal{F})|_{C_i}$ are locally constant. We will call such a partition *subordinate* to $\mathcal{F}$.

(b) For each $\mathbf{x} \in \mathbf{X}$, the stalk $\mathcal{F}_\mathbf{x}$ has the following properties:

  (i) for each $j \in \mathbb{Z}$, the cohomology groups $\mathrm{H}^j(\mathcal{F}_\mathbf{x})$ are finitely generated, and

  (ii) there exists $N$ such that $\mathrm{H}^j(\mathcal{F}_\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbf{X}$ and $|j| > N$.

We will denote the category of constructible sheaves on $\mathbf{X}$ by $\mathbf{D}^b_{\mathrm{sa}}(\mathbf{X})$, and denote by

$$\mathcal{CS}(X) := \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{X})).$$

# Constructible sheaves

## Definition (Constructible Sheaves)

Let $\mathbf{X}$ be a locally closed semi-algebraic set. Following [Kashiwara-Schapira], an object $\mathcal{F} \in \mathrm{Ob}(\mathbf{D}^b(\mathbf{X}))$ is said to be *constructible* if it satisfies the following two conditions:

(a) There exists a finite partition $\mathbf{X} = \coprod_{i \in I} C_i$ of $\mathbf{X}$ by locally closed semi-algebraic subsets such that for $j \in \mathbb{Z}$ and $i \in I$, the $\mathrm{H}^j(\mathcal{F})|_{C_i}$ are locally constant. We will call such a partition *subordinate* to $\mathcal{F}$.

(b) For each $\mathbf{x} \in \mathbf{X}$, the stalk $\mathcal{F}_{\mathbf{x}}$ has the following properties:
   (i) for each $j \in \mathbb{Z}$, the cohomology groups $\mathrm{H}^j(\mathcal{F}_{\mathbf{x}})$ are finitely generated, and
   (ii) there exists $N$ such that $\mathrm{H}^j(\mathcal{F}_{\mathbf{x}}) = 0$ for all $\mathbf{x} \in \mathbf{X}$ and $|j| > N$.

We will denote the category of constructible sheaves on $\mathbf{X}$ by $\mathbf{D}^b_{\mathrm{sa}}(\mathbf{X})$, and denote by

$$\mathcal{CS}(X) := \mathrm{Ob}(\mathbf{D}^b_{\mathrm{sa}}(\mathbf{X})).$$

# Sheaf-theoretic version of Tarski-Seidenberg

Theorem (Kashiwara (1975), Kashiwara-Schapira (1979))

*Let* $\mathbf{X} \xrightarrow{f} \mathbf{Y}$ *be a continuous semi-algebraic map. Then for* $\mathcal{F} \in \mathcal{CS}(\mathbf{X})$ *and* $\mathcal{G} \in \mathcal{CS}(\mathbf{Y})$*, then*

$$f^{-1}(\mathcal{G}) \in \mathcal{CS}(\mathbf{X})$$

*and*

$$Rf_*(\mathcal{F}) \in \mathcal{CS}(\mathbf{Y}).$$

*More generally, the category of constructible sheaves is closed under the six operations of Grothendieck – namely,* $Rf_*, Rf_!, f^{-1}, f^!, \otimes, R\mathcal{H}om$ *– where f is a continuous semi-algebraic map.*

End of "Qualitative background". Next, "Quantitative" ....

# Sheaf-theoretic version of Tarski-Seidenberg

Theorem (Kashiwara (1975), Kashiwara-Schapira (1979))

*Let* $\mathbf{X} \xrightarrow{f} \mathbf{Y}$ *be a continuous semi-algebraic map. Then for* $\mathcal{F} \in \mathcal{CS}(\mathbf{X})$ *and* $\mathcal{G} \in \mathcal{CS}(\mathbf{Y})$, *then*

$$f^{-1}(\mathcal{G}) \in \mathcal{CS}(\mathbf{X})$$

*and*

$$Rf_*(\mathcal{F}) \in \mathcal{CS}(\mathbf{Y}).$$

*More generally, the category of constructible sheaves is closed under the six operations of Grothendieck – namely,* $Rf_*, Rf_!, f^{-1}, f^!, \otimes, R\mathcal{H}om$ *– where* $f$ *is a continuous semi-algebraic map.*

End of "Qualitative background". Next, "Quantitative" ....

# Sheaf-theoretic version of Tarski-Seidenberg

Theorem (Kashiwara (1975), Kashiwara-Schapira (1979))

*Let* $\mathbf{X} \xrightarrow{f} \mathbf{Y}$ *be a continuous semi-algebraic map. Then for* $\mathcal{F} \in \mathcal{CS}(\mathbf{X})$ *and* $\mathcal{G} \in \mathcal{CS}(\mathbf{Y})$*, then*

$$f^{-1}(\mathcal{G}) \in \mathcal{CS}(\mathbf{X})$$

*and*

$$Rf_*(\mathcal{F}) \in \mathcal{CS}(\mathbf{Y}).$$

*More generally, the category of constructible sheaves is closed under the six operations of Grothendieck – namely,* $Rf_*, Rf_!, f^{-1}, f^!, \otimes, R\mathcal{H}om$ *– where f is a continuous semi-algebraic map.*

End of "Qualitative background". Next, "Quantitative" ....

# Sheaf-theoretic version of Tarski-Seidenberg

**Theorem (Kashiwara (1975), Kashiwara-Schapira (1979))**

*Let* $\mathbf{X} \xrightarrow{f} \mathbf{Y}$ *be a continuous semi-algebraic map. Then for* $\mathcal{F} \in \mathcal{CS}(\mathbf{X})$ *and* $\mathcal{G} \in \mathcal{CS}(\mathbf{Y})$*, then*

$$f^{-1}(\mathcal{G}) \in \mathcal{CS}(\mathbf{X})$$

*and*

$$Rf_*(\mathcal{F}) \in \mathcal{CS}(\mathbf{Y}).$$

*More generally, the category of constructible sheaves is closed under the six operations of Grothendieck – namely,* $Rf_*, Rf_!, f^{-1}, f^!, \otimes, R\mathcal{H}om$ *– where f is a continuous semi-algebraic map.*

End of "Qualitative background". Next, "Quantitative" ....

# Complexity of real quantifier elimination

- Long history, starting with non-elementary-recursive bound of Tarski's original algorithm, doubly exponential algorithm due to Collins (1975) (and also Wuthrich (1976)) using Cylindrical Algebraic Decomposition.
- For each $n \geq 0$, let $\pi_n : \mathbb{R}^n \to \mathbb{R}^{[n/2]}$ denote the projection map forgetting the last $n - [n/2]$ coordinates.
- A new ingredient – *critical point method* – gives:

Theorem (Grigoriev-Vorobjov (1988), Renegar (1992))

*The complexity (both quantitative and algorithmic) for the function*

*is bounded singly exponentially.*

Later improvements and more precise estimates by B.-Pollack-Roy (1996) and B. (1999).

# Complexity of real quantifier elimination

- Long history, starting with non-elementary-recursive bound of Tarski's original algorithm, doubly exponential algorithm due to Collins (1975) (and also Wuthrich (1976)) using Cylindrical Algebraic Decomposition.
- For each $n \geq 0$, let $\pi_n : \mathbb{R}^n \to \mathbb{R}^{[n/2]}$ denote the projection map forgetting the last $n - [n/2]$ coordinates.
- A new ingredient – *critical point method* – gives:

Theorem (Grigoriev-Vorobjov (1988), Renegar (1992))

*The complexity (both quantitative and algorithmic) of the functors*

$$\pi_{n,*}, \pi_n^{*} : S(\mathbb{R}^n) \to S(\mathbb{R}^{[n/2]})$$

*is bounded singly exponentially.*

Later improvements and more precise estimates by B.-Pollack-Roy (1996) and B. (1999).

# Complexity of real quantifier elimination

- Long history, starting with non-elementary-recursive bound of Tarski's original algorithm, doubly exponential algorithm due to Collins (1975) (and also Wuthrich (1976)) using Cylindrical Algebraic Decomposition.
- For each $n \geq 0$, let $\pi_n : \mathbb{R}^n \to \mathbb{R}^{[n/2]}$ denote the projection map forgetting the last $n - [n/2]$ coordinates.
- A new ingredient – *critical point method* – gives:

Theorem (Grigoriev-Vorobjov (1988), Renegar (1992))

*The complexity (both quantitative and algorithmic) of the functors*

$$\pi_n^{\exists}, \pi_n^{\forall} : S(\mathbb{R}^n) \to S(\mathbb{R}^{[n/2]})$$

*is bounded singly exponentially.*

Later improvements and more precise estimates by B.-Pollack-Roy (1996) and B. (1999).

# Complexity of real quantifier elimination

- Long history, starting with non-elementary-recursive bound of Tarski's original algorithm, doubly exponential algorithm due to Collins (1975) (and also Wuthrich (1976)) using Cylindrical Algebraic Decomposition.
- For each $n \geq 0$, let $\pi_n : \mathbb{R}^n \to \mathbb{R}^{[n/2]}$ denote the projection map forgetting the last $n - [n/2]$ coordinates.
- A new ingredient – *critical point method* – gives:

Theorem (Grigoriev-Vorobjov (1988), Renegar (1992))

*The complexity (both quantitative and algorithmic) of the functors*

$$\pi_n^{\exists}, \pi_n^{\forall} : S(\mathbb{R}^n) \to S(\mathbb{R}^{[n/2]})$$

*is bounded singly exponentially.*

Later improvements and more precise estimates by B.-Pollack-Roy (1996) and B. (1999).

# Complexity of real quantifier elimination

- Long history, starting with non-elementary-recursive bound of Tarski's original algorithm, doubly exponential algorithm due to Collins (1975) (and also Wuthrich (1976)) using Cylindrical Algebraic Decomposition.
- For each $n \geq 0$, let $\pi_n : \mathbb{R}^n \to \mathbb{R}^{[n/2]}$ denote the projection map forgetting the last $n - [n/2]$ coordinates.
- A new ingredient – *critical point method* – gives:

**Theorem (Grigoriev-Vorobjov (1988), Renegar (1992))**

*The complexity (both quantitative and algorithmic) of the functors*

$$\pi_n^{\exists}, \pi_n^{\forall} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^{[n/2]})$$

*is bounded singly exponentially.*

Later improvements and more precise estimates by B.-Pollack-Roy (1996) and B. (1999).

# Complexity of real quantifier elimination

- Long history, starting with non-elementary-recursive bound of Tarski's original algorithm, doubly exponential algorithm due to Collins (1975) (and also Wuthrich (1976)) using Cylindrical Algebraic Decomposition.
- For each $n \geq 0$, let $\pi_n : \mathbb{R}^n \to \mathbb{R}^{[n/2]}$ denote the projection map forgetting the last $n - [n/2]$ coordinates.
- A new ingredient – *critical point method* – gives:

**Theorem (Grigoriev-Vorobjov (1988), Renegar (1992))**

*The complexity (both quantitative and algorithmic) of the functors*

$$\pi_n^{\exists}, \pi_n^{\forall} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^{[n/2]})$$

*is bounded singly exponentially.*

Later improvements and more precise estimates by B.-Pollack-Roy (1996) and B. (1999).

# Complexity of real quantifier elimination

- Long history, starting with non-elementary-recursive bound of Tarski's original algorithm, doubly exponential algorithm due to Collins (1975) (and also Wuthrich (1976)) using Cylindrical Algebraic Decomposition.
- For each $n \geq 0$, let $\pi_n : \mathbb{R}^n \to \mathbb{R}^{[n/2]}$ denote the projection map forgetting the last $n - [n/2]$ coordinates.
- A new ingredient – *critical point method* – gives:

**Theorem (Grigoriev-Vorobjov (1988), Renegar (1992))**

*The complexity (both quantitative and algorithmic) of the functors*

$$\pi_n^{\exists}, \pi_n^{\forall} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^{[n/2]})$$

*is bounded singly exponentially.*

Later improvements and more precise estimates by B.-Pollack-Roy (1996) and B. (1999).

# Bounding the number of distinct types of fibers

**Theorem (B., Vorobjov)**

*The semi-algebraic partition in Hardt triviality theorem has at most doubly exponential complexity.*

Unknown, whether it is actually singly exponential.

However, ...

**Theorem (B., Vorobjov)**

*The number of homotopy types of fibers is bounded singly exponentially. More precisely, if $S \subset \mathbb{R}^n \times \mathbb{R}^m$ is a semi-algebraic set defined by $s$ polynomials of degrees at most $d$, and $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ the projection to the second factor, then the number of homotopy types amongst the fibers $S_{\mathbf{y}}, \mathbf{y} \in \mathbb{R}^m$ (where $S_{\mathbf{y}} = S \cap \pi^{-1}(\mathbf{y})$) is bounded by $(sd)^{O(mn)}$.*

# Bounding the number of distinct types of fibers

**Theorem (B., Vorobjov)**

*The semi-algebraic partition in Hardt triviality theorem has at most doubly exponential complexity.*

Unknown, whether it is actually singly exponential.

However, ...

**Theorem (B., Vorobjov)**

*The number of homotopy types of fibers is bounded singly exponentially. More precisely, if $S \subset \mathbb{R}^n \times \mathbb{R}^m$ is a semi-algebraic set defined by $s$ polynomials of degrees at most $d$, and $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ the projection to the second factor, then the number of homotopy types amongst the fibers $S_{\mathbf{y}}, \mathbf{y} \in \mathbb{R}^m$ (where $S_{\mathbf{y}} = S \cap \pi^{-1}(\mathbf{y})$) is bounded by $(sd)^{O(mn)}$.*

# Bounding the number of distinct types of fibers

**Theorem (B., Vorobjov)**

*The semi-algebraic partition in Hardt triviality theorem has at most doubly exponential complexity.*

Unknown, whether it is actually singly exponential.
However, ...

**Theorem (B., Vorobjov)**

*The number of homotopy types of fibers is bounded singly exponentially. More precisely, if $S \subset \mathbb{R}^n \times \mathbb{R}^m$ is a semi-algebraic set defined by $s$ polynomials of degrees at most $d$, and $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ the projection to the second factor, then the number of homotopy types amongst the fibers $S_\mathbf{y}, \mathbf{y} \in \mathbb{R}^m$ (where $S_\mathbf{y} = S \cap \pi^{-1}(\mathbf{y})$) is bounded by $(sd)^{O(mn)}$.*

# Bounding the number of distinct types of fibers

**Theorem (B., Vorobjov)**

*The semi-algebraic partition in Hardt triviality theorem has at most doubly exponential complexity.*

Unknown, whether it is actually singly exponential.
However, ...

**Theorem (B., Vorobjov)**

*The number of homotopy types of fibers is bounded singly exponentially. More precisely, if $S \subset \mathbb{R}^n \times \mathbb{R}^m$ is a semi-algebraic set defined by s polynomials of degrees at most d, and $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ the projection to the second factor, then the number of homotopy types amongst the fibers $S_{\mathbf{y}}, \mathbf{y} \in \mathbb{R}^m$ (where $S_{\mathbf{y}} = S \cap \pi^{-1}(\mathbf{y})$) is bounded by $(sd)^{O(mn)}$.*

# Bounding the number of distinct types of fibers

**Theorem (B., Vorobjov)**

*The semi-algebraic partition in Hardt triviality theorem has at most doubly exponential complexity.*

Unknown, whether it is actually singly exponential.
However, ...

**Theorem (B., Vorobjov)**

*The number of homotopy types of fibers is bounded singly exponentially. More precisely, if $S \subset \mathbb{R}^n \times \mathbb{R}^m$ is a semi-algebraic set defined by s polynomials of degrees at most d, and $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ the projection to the second factor, then the number of homotopy types amongst the fibers $S_{\mathbf{y}}, \mathbf{y} \in \mathbb{R}^m$ (where $S_{\mathbf{y}} = S \cap \pi^{-1}(\mathbf{y})$) is bounded by $(sd)^{O(mn)}$.*

# Bounding the number of distinct types of fibers

**Theorem (B., Vorobjov)**

*The semi-algebraic partition in Hardt triviality theorem has at most doubly exponential complexity.*

Unknown, whether it is actually singly exponential.
However, ...

**Theorem (B., Vorobjov)**

*The number of homotopy types of fibers is bounded singly exponentially. More precisely, if $S \subset \mathbb{R}^n \times \mathbb{R}^m$ is a semi-algebraic set defined by s polynomials of degrees at most d, and $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ the projection to the second factor, then the number of homotopy types amongst the fibers $S_{\mathbf{y}}, \mathbf{y} \in \mathbb{R}^m$ (where $S_{\mathbf{y}} = S \cap \pi^{-1}(\mathbf{y})$) is bounded by $(sd)^{O(mn)}$.*

# Complexity of the direct image functor

**Theorem (B. 2014)**

*The complexity (both quantitative and algorithmic) of the (direct image) functor $R\pi_{n,*} : \mathcal{CS}(\mathbb{R}^n) \to \mathcal{CS}(\mathbb{R}^{[n/2]})$ is bounded singly exponentially.*

More precisely:

Let $F \in \mathcal{CS}(\mathbb{R}^n)$ have compact support, and such that there exists a semi-algebraic partition of $\mathbb{R}^n$ subordinate to $F$ defined by the sign conditions on $s$ polynomials of degree at most $d$, then

(a) there exists a semi-algebraic partition of $\mathbb{R}^{[n/2]}$ subordinate to $R\pi_*(F)$ having complexity $(sd)^{n^{O(1)}}$;

(b) and moreover there exists an algorithm to obtain this partition from the given partition with the same complexity;

(c) if $\dim_{\mathbb{Q}} \mathrm{H}^*(F_x) \leq N$ for all $x \in \mathbb{R}^n$, then $\dim_{\mathbb{Q}} \mathrm{H}^*((R\pi_{n,*}(F))_y) \leq N(sd)^{n^{O(1)}}$ for all $y \in \mathbb{R}^{[n/2]}$.

# Complexity of the direct image functor

**Theorem (B. 2014)**

*The complexity (both quantitative and algorithmic) of the (direct image) functor* $R\pi_{n,*} : \mathcal{CS}(\mathbb{R}^n) \to \mathcal{CS}(\mathbb{R}^{[n/2]})$ *is bounded singly exponentially.*

More precisely:

Let $F \in \mathcal{CS}(\mathbb{R}^n)$ have compact support, and such that there exists a semi-algebraic partition of $\mathbb{R}^n$ subordinate to $F$ defined by the sign conditions on $s$ polynomials of degree at most $d$, then

(a) there exists a semi-algebraic partition of $\mathbb{R}^{[n/2]}$ subordinate to $R\pi_*(F)$ having complexity $(sd)^{n^{O(1)}}$;

(b) and moreover there exists an algorithm to obtain this partition from the given partition with the same complexity;

(c) if $\dim_{\mathbb{Q}} \mathrm{H}^*(F_x) \leq N$ for all $x \in \mathbb{R}^n$, then $\dim_{\mathbb{Q}} \mathrm{H}^*((R\pi_{n,*}(F))_y) \leq N(sd)^{n^{O(1)}}$ for all $y \in \mathbb{R}^{[n/2]}$.

# Complexity of the direct image functor

**Theorem (B. 2014)**

*The complexity (both quantitative and algorithmic) of the (direct image) functor $R\pi_{n,*} : \mathcal{CS}(\mathbb{R}^n) \to \mathcal{CS}(\mathbb{R}^{[n/2]})$ is bounded singly exponentially.*

More precisely:

Let $F \in \mathcal{CS}(\mathbb{R}^n)$ have compact support, and such that there exists a semi-algebraic partition of $\mathbb{R}^n$ subordinate to $F$ defined by the sign conditions on $s$ polynomials of degree at most $d$, then

(a) there exists a semi-algebraic partition of $\mathbb{R}^{[n/2]}$ subordinate to $R\pi_*(F)$ having complexity $(sd)^{n^{O(1)}}$;

(b) and moreover there exists an algorithm to obtain this partition from the given partition with the same complexity;

(c) if $\dim_{\mathbb{Q}} \mathrm{H}^*(F_x) \le N$ for all $x \in \mathbb{R}^n$, then $\dim_{\mathbb{Q}} \mathrm{H}^*((R\pi_{n,*}(F))_y) \le N(sd)^{n^{O(1)}}$ for all $y \in \mathbb{R}^{[n/2]}$.

# Complexity of the direct image functor

**Theorem (B. 2014)**

*The complexity (both quantitative and algorithmic) of the (direct image) functor* $R\pi_{n,*} : \mathcal{CS}(\mathbb{R}^n) \to \mathcal{CS}(\mathbb{R}^{[n/2]})$ *is bounded singly exponentially.*

More precisely:

Let $F \in \mathcal{CS}(\mathbb{R}^n)$ have compact support, and such that there exists a semi-algebraic partition of $\mathbb{R}^n$ subordinate to $F$ defined by the sign conditions on $s$ polynomials of degree at most $d$, then

(a) there exists a semi-algebraic partition of $\mathbb{R}^{[n/2]}$ subordinate to $R\pi_*(F)$ having complexity $(sd)^{n^{O(1)}}$;

(b) and moreover there exists an algorithm to obtain this partition from the given partition with the same complexity;

(c) if $\dim_{\mathbb{Q}} \mathrm{H}^*(F_x) \le N$ for all $x \in \mathbb{R}^n$, then $\dim_{\mathbb{Q}} \mathrm{H}^*((R\pi_{n,*}(F))_y) \le N(sd)^{n^{O(1)}}$ for all $y \in \mathbb{R}^{[n/2]}$.

# Complexity of the direct image functor

**Theorem (B. 2014)**

*The complexity (both quantitative and algorithmic) of the (direct image) functor $R\pi_{n,*} : \mathcal{CS}(\mathbb{R}^n) \to \mathcal{CS}(\mathbb{R}^{[n/2]})$ is bounded singly exponentially.*

More precisely:

Let $F \in \mathcal{CS}(\mathbb{R}^n)$ have compact support, and such that there exists a semi-algebraic partition of $\mathbb{R}^n$ subordinate to $F$ defined by the sign conditions on $s$ polynomials of degree at most $d$, then

(a) there exists a semi-algebraic partition of $\mathbb{R}^{[n/2]}$ subordinate to $R\pi_*(F)$ having complexity $(sd)^{n^{O(1)}}$;

(b) and moreover there exists an algorithm to obtain this partition from the given partition with the same complexity;

(c) if $\dim_{\mathbb{Q}} \mathrm{H}^*(F_{\mathrm{x}}) \leq N$ for all $\mathrm{x} \in \mathbb{R}^n$, then $\dim_{\mathbb{Q}} \mathrm{H}^*((R\pi_{n,*}(F))_{\mathrm{y}}) \leq N(sd)^{n^{O(1)}}$ for all $\mathrm{y} \in \mathbb{R}^{[n/2]}$.

# Complexity of the direct image functor

**Theorem (B. 2014)**

*The complexity (both quantitative and algorithmic) of the (direct image) functor $R\pi_{n,*} : \mathcal{CS}(\mathbb{R}^n) \to \mathcal{CS}(\mathbb{R}^{[n/2]})$ is bounded singly exponentially.*

More precisely:

Let $F \in \mathcal{CS}(\mathbb{R}^n)$ have compact support, and such that there exists a semi-algebraic partition of $\mathbb{R}^n$ subordinate to $F$ defined by the sign conditions on $s$ polynomials of degree at most $d$, then

(a) there exists a semi-algebraic partition of $\mathbb{R}^{[n/2]}$ subordinate to $R\pi_*(F)$ having complexity $(sd)^{n^{O(1)}}$;

(b) and moreover there exists an algorithm to obtain this partition from the given partition with the same complexity;

(c) if $\dim_{\mathbb{Q}} \mathrm{H}^*(F_{\mathrm{x}}) \leq N$ for all $\mathrm{x} \in \mathbb{R}^n$, then $\dim_{\mathbb{Q}} \mathrm{H}^*((R\pi_{n,*}(F))_{\mathrm{y}}) \leq N(sd)^{n^{O(1)}}$ for all $\mathrm{y} \in \mathbb{R}^{[n/2]}$.

# Complexity of the direct image functor

**Theorem (B. 2014)**

*The complexity (both quantitative and algorithmic) of the (direct image) functor $R\pi_{n,*} : \mathcal{CS}(\mathbb{R}^n) \to \mathcal{CS}(\mathbb{R}^{[n/2]})$ is bounded singly exponentially.*

More precisely:

Let $F \in \mathcal{CS}(\mathbb{R}^n)$ have compact support, and such that there exists a semi-algebraic partition of $\mathbb{R}^n$ subordinate to $F$ defined by the sign conditions on $s$ polynomials of degree at most $d$, then

(a) there exists a semi-algebraic partition of $\mathbb{R}^{[n/2]}$ subordinate to $R\pi_*(F)$ having complexity $(sd)^{n^{O(1)}}$;

(b) and moreover there exists an algorithm to obtain this partition from the given partition with the same complexity;

(c) if $\dim_{\mathbb{Q}} \mathrm{H}^*(F_{\mathbf{x}}) \leq N$ for all $\mathbf{x} \in \mathbb{R}^n$, then $\dim_{\mathbb{Q}} \mathrm{H}^*((R\pi_{n,*}(F))_{\mathbf{y}}) \leq N(sd)^{n^{O(1)}}$ for all $\mathbf{y} \in \mathbb{R}^{[n/2]}$.

# Proof ingredients

- Several ingredients recently developed for studying algorithmic and quantitative questions in semi-algebraic geometry.
- Ideas used to prove singly exponential bounds on the number of homotopy types of the fibers of definable maps (B.-Vorobjov (2007)).
- Singly exponential sized covering by contractibles (B.-Pollack-Roy (2008)).
- Delicate infinitesimal thickening and shrinking arguments.
- Certain arguments using spectral sequences – Leray and Mayer-Vietoris.
- Proper base change theorem for constructible sheaves.

End of "Quantitative". Next, "Complexity-theoretic" ....

# Proof ingredients

- Several ingredients recently developed for studying algorithmic and quantitative questions in semi-algebraic geometry.
- Ideas used to prove singly exponential bounds on the number of homotopy types of the fibers of definable maps (B.-Vorobjov (2007)).
- Singly exponential sized covering by contractibles (B.-Pollack-Roy (2008)).
- Delicate infinitesimal thickening and shrinking arguments.
- Certain arguments using spectral sequences – Leray and Mayer-Vietoris.
- Proper base change theorem for constructible sheaves.

End of "Quantitative". Next, "Complexity-theoretic" ....

# Proof ingredients

- Several ingredients recently developed for studying algorithmic and quantitative questions in semi-algebraic geometry.
- Ideas used to prove singly exponential bounds on the number of homotopy types of the fibers of definable maps (B.-Vorobjov (2007)).
- Singly exponential sized covering by contractibles (B.-Pollack-Roy (2008)).
- Delicate infinitesimal thickening and shrinking arguments.
- Certain arguments using spectral sequences – Leray and Mayer-Vietoris.
- Proper base change theorem for constructible sheaves.

End of "Quantitative". Next, "Complexity-theoretic" ....

# Proof ingredients

- Several ingredients recently developed for studying algorithmic and quantitative questions in semi-algebraic geometry.
- Ideas used to prove singly exponential bounds on the number of homotopy types of the fibers of definable maps (B.-Vorobjov (2007)).
- Singly exponential sized covering by contractibles (B.-Pollack-Roy (2008)).
- Delicate infinitesimal thickening and shrinking arguments.
- Certain arguments using spectral sequences – Leray and Mayer-Vietoris.
- Proper base change theorem for constructible sheaves.

End of "Quantitative". Next, "Complexity-theoretic" ....

# Proof ingredients

- Several ingredients recently developed for studying algorithmic and quantitative questions in semi-algebraic geometry.
- Ideas used to prove singly exponential bounds on the number of homotopy types of the fibers of definable maps (B.-Vorobjov (2007)).
- Singly exponential sized covering by contractibles (B.-Pollack-Roy (2008)).
- Delicate infinitesimal thickening and shrinking arguments.
- Certain arguments using spectral sequences – Leray and Mayer-Vietoris.
- Proper base change theorem for constructible sheaves.

End of "Quantitative". Next, "Complexity-theoretic" ....

# Proof ingredients

- Several ingredients recently developed for studying algorithmic and quantitative questions in semi-algebraic geometry.
- Ideas used to prove singly exponential bounds on the number of homotopy types of the fibers of definable maps (B.-Vorobjov (2007)).
- Singly exponential sized covering by contractibles (B.-Pollack-Roy (2008)).
- Delicate infinitesimal thickening and shrinking arguments.
- Certain arguments using spectral sequences – Leray and Mayer-Vietoris.
- Proper base change theorem for constructible sheaves.

End of "Quantitative". Next, "Complexity-theoretic" ....

# Proof ingredients

- Several ingredients recently developed for studying algorithmic and quantitative questions in semi-algebraic geometry.
- Ideas used to prove singly exponential bounds on the number of homotopy types of the fibers of definable maps (B.-Vorobjov (2007)).
- Singly exponential sized covering by contractibles (B.-Pollack-Roy (2008)).
- Delicate infinitesimal thickening and shrinking arguments.
- Certain arguments using spectral sequences – Leray and Mayer-Vietoris.
- Proper base change theorem for constructible sheaves.

End of "Quantitative". Next, "Complexity-theoretic" ....

# Proof ingredients

- Several ingredients recently developed for studying algorithmic and quantitative questions in semi-algebraic geometry.
- Ideas used to prove singly exponential bounds on the number of homotopy types of the fibers of definable maps (B.-Vorobjov (2007)).
- Singly exponential sized covering by contractibles (B.-Pollack-Roy (2008)).
- Delicate infinitesimal thickening and shrinking arguments.
- Certain arguments using spectral sequences – Leray and Mayer-Vietoris.
- Proper base change theorem for constructible sheaves.

End of "Quantitative". Next, "Complexity-theoretic" ....

# Blum-Shub-Smale complexity classes over $\mathbb{R}$

- Let $\mathcal{S}$ denote the (poset) category of sequences $(S_n \in \mathcal{S}(\mathbb{R}^{m(n)}))_{n>0}$ where each $m(n)$ is a non-negative integer valued function.
- We say that $\mathbf{L} \in \mathcal{S}$ is in $\mathbf{P}_{\mathbb{R}}$, iff there exists a B-S-S machine recognizing $\mathbf{L}$ in polynomial time.
- Recall that we also have sequences of maps:

$$\left( \mathcal{S}(\mathbb{R}^m) \underset{\xrightarrow{\pi_{m,\forall}}}{\overset{\xrightarrow{\pi_{m,\exists}}}{\xleftarrow{\pi_m^*}}} \mathcal{S}(\mathbb{R}^{[m/2]}) \right)_{m>0}.$$

- The class $\mathbf{P}_{\mathbb{R}}$ is stable under taking products, unions, intersections, and pull-backs ($\pi_m^*$). But what about stability under $\pi_{m,\exists}, \pi_{m,\forall}$ ?

# Blum-Shub-Smale complexity classes over $\mathbb{R}$

- Let $\boldsymbol{\mathcal{S}}$ denote the (poset) category of sequences $(S_n \in \mathcal{S}(\mathbb{R}^{m(n)}))_{n>0}$ where each $m(n)$ is a non-negative integer valued function.
- We say that $\mathbf{L} \in \boldsymbol{\mathcal{S}}$ is in $\mathbf{P}_{\mathbb{R}}$, iff there exists a B-S-S machine recognizing $\mathbf{L}$ in polynomial time.
- Recall that we also have sequences of maps:

$$\left( \mathcal{S}(\mathbb{R}^m) \underset{\xrightarrow{\pi_{m,\forall}}}{\overset{\xrightarrow{\pi_{m,\exists}}}{\xleftarrow{\pi_m^*}}} \mathcal{S}(\mathbb{R}^{[m/2]}) \right)_{m>0}.$$

- The class $\mathbf{P}_{\mathbb{R}}$ is stable under taking products, unions, intersections, and pull-backs ($\pi_m^*$). But what about stability under $\pi_{m,\exists}, \pi_{m,\forall}$ ?

# Blum-Shub-Smale complexity classes over $\mathbb{R}$

- Let $\mathcal{S}$ denote the (poset) category of sequences $(S_n \in \mathcal{S}(\mathbb{R}^{m(n)}))_{n>0}$ where each $m(n)$ is a non-negative integer valued function.
- We say that $\mathbf{L} \in \mathcal{S}$ is in $\mathbf{P}_\mathbb{R}$, iff there exists a B-S-S machine recognizing $\mathbf{L}$ in polynomial time.
- Recall that we also have sequences of maps:

$$\left( \mathcal{S}(\mathbb{R}^m) \begin{array}{c} \xrightarrow{\pi_{m,\exists}} \\ \xleftarrow{\pi_m^*} \\ \xrightarrow{\pi_{m,\forall}} \end{array} \mathcal{S}(\mathbb{R}^{[m/2]}) \right)_{m>0} .$$

- The class $\mathbf{P}_\mathbb{R}$ is stable under taking products, unions, intersections, and pull-backs ($\pi_m^*$). But what about stability under $\pi_{m,\exists}, \pi_{m,\forall}$ ?

# Blum-Shub-Smale complexity classes over $\mathbb{R}$

- Let $\mathcal{S}$ denote the (poset) category of sequences $(S_n \in \mathcal{S}(\mathbb{R}^{m(n)}))_{n>0}$ where each $m(n)$ is a non-negative integer valued function.
- We say that $\mathbf{L} \in \mathcal{S}$ is in $\mathbf{P}_{\mathbb{R}}$, iff there exists a B-S-S machine recognizing $\mathbf{L}$ in polynomial time.
- Recall that we also have sequences of maps:

$$\left( \mathcal{S}(\mathbb{R}^m) \underset{\xrightarrow{\pi_{m,\forall}}}{\overset{\xrightarrow{\pi_{m,\exists}}}{\xleftarrow{\pi_m^*}}} \mathcal{S}(\mathbb{R}^{[m/2]}) \right)_{m>0} .$$

- The class $\mathbf{P}_{\mathbb{R}}$ is stable under taking products, unions, intersections, and pull-backs $(\pi_m^*)$. But what about stability under $\pi_{m,\exists}, \pi_{m,\forall}$ ?

# $\mathbf{NP}_{\mathbb{R}}$, $\mathbf{co\text{-}NP}_{\mathbb{R}}$, $\mathbf{PH}_{\mathbb{R}}$ and all that ...

$\pi_m^*, \pi_{m,\exists}, \pi_{m,\forall}$ induce in a natural way the following endo-functors

$$\mathcal{S} \begin{array}{c} \xrightarrow{\pi_\exists} \\ \xleftarrow{\pi^*} \\ \xrightarrow{\pi_\forall} \end{array} \mathcal{S}.$$

# $\mathbf{NP}_{\mathbb{R}}$, $\mathbf{co\text{-}NP}_{\mathbb{R}}$, $\mathbf{PH}_{\mathbb{R}}$ and all that ...

$\pi_m^*, \pi_{m,\exists}, \pi_{m,\forall}$ induce in a natural way the following endo-functors

$$\mathcal{S} \quad \begin{array}{c} \xrightarrow{\boldsymbol{\pi}_\exists} \\ \xleftarrow{\boldsymbol{\pi}^*} \\ \xrightarrow{\boldsymbol{\pi}_\forall} \end{array} \quad \mathcal{S}.$$

(Aside) As mentioned before the pairs $(\boldsymbol{\pi}_\exists, \boldsymbol{\pi}^*), (\boldsymbol{\pi}^*, \boldsymbol{\pi}_\forall)$ are not quite pairs of inverse functors, but they form an adjoint triple:

$$\boldsymbol{\pi}_\exists \dashv \boldsymbol{\pi}^* \dashv \boldsymbol{\pi}_\forall.$$

# $\mathbf{NP}_{\mathbb{R}}$, $\mathbf{co\text{-}NP}_{\mathbb{R}}$, $\mathbf{PH}_{\mathbb{R}}$ and all that ...

$\pi_m^*, \pi_{m,\exists}, \pi_{m,\forall}$ induce in a natural way the following endo-functors

$$\mathcal{S} \quad \begin{matrix} \xrightarrow{\boldsymbol{\pi}_\exists} \\ \xleftarrow{\boldsymbol{\pi}^*} \\ \xrightarrow{\boldsymbol{\pi}_\forall} \end{matrix} \quad \mathcal{S}.$$

We have the following obvious equality and inclusions:

$$\mathbf{P}_{\mathbb{R}} = \boldsymbol{\pi}^*(\mathbf{P}_{\mathbb{R}}),$$

$$\mathbf{P}_{\mathbb{R}} \subset \boldsymbol{\pi}_\exists(\mathbf{P}_{\mathbb{R}}),$$

$$\mathbf{P}_{\mathbb{R}} \subset \boldsymbol{\pi}_\forall(\mathbf{P}_{\mathbb{R}}).$$

# $\mathbf{NP}_{\mathbb{R}}$, $\mathbf{co\text{-}NP}_{\mathbb{R}}$, $\mathbf{PH}_{\mathbb{R}}$ and all that ...

$\pi_m^*, \pi_{m,\exists}, \pi_{m,\forall}$ induce in a natural way the following endo-functors

$$\mathcal{S} \quad \begin{matrix} \xrightarrow{\boldsymbol{\pi}_\exists} \\ \xleftarrow{\boldsymbol{\pi}^*} \\ \xrightarrow{\boldsymbol{\pi}_\forall} \end{matrix} \quad \mathcal{S}.$$

For historical reasons it is traditional to denote

$$\mathbf{NP}_{\mathbb{R}} := \boldsymbol{\pi}_\exists(\mathbf{P}_{\mathbb{R}}).$$

# $\mathbf{NP}_{\mathbb{R}}$, $\mathbf{co\text{-}NP}_{\mathbb{R}}$, $\mathbf{PH}_{\mathbb{R}}$ and all that ...

$\pi_m^*, \pi_{m,\exists}, \pi_{m,\forall}$ induce in a natural way the following endo-functors

$$\mathcal{S} \quad \begin{array}{c} \xrightarrow{\pi_\exists} \\ \xleftarrow{\pi^*} \\ \xrightarrow{\pi_\forall} \end{array} \quad \mathcal{S}.$$

For historical reasons it is traditional to denote

$$\mathbf{NP}_{\mathbb{R}} := \pi_\exists(\mathbf{P}_{\mathbb{R}}).$$

And similarly ...

$$\mathbf{co\text{-}NP}_{\mathbb{R}} := \pi_\forall(\mathbf{P}_{\mathbb{R}}).$$

# $\mathbf{NP}_{\mathbb{R}}$, $\mathbf{co\text{-}NP}_{\mathbb{R}}$, $\mathbf{PH}_{\mathbb{R}}$ and all that ...

$\pi_m^*, \pi_{m,\exists}, \pi_{m,\forall}$ induce in a natural way the following endo-functors

$$\mathcal{S} \quad \begin{array}{c} \xrightarrow{\boldsymbol{\pi}_\exists} \\ \xleftarrow{\boldsymbol{\pi}^*} \\ \xrightarrow{\boldsymbol{\pi}_\forall} \end{array} \quad \mathcal{S}.$$

For historical reasons it is traditional to denote

$$\mathbf{NP}_{\mathbb{R}} := \boldsymbol{\pi}_\exists(\mathbf{P}_{\mathbb{R}}).$$

And similarly ...

$$\mathbf{co\text{-}NP}_{\mathbb{R}} := \boldsymbol{\pi}_\forall(\mathbf{P}_{\mathbb{R}}).$$

Definition of $\mathbf{PH}_{\mathbb{R}}$

$$\mathbf{PH}_{\mathbb{R}} := \mathbf{P}_{\mathbb{R}} \cup \boldsymbol{\pi}_\exists(\mathbf{P}_{\mathbb{R}}) \cup \boldsymbol{\pi}_\forall(\mathbf{P}_{\mathbb{R}}) \cup \boldsymbol{\pi}_{\exists\forall}(\mathbf{P}_{\mathbb{R}}) \cup \boldsymbol{\pi}_{\forall\exists}(\mathbf{P}_{\mathbb{R}}) \cup \cdots$$

where the endo-functor $\boldsymbol{\pi}_{\exists\forall}$ is induced by $(\pi_{[m/2],\exists} \circ \pi_{m,\forall})_{m>0}$ and so on ...

# $\mathbf{NP}_{\mathbb{R}}$, $\mathbf{co\text{-}NP}_{\mathbb{R}}$, $\mathbf{PH}_{\mathbb{R}}$ and all that ...

$\pi_m^*, \pi_{m,\exists}, \pi_{m,\forall}$ induce in a natural way the following endo-functors

$$\mathcal{S} \quad \begin{matrix} \xrightarrow{\pi_\exists} \\ \xleftarrow{\pi^*} \\ \xrightarrow{\pi_\forall} \end{matrix} \quad \mathcal{S}.$$

For historical reasons it is traditional to denote

$$\mathbf{NP}_{\mathbb{R}} := \pi_\exists(\mathbf{P}_{\mathbb{R}}).$$

And similarly ...

$$\mathbf{co\text{-}NP}_{\mathbb{R}} := \pi_\forall(\mathbf{P}_{\mathbb{R}}).$$

Definition of $\mathbf{PH}_{\mathbb{R}}$

$$\mathbf{PH}_{\mathbb{R}} := \mathbf{P}_{\mathbb{R}} \cup \pi_\exists(\mathbf{P}_{\mathbb{R}}) \cup \pi_\forall(\mathbf{P}_{\mathbb{R}}) \cup \pi_{\exists\forall}(\mathbf{P}_{\mathbb{R}}) \cup \pi_{\forall\exists}(\mathbf{P}_{\mathbb{R}}) \cup \cdots$$

where the endo-functor $\pi_{\exists\forall}$ is induced by $(\pi_{[m/2],\exists} \circ \pi_{m,\forall})_{m>0}$ and so on ...

# Complexity classes of constructible sheaves

Definition (Informal definition of the class $\mathcal{P}_{\mathbb{R}}$)

Informally we define the class $\mathcal{P}_{\mathbb{R}}$ as the set of sequences $\left(F_n \in \mathcal{CS}(\mathbb{R}^{m(n)})\right)_{n>0}$ such that

(a) there exists a corresponding sequence of semi-algebraic partitions of $\mathbb{R}^{m(n)}$, subordinate to $F_n$, in which *point location can be performed efficiently*;

(b) The Poincaré polynomial of the stalks $(F_n)_{\mathbf{x}}, \mathbf{x} \in \mathbb{R}^{m(n)}$ (i.e. the polynomial $P_{(F_n)_{\mathbf{x}}}(T) = \sum_i \dim_i \mathrm{H}^i((F_n)_{\mathbf{x}})T^i$) can be computed efficiently.

# Complexity classes of constructible sheaves

Definition (Informal definition of the class $\mathcal{P}_{\mathbb{R}}$)

Informally we define the class $\mathcal{P}_{\mathbb{R}}$ as the set of sequences $\left(F_n \in \mathcal{CS}(\mathbb{R}^{m(n)})\right)_{n>0}$ such that

(a) there exists a corresponding sequence of semi-algebraic partitions of $\mathbb{R}^{m(n)}$, subordinate to $F_n$, in which *point location can be performed efficiently*;

(b) The Poincaré polynomial of the stalks $(F_n)_{\mathbf{x}}, \mathbf{x} \in \mathbb{R}^{m(n)}$ (i.e. the polynomial $P_{(F_n)_{\mathbf{x}}}(T) = \sum_i \dim_i \mathrm{H}^i((F_n)_{\mathbf{x}}) T^i$) can be computed efficiently.

# Complexity classes of constructible sheaves

**Definition (Informal definition of the class $\mathcal{P}_{\mathbb{R}}$)**

Informally we define the class $\mathcal{P}_{\mathbb{R}}$ as the set of sequences $\left(F_n \in \mathcal{CS}(\mathbb{R}^{m(n)})\right)_{n>0}$ such that

(a) there exists a corresponding sequence of semi-algebraic partitions of $\mathbb{R}^{m(n)}$, subordinate to $F_n$, in which *point location can be performed efficiently*;

(b) The Poincaré polynomial of the stalks $(F_n)_{\mathbf{x}}, \mathbf{x} \in \mathbb{R}^{m(n)}$ (i.e. the polynomial $P_{(F_n)_{\mathbf{x}}}(T) = \sum_i \dim_i \mathrm{H}^i((F_n)_{\mathbf{x}}) T^i$) can be computed efficiently.

# The class $\mathcal{P}_{\mathbb{R}}$ (formally))

## Definition of $\mathcal{P}_{\mathbb{R}}$ [B. 2014]

The class $\mathcal{P}_{\mathbb{R}}$ of constructible sheaves consists of sequences $\mathbf{F} = \left(F_n \in \mathcal{CS}(\mathbb{R}^{m(n)})\right)_{n>0}$, where $m(n)$ is a non-negative (polynomially bounded) function satisfying the following conditions. There exists a non-negative (polynomially bounded) function $m_1(n)$ such that:

(a) Each $F_n$ has compact support.

(b) For each $n > 0$, there is an index set $I_n$ of cardinality $2^{m_1(n)}$, and a semi-algebraic partition, $(S_{n,i})_{i \in I_n}$, of $\mathbb{R}^{m(n)}$ into locally closed semi-algebraic subsets $S_{n,i}$ indexed by $I_n$, which is subordinate to $F_n$.

(c) For each $n > 0$ and each $\mathbf{x} \in \mathbb{R}^{m(n)}$,

   (i) The dimensions $\dim_{\mathbb{Q}} \mathrm{H}^j((F_n)_{\mathbf{x}})$ are bounded by $2^{m_1(n)}$;

   (ii) $\mathrm{H}^j((F_n)_{\mathbf{x}}) = 0$ for all $j$ with $|j| > m_1(n)$.

# The class $\mathcal{P}_{\mathbb{R}}$ (formally))

## Definition of $\mathcal{P}_{\mathbb{R}}$ [B. 2014]

The class $\mathcal{P}_{\mathbb{R}}$ of constructible sheaves consists of sequences $\mathbf{F} = \left(F_n \in \mathcal{CS}(\mathbb{R}^{m(n)})\right)_{n>0}$, where $m(n)$ is a non-negative (polynomially bounded) function satisfying the following conditions. There exists a non-negative (polynomially bounded) function $m_1(n)$ such that:

(a) Each $F_n$ has compact support.

(b) For each $n > 0$, there is an index set $I_n$ of cardinality $2^{m_1(n)}$, and a semi-algebraic partition, $(S_{n,i})_{i \in I_n}$, of $\mathbb{R}^{m(n)}$ into locally closed semi-algebraic subsets $S_{n,i}$ indexed by $I_n$, which is subordinate to $F_n$.

(c) For each $n > 0$ and each $\mathbf{x} \in \mathbb{R}^{m(n)}$,

(i) The dimensions $\dim_{\mathbb{Q}} \mathrm{H}^j((F_n)_{\mathbf{x}})$ are bounded by $2^{m_1(n)}$;

(ii) $\mathrm{H}^j((F_n)_{\mathbf{x}}) = 0$ for all $j$ with $|j| > m_1(n)$.

# The class $\mathcal{P}_{\mathbb{R}}$ (formally))

## Definition of $\mathcal{P}_{\mathbb{R}}$ [B. 2014]

The class $\mathcal{P}_{\mathbb{R}}$ of constructible sheaves consists of sequences $\mathbf{F} = \left(F_n \in \mathcal{CS}(\mathbb{R}^{m(n)})\right)_{n>0}$, where $m(n)$ is a non-negative (polynomially bounded) function satisfying the following conditions. There exists a non-negative (polynomially bounded) function $m_1(n)$ such that:

(a) Each $F_n$ has compact support.

(b) For each $n > 0$, there is an index set $I_n$ of cardinality $2^{m_1(n)}$, and a semi-algebraic partition, $(S_{n,i})_{i \in I_n}$, of $\mathbb{R}^{m(n)}$ into locally closed semi-algebraic subsets $S_{n,i}$ indexed by $I_n$, which is subordinate to $F_n$.

(c) For each $n > 0$ and each $\mathbf{x} \in \mathbb{R}^{m(n)}$,

(i) The dimensions $\dim_{\mathbb{Q}} \mathrm{H}^j((F_n)_{\mathbf{x}})$ are bounded by $2^{m_1(n)}$;

(ii) $\mathrm{H}^j((F_n)_{\mathbf{x}}) = 0$ for all $j$ with $|j| > m_1(n)$.

# The class $\mathcal{P}_{\mathbb{R}}$ (formally))

## Definition of $\mathcal{P}_{\mathbb{R}}$ [B. 2014]

The class $\mathcal{P}_{\mathbb{R}}$ of constructible sheaves consists of sequences $\mathbf{F} = \left(F_n \in \mathcal{CS}(\mathbb{R}^{m(n)})\right)_{n>0}$, where $m(n)$ is a non-negative (polynomially bounded) function satisfying the following conditions. There exists a non-negative (polynomially bounded) function $m_1(n)$ such that:

(a) Each $F_n$ has compact support.

(b) For each $n > 0$, there is an index set $I_n$ of cardinality $2^{m_1(n)}$, and a semi-algebraic partition, $(S_{n,i})_{i \in I_n}$, of $\mathbb{R}^{m(n)}$ into locally closed semi-algebraic subsets $S_{n,i}$ indexed by $I_n$, which is subordinate to $F_n$.

(c) For each $n > 0$ and each $\mathbf{x} \in \mathbb{R}^{m(n)}$,

  (i) The dimensions $\dim_{\mathbb{Q}} \mathrm{H}^j((F_n)_{\mathbf{x}})$ are bounded by $2^{m_1(n)}$;

  (ii) $\mathrm{H}^j((F_n)_{\mathbf{x}}) = 0$ for all $j$ with $|j| > m_1(n)$.

# The class $\mathcal{P}_{\mathbb{R}}$ (formally))

Definition of $\mathcal{P}_{\mathbb{R}}$ [B. 2014]

The class $\mathcal{P}_{\mathbb{R}}$ of constructible sheaves consists of sequences $\mathbf{F} = \left(F_n \in \mathcal{CS}(\mathbb{R}^{m(n)})\right)_{n>0}$, where $m(n)$ is a non-negative (polynomially bounded) function satisfying the following conditions. There exists a non-negative (polynomially bounded) function $m_1(n)$ such that:

(a) Each $F_n$ has compact support.

(b) For each $n > 0$, there is an index set $I_n$ of cardinality $2^{m_1(n)}$, and a semi-algebraic partition, $(S_{n,i})_{i \in I_n}$, of $\mathbb{R}^{m(n)}$ into locally closed semi-algebraic subsets $S_{n,i}$ indexed by $I_n$, which is subordinate to $F_n$.

(c) For each $n > 0$ and each $\mathbf{x} \in \mathbb{R}^{m(n)}$,

  (i) The dimensions $\dim_{\mathbb{Q}} \mathrm{H}^j((F_n)_{\mathbf{x}})$ are bounded by $2^{m_1(n)}$;

  (ii) $\mathrm{H}^j((F_n)_{\mathbf{x}}) = 0$ for all $j$ with $|j| > m_1(n)$.

# The class $\mathcal{P}_{\mathbb{R}}$ (formally))

## Definition of $\mathcal{P}_{\mathbb{R}}$ [B. 2014]

The class $\mathcal{P}_{\mathbb{R}}$ of constructible sheaves consists of sequences $\mathbf{F} = \left(F_n \in \mathcal{CS}(\mathbb{R}^{m(n)})\right)_{n>0}$, where $m(n)$ is a non-negative (polynomially bounded) function satisfying the following conditions. There exists a non-negative (polynomially bounded) function $m_1(n)$ such that:

(a) Each $F_n$ has compact support.

(b) For each $n > 0$, there is an index set $I_n$ of cardinality $2^{m_1(n)}$, and a semi-algebraic partition, $(S_{n,i})_{i \in I_n}$, of $\mathbb{R}^{m(n)}$ into locally closed semi-algebraic subsets $S_{n,i}$ indexed by $I_n$, which is subordinate to $F_n$.

(c) For each $n > 0$ and each $\mathbf{x} \in \mathbb{R}^{m(n)}$,

  (i) The dimensions $\dim_{\mathbb{Q}} \mathrm{H}^j((F_n)_{\mathbf{x}})$ are bounded by $2^{m_1(n)}$;

  (ii) $\mathrm{H}^j((F_n)_{\mathbf{x}}) = 0$ for all $j$ with $|j| > m_1(n)$.

# The class $\mathcal{P}_{\mathbb{R}}$ (cont).

Definition of $\mathcal{P}_{\mathbb{R}}$ (cont).

The two sequences of functions $(i_n : \mathbb{R}^{m(n)} \to I_n)_{n>0}$, and $(p_n : \mathbb{R}^{m(n)} \to \mathbb{Z}[T, T^{-1}])$ defined by

$$\begin{aligned} i_n(\mathbf{x}) &= i \in I_n, \text{ such that, } \mathbf{x} \in S_{n,i} \\ p_n(\mathbf{x}) &= P_{(F_n)_{\mathbf{x}}} \end{aligned}$$

are computable by B-S-S machines with complexity polynomial in $n$.

Notice that the number of bits needed to represent elements of $I_n$, and the coefficients of $P_{(F_n)_{\mathbf{x}}}$ are bounded polynomially in $n$.

# The class $\mathcal{P}_{\mathbb{R}}$ (cont).

Definition of $\mathcal{P}_{\mathbb{R}}$ (cont).

The two sequences of functions $(i_n : \mathbb{R}^{m(n)} \rightarrow I_n)_{n>0}$, and $(p_n : \mathbb{R}^{m(n)} \rightarrow \mathbb{Z}[T, T^{-1}])$ defined by

$$\begin{aligned} i_n(\mathbf{x}) &= i \in I_n, \text{ such that, } \mathbf{x} \in S_{n,i} \\ p_n(\mathbf{x}) &= P_{(F_n)_{\mathbf{x}}} \end{aligned}$$

are computable by B-S-S machines with complexity polynomial in $n$.

Notice that the number of bits needed to represent elements of $I_n$, and the coefficients of $P_{(F_n)_{\mathbf{x}}}$ are bounded polynomially in $n$.

# The class $\mathcal{P}_{\mathbb{R}}$ (cont).

## Definition of $\mathcal{P}_{\mathbb{R}}$ (cont).

The two sequences of functions $(i_n : \mathbb{R}^{m(n)} \to I_n)_{n>0}$, and $(p_n : \mathbb{R}^{m(n)} \to \mathbb{Z}[T, T^{-1}])$ defined by

$$\begin{aligned} i_n(\mathbf{x}) &= i \in I_n, \text{ such that, } \mathbf{x} \in S_{n,i} \\ p_n(\mathbf{x}) &= P_{(F_n)_{\mathbf{x}}} \end{aligned}$$

are computable by B-S-S machines with complexity polynomial in $n$.

Notice that the number of bits needed to represent elements of $I_n$, and the coefficients of $P_{(F_n)_{\mathbf{x}}}$ are bounded polynomially in $n$.

# Example 0

**Constant sheaf on compact sequences in $\mathbf{P}_{\mathbb{R}}$**

Let $(S_n \in \mathcal{S}(\mathbb{R}^{m(n)}))_{n>0} \in \mathbf{P}^c_{\mathbb{R}}$. Let $j_n : S_n \hookrightarrow \mathbb{R}^n$ be the inclusion map. Then,

$$(j_{n,*}\mathbb{Q}_{S_n})_{n>0} \in \boldsymbol{\mathcal{P}}_{\mathbb{R}}.$$

# Example 1

## Systems of few quadrics

Let $s > 0$ be fixed, and consider for each $n > 0$, the compact real algebraic set $V_n \subset (\mathbf{S}^{\binom{n+1}{2}-1})^s \times \mathbf{S}^n$ defined by

$$V_n = \{(P_1, \ldots, P_s, \mathbf{x}) \mid \mathbf{x} \in \mathbf{S}^n, P_i \in \mathbf{S}^{\binom{n+1}{2}-1}, P_i(\mathbf{x}) = 0, 1 \leq i \leq s\}.$$

Let $\pi_n : (\mathbf{S}^{\binom{n+1}{2}-1})^s \times \mathbf{S}^n \to (\mathbf{S}^{\binom{n+1}{2}-1})^s \hookrightarrow \mathbb{R}^{s\binom{n+1}{2}}$ be the projection to the first factor composed with the natural inclusion.

Using prior results of B.-Kettner (2008), B. (2008), B.-Pasechnik-Roy (2009):

Proposition (B. (2014))

- $(V_n)_{n>0} \in \mathbf{P}_{\mathbb{R}}$.

# Example 1

## Systems of few quadrics

Let $s > 0$ be fixed, and consider for each $n > 0$, the compact real algebraic set $V_n \subset (\mathbf{S}^{\binom{n+1}{2}-1})^s \times \mathbf{S}^n$ defined by

$$V_n = \{(P_1, \ldots, P_s, \mathbf{x}) \mid \mathbf{x} \in \mathbf{S}^n, P_i \in \mathbf{S}^{\binom{n+1}{2}-1}, P_i(\mathbf{x}) = 0, 1 \leq i \leq s\}.$$

Let $\pi_n : (\mathbf{S}^{\binom{n+1}{2}-1})^s \times \mathbf{S}^n \to (\mathbf{S}^{\binom{n+1}{2}-1})^s \hookrightarrow \mathbb{R}^{s\binom{n+1}{2}}$ be the projection to the first factor composed with the natural inclusion.

Using prior results of B.-Kettner (2008), B. (2008), B.-Pasechnik-Roy (2009):

Proposition (B. (2014))

- $(V_n)_{n>0} \in \mathbf{P}_{\mathbb{R}}$.
- $(0_{V_n})_{n>0} \in \mathcal{P}_{\mathbb{R}}$. $\left(R\pi_{n,*}(0_{V_n}) \in \mathbf{CS}(\mathbb{R}^{s\binom{n+1}{2}})\right)_{n>0} \in \mathcal{P}_{\mathbb{R}}$.

# Example 1

## Systems of few quadrics

Let $s > 0$ be fixed, and consider for each $n > 0$, the compact real algebraic set $V_n \subset (\mathbf{S}^{\binom{n+1}{2}-1})^s \times \mathbf{S}^n$ defined by

$$V_n = \{(P_1, \ldots, P_s, \mathbf{x}) \mid \mathbf{x} \in \mathbf{S}^n, P_i \in \mathbf{S}^{\binom{n+1}{2}-1}, P_i(\mathbf{x}) = 0, 1 \leq i \leq s\}.$$

Let $\pi_n : (\mathbf{S}^{\binom{n+1}{2}-1})^s \times \mathbf{S}^n \to (\mathbf{S}^{\binom{n+1}{2}-1})^s \hookrightarrow \mathbb{R}^{s\binom{n+1}{2}}$ be the projection to the first factor composed with the natural inclusion.

Using prior results of B.-Kettner (2008), B. (2008), B.-Pasechnik-Roy (2009):

Proposition (B. (2014))

# Example 1

## Systems of few quadrics

Let $s > 0$ be fixed, and consider for each $n > 0$, the compact real algebraic set $V_n \subset (\mathbf{S}^{\binom{n+1}{2}-1})^s \times \mathbf{S}^n$ defined by

$$V_n = \{(P_1, \ldots, P_s, \mathbf{x}) \mid \mathbf{x} \in \mathbf{S}^n, P_i \in \mathbf{S}^{\binom{n+1}{2}-1}, P_i(\mathbf{x}) = 0, 1 \leq i \leq s\}.$$

Let $\pi_n : (\mathbf{S}^{\binom{n+1}{2}-1})^s \times \mathbf{S}^n \to (\mathbf{S}^{\binom{n+1}{2}-1})^s \hookrightarrow \mathbb{R}^{s\binom{n+1}{2}}$ be the projection to the first factor composed with the natural inclusion.
Using prior results of B.-Kettner (2008), B. (2008), B.-Pasechnik-Roy (2009):

## Proposition (B. (2014))

- $(V_n)_{n>0} \in \mathbf{P}_\mathbb{R}$,
- $(j_{n,*}\mathbb{Q}_{V_n})_{n>0} \in \mathcal{P}_\mathbb{R}$, $\left(R\pi_{n,*}(j_{n,*}\mathbb{Q}_{V_n}) \in \mathcal{CS}(\mathbb{R}^{s\binom{n+1}{2}})\right)_{n>0} \in \mathcal{P}_\mathbb{R}$.

# Example 1

## Systems of few quadrics

Let $s > 0$ be fixed, and consider for each $n > 0$, the compact real algebraic set $V_n \subset (\mathbf{S}^{\binom{n+1}{2}-1})^s \times \mathbf{S}^n$ defined by

$$V_n = \{(P_1, \ldots, P_s, \mathbf{x}) \mid \mathbf{x} \in \mathbf{S}^n, P_i \in \mathbf{S}^{\binom{n+1}{2}-1}, P_i(\mathbf{x}) = 0, 1 \leq i \leq s\}.$$

Let $\pi_n : (\mathbf{S}^{\binom{n+1}{2}-1})^s \times \mathbf{S}^n \to (\mathbf{S}^{\binom{n+1}{2}-1})^s \hookrightarrow \mathbb{R}^{s\binom{n+1}{2}}$ be the projection to the first factor composed with the natural inclusion.
Using prior results of B.-Kettner (2008), B. (2008), B.-Pasechnik-Roy (2009):

## Proposition (B. (2014))

- $(V_n)_{n>0} \in \mathbf{P}_{\mathbb{R}}$,
- $(j_{n,*}\mathbb{Q}_{V_n})_{n>0} \in \mathcal{P}_{\mathbb{R}}$, $\left(R\pi_{n,*}(j_{n,*}\mathbb{Q}_{V_n}) \in \mathcal{CS}(\mathbb{R}^{s\binom{n+1}{2}})\right)_{n>0} \in \mathcal{P}_{\mathbb{R}}$.

# Example 1

## Systems of few quadrics

Let $s > 0$ be fixed, and consider for each $n > 0$, the compact real algebraic set $V_n \subset (\mathbf{S}^{\binom{n+1}{2}-1})^s \times \mathbf{S}^n$ defined by

$$V_n = \{(P_1, \ldots, P_s, \mathbf{x}) \mid \mathbf{x} \in \mathbf{S}^n, P_i \in \mathbf{S}^{\binom{n+1}{2}-1}, P_i(\mathbf{x}) = 0, 1 \le i \le s\}.$$

Let $\pi_n : (\mathbf{S}^{\binom{n+1}{2}-1})^s \times \mathbf{S}^n \to (\mathbf{S}^{\binom{n+1}{2}-1})^s \hookrightarrow \mathbb{R}^{s\binom{n+1}{2}}$ be the projection to the first factor composed with the natural inclusion.
Using prior results of B.-Kettner (2008), B. (2008), B.-Pasechnik-Roy (2009):

## Proposition (B. (2014))

- $(V_n)_{n>0} \in \mathbf{P}_\mathbb{R}$,
- $(j_{n,*}\mathbb{Q}_{V_n})_{n>0} \in \boldsymbol{\mathcal{P}}_\mathbb{R}$, $\left(R\pi_{n,*}(j_{n,*}\mathbb{Q}_{V_n}) \in \mathcal{CS}(\mathbb{R}^{s\binom{n+1}{2}})\right)_{n>0} \in \boldsymbol{\mathcal{P}}_\mathbb{R}$.

# Example 2

## Rank stratification sheaf

For each $n > 0$, let $V_n \subset \mathbf{S}^{n-1} \times \mathbf{S}^{n^2-1}$ be the semi-algebraic set defined by

$$V_n = \{(\mathbf{x}, A) \mid \mathbf{x} \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}, A \cdot \mathbf{x} = \mathbf{0}, ||A||^2 = 1, ||\mathbf{x}||^2 = 1\}.$$

Let $\pi_n : \mathbb{R}^n \times \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ denote the projection to the second factor composed with the natural inclusion.

### Proposition

$$\left( R\pi_{n,*}\mathbb{Q}_{V_n} \in \mathcal{CS}(\mathbb{R}^{n^2}) \right)_{n>0} \in \mathcal{P}_{\mathbb{R}}.$$

# Example 2

## Rank stratification sheaf

For each $n > 0$, let $V_n \subset \mathbf{S}^{n-1} \times \mathbf{S}^{n^2-1}$ be the semi-algebraic set defined by

$$V_n = \{(\mathbf{x}, A) \mid \mathbf{x} \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}, A \cdot \mathbf{x} = \mathbf{0}, ||A||^2 = 1, ||\mathbf{x}||^2 = 1\}.$$

Let $\pi_n : \mathbb{R}^n \times \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ denote the projection to the second factor composed with the natural inclusion.

## Proposition

$$\left(R\pi_{n,*}\mathbb{Q}_{V_n} \in \mathcal{CS}(\mathbb{R}^{n^2})\right)_{n>0} \in \boldsymbol{\mathcal{P}}_{\mathbb{R}}.$$

# Stability properties of $\mathcal{P}_{\mathbb{R}}$

Reminiscent of the classical B-S-S complexity class $\mathbf{P}_{\mathbb{R}}$ ...

- The class $\mathcal{P}_{\mathbb{R}}$ is stable under various sheaf operations – direct sums, tensor products, truncation functors.
- The class $\mathcal{P}_{\mathbb{R}}$ is also stable under the induced functor $\pi^{-1}$.

But what about the sequence of direct image functor $R\pi_*$ ?

# Stability properties of $\mathcal{P}_{\mathbb{R}}$

Reminiscent of the classical B-S-S complexity class $\mathbf{P}_{\mathbb{R}}$ ...

- The class $\mathcal{P}_{\mathbb{R}}$ is stable under various sheaf operations – direct sums, tensor products, truncation functors.
- The class $\mathcal{P}_{\mathbb{R}}$ is also stable under the induced functor $\pi^{-1}$.

But what about the sequence of direct image functor $R\pi_*$ ?

# Stability properties of $\mathcal{P}_{\mathbb{R}}$

Reminiscent of the classical B-S-S complexity class $\mathbf{P}_{\mathbb{R}}$ ...

- The class $\mathcal{P}_{\mathbb{R}}$ is stable under various sheaf operations – direct sums, tensor products, truncation functors.
- The class $\mathcal{P}_{\mathbb{R}}$ is also stable under the induced functor $\pi^{-1}$.

But what about the sequence of direct image functor $R\pi_*$ ?

# Stability properties of $\mathcal{P}_{\mathbb{R}}$

Reminiscent of the classical B-S-S complexity class $\mathbf{P}_{\mathbb{R}}$ ...

- The class $\mathcal{P}_{\mathbb{R}}$ is stable under various sheaf operations – direct sums, tensor products, truncation functors.
- The class $\mathcal{P}_{\mathbb{R}}$ is also stable under the induced functor $\pi^{-1}$.

But what about the sequence of direct image functor $R\pi_*$ ?

# Stability properties of $\mathcal{P}_{\mathbb{R}}$

Reminiscent of the classical B-S-S complexity class $\mathbf{P}_{\mathbb{R}}$ ...

- The class $\mathcal{P}_{\mathbb{R}}$ is stable under various sheaf operations – direct sums, tensor products, truncation functors.
- The class $\mathcal{P}_{\mathbb{R}}$ is also stable under the induced functor $\pi^{-1}$.

But what about the sequence of direct image functor $R\pi_*$ ?

# Complexity classes of constructible sheaves (cont).

- The functors $\pi_m^{-1}, R\pi_{m,*}$ induce in a natural way endo-functors

$$\mathcal{CS} \underset{R\pi_*}{\overset{\pi^{-1}}{\rightleftarrows}} \mathcal{CS}.$$

where $\mathcal{CS}$ is the category of sequences $(F_n \in \mathcal{CS}(\mathbb{R}^{m(n)}))_{n>0}$ .

- We have the adjunction: $\pi^{-1} \dashv R\pi_*$.
- Similar to the set-theoretic case, the following equality and containment can be checked easily.

$$\pi^{-1}(\mathcal{P}_\mathbb{R}) = \mathcal{P}_\mathbb{R},$$

$$\mathcal{P}_\mathbb{R} \subset R\pi_*(\mathcal{P}_\mathbb{R}).$$

- We define: $\Lambda_\mathbb{R}$ as the closure of the class $R\pi_*(\mathcal{P}_\mathbb{R})$ under the "easy" sheaf operations (namely, truncations, tensor products, direct sums and pull-backs), and define $\mathcal{PH}_\mathbb{R}$ by iteration as before.

# Complexity classes of constructible sheaves (cont).

- The functors $\pi_m^{-1}, R\pi_{m,*}$ induce in a natural way endo-functors

$$\mathcal{CS} \underset{R\pi_*}{\overset{\pi^{-1}}{\rightleftarrows}} \mathcal{CS}.$$

  where $\mathcal{CS}$ is the category of sequences $(F_n \in \mathcal{CS}(\mathbb{R}^{m(n)}))_{n>0}$ .
- We have the adjunction: $\pi^{-1} \dashv R\pi_*$.
- Similar to the set-theoretic case, the following equality and containment can be checked easily.

$$\pi^{-1}(\mathcal{P}_\mathbb{R}) = \mathcal{P}_\mathbb{R},$$

$$\mathcal{P}_\mathbb{R} \subset R\pi_*(\mathcal{P}_\mathbb{R}).$$

- We define: $\Lambda_\mathbb{R}$ as the closure of the class $R\pi_*(\mathcal{P}_\mathbb{R})$ under the "easy" sheaf operations (namely, truncations, tensor products, direct sums and pull-backs), and define $\mathcal{PH}_\mathbb{R}$ by iteration as before.

# Complexity classes of constructible sheaves (cont).

- The functors $\pi_m^{-1}$, $R\pi_{m,*}$ induce in a natural way endo-functors

$$\mathcal{CS} \underset{R\pi_*}{\overset{\pi^{-1}}{\leftrightarrows}} \mathcal{CS}.$$

where $\mathcal{CS}$ is the category of sequences $(F_n \in \mathcal{CS}(\mathbb{R}^{m(n)}))_{n>0}$ .

- We have the adjunction: $\pi^{-1} \dashv R\pi_*$.
- Similar to the set-theoretic case, the following equality and containment can be checked easily.

$$\pi^{-1}(\mathcal{P}_{\mathbb{R}}) = \mathcal{P}_{\mathbb{R}},$$

$$\mathcal{P}_{\mathbb{R}} \subset R\pi_*(\mathcal{P}_{\mathbb{R}}).$$

- We define: $\Lambda_{\mathbb{R}}$ as the closure of the class $R\pi_*(\mathcal{P}_{\mathbb{R}})$ under the "easy" sheaf operations (namely, truncations, tensor products, direct sums and pull-backs), and define $\mathcal{PH}_{\mathbb{R}}$ by iteration as before.

# Complexity classes of constructible sheaves (cont).

- The functors $\pi_m^{-1}, R\pi_{m,*}$ induce in a natural way endo-functors

$$\mathcal{CS} \underset{R\pi_*}{\overset{\pi^{-1}}{\leftrightarrows}} \mathcal{CS}.$$

where $\mathcal{CS}$ is the category of sequences $(F_n \in \mathcal{CS}(\mathbb{R}^{m(n)}))_{n>0}$ .

- We have the adjunction: $\pi^{-1} \dashv R\pi_*$.
- Similar to the set-theoretic case, the following equality and containment can be checked easily.

$$\pi^{-1}(\mathcal{P}_{\mathbb{R}}) = \mathcal{P}_{\mathbb{R}},$$

$$\mathcal{P}_{\mathbb{R}} \subset R\pi_*(\mathcal{P}_{\mathbb{R}}).$$

- We define: $\Lambda_{\mathbb{R}}$ as the closure of the class $R\pi_*(\mathcal{P}_{\mathbb{R}})$ under the "easy" sheaf operations (namely, truncations, tensor products, direct sums and pull-backs), and define $\mathcal{PH}_{\mathbb{R}}$ by iteration as before.

# Complexity classes of constructible sheaves (cont).

- The functors $\pi_m^{-1}, R\pi_{m,*}$ induce in a natural way endo-functors

$$\mathcal{CS} \underset{R\pi_*}{\overset{\pi^{-1}}{\leftrightarrows}} \mathcal{CS}.$$

where $\mathcal{CS}$ is the category of sequences $(F_n \in \mathcal{CS}(\mathbb{R}^{m(n)}))_{n>0}$ .

- We have the adjunction: $\pi^{-1} \dashv R\pi_*$.
- Similar to the set-theoretic case, the following equality and containment can be checked easily.

$$\pi^{-1}(\mathcal{P}_\mathbb{R}) = \mathcal{P}_\mathbb{R},$$

$$\mathcal{P}_\mathbb{R} \subset R\pi_*(\mathcal{P}_\mathbb{R}).$$

- We define: $\Lambda_\mathbb{R}$ as the closure of the class $R\pi_*(\mathcal{P}_\mathbb{R})$ under the "easy" sheaf operations (namely, truncations, tensor products, direct sums and pull-backs), and define $\mathcal{PH}_\mathbb{R}$ by iteration as before.

# Complexity classes of constructible sheaves (cont).

- The functors $\pi_m^{-1}, R\pi_{m,*}$ induce in a natural way endo-functors

$$\mathcal{CS} \underset{R\pi_*}{\overset{\pi^{-1}}{\rightleftarrows}} \mathcal{CS}.$$

where $\mathcal{CS}$ is the category of sequences $(F_n \in \mathcal{CS}(\mathbb{R}^{m(n)}))_{n>0}$ .

- We have the adjunction: $\pi^{-1} \dashv R\pi_*$.
- Similar to the set-theoretic case, the following equality and containment can be checked easily.

$$\pi^{-1}(\mathcal{P}_{\mathbb{R}}) = \mathcal{P}_{\mathbb{R}},$$

$$\mathcal{P}_{\mathbb{R}} \subset R\pi_*(\mathcal{P}_{\mathbb{R}}).$$

- We define: $\Lambda_{\mathbb{R}}$ as the closure of the class $R\pi_*(\mathcal{P}_{\mathbb{R}})$ under the "easy" sheaf operations (namely, truncations, tensor products, direct sums and pull-backs), and define $\mathcal{PH}_{\mathbb{R}}$ by iteration as before.

# Examples of sequences in $\mathbf{\Lambda}_{\mathbb{R}}$

Suppose that $\left(j_n : S_n \hookrightarrow \mathbb{R}^{m(n)}\right)_{n>0}$ belong to $\mathbf{NP}^c_{\mathbb{R}}$ or to $\textbf{co-NP}^c_{\mathbb{R}}$.

**Proposition**

*Then,*

$$\left(j_{n,*}\mathbb{Q}_{S_n} \in \mathcal{CS}(\mathbb{R}^{m(n)})\right)_{n>0} \in \mathbf{\Lambda}_{\mathbb{R}}.$$

# Another example

Let $V_n \subset \mathbf{S}^{N_{n,4}-1} \times \mathbf{S}^n$ defined by

$$V_n = \{(P, \mathbf{x}) \mid \mathbf{x} \in \mathbf{S}^n, P \in \mathbf{S}^{N_{n,4}-1}, P(\mathbf{x}) = 0\},$$

and let $\pi_n : \mathbf{S}^{N_{n,4}-1} \times \mathbf{S}^n \to \mathbf{S}^{N_{n,4}-1} \hookrightarrow \mathbb{R}^{N_{n,4}}$ be the projection to the first factor composed with the natural inclusion.

## Proposition

- $(V_n)_{n>0} \in \mathbf{P}^c_{\mathbb{R}}$.
- $(\pi_n(V_n))_{n>0} \in \mathbf{NP}^c_{\mathbb{R}}$.
- $(j_{n,*}\mathbb{Q}_{V_n})_{n>0} \in \mathcal{P}_{\mathbb{R}}$.
- $\left(R\pi_{n,*}(j_{n,*}\mathbb{Q}_{V_n}) \in \mathbf{CS}(\mathbb{R}^{N_{n,4}})\right)_{n>0} \in \Lambda_{\mathbb{R}}$.

# Another example

Let $V_n \subset \mathbf{S}^{N_{n,4}-1} \times \mathbf{S}^n$ defined by

$$V_n = \{(P, \mathbf{x}) \mid \mathbf{x} \in \mathbf{S}^n, P \in \mathbf{S}^{N_{n,4}-1}, P(\mathbf{x}) = 0\},$$

and let $\pi_n : \mathbf{S}^{N_{n,4}-1} \times \mathbf{S}^n \to \mathbf{S}^{N_{n,4}-1} \hookrightarrow \mathbb{R}^{N_{n,4}}$ be the projection to the first factor composed with the natural inclusion.

# Another example

Let $V_n \subset \mathbf{S}^{N_{n,4}-1} \times \mathbf{S}^n$ defined by

$$V_n = \{(P, \mathbf{x}) \mid \mathbf{x} \in \mathbf{S}^n, P \in \mathbf{S}^{N_{n,4}-1}, P(\mathbf{x}) = 0\},$$

and let $\pi_n : \mathbf{S}^{N_{n,4}-1} \times \mathbf{S}^n \to \mathbf{S}^{N_{n,4}-1} \hookrightarrow \mathbb{R}^{N_{n,4}}$ be the projection to the first factor composed with the natural inclusion.

## Proposition

- $(V_n)_{n>0} \in \mathbf{P}^c_{\mathbb{R}}$.
- $(\pi_n(V_n))_{n>0} \in \mathbf{NP}^c_{\mathbb{R}}$.
- $(j_{n,*}\mathbb{Q}_{V_n})_{n>0} \in \mathcal{P}_{\mathbb{R}}$.
- $\left(R\pi_{n,*}(j_{n,*}\mathbb{Q}_{V_n}) \in \mathcal{CS}(\mathbb{R}^{N_{n,4}})\right)_{n>0} \in \mathbf{\Lambda}_{\mathbb{R}}$.

# Another example

Let $V_n \subset \mathbf{S}^{N_{n,4}-1} \times \mathbf{S}^n$ defined by

$$V_n = \{(P, \mathbf{x}) \mid \mathbf{x} \in \mathbf{S}^n, P \in \mathbf{S}^{N_{n,4}-1}, P(\mathbf{x}) = 0\},$$

and let $\pi_n : \mathbf{S}^{N_{n,4}-1} \times \mathbf{S}^n \to \mathbf{S}^{N_{n,4}-1} \hookrightarrow \mathbb{R}^{N_{n,4}}$ be the projection to the first factor composed with the natural inclusion.

## Proposition

- $(V_n)_{n>0} \in \mathbf{P}^c_{\mathbb{R}}$.
- $(\pi_n(V_n))_{n>0} \in \mathbf{NP}^c_{\mathbb{R}}$.
- $(j_{n,*}\mathbb{Q}_{V_n})_{n>0} \in \mathcal{P}_{\mathbb{R}}$.
- $\left(R\pi_{n,*}(j_{n,*}\mathbb{Q}_{V_n}) \in \mathcal{CS}(\mathbb{R}^{N_{n,4}})\right)_{n>0} \in \mathbf{\Lambda}_{\mathbb{R}}$.

# Another example

Let $V_n \subset \mathbf{S}^{N_{n,4}-1} \times \mathbf{S}^n$ defined by

$$V_n = \{(P, \mathbf{x}) \mid \mathbf{x} \in \mathbf{S}^n, P \in \mathbf{S}^{N_{n,4}-1}, P(\mathbf{x}) = 0\},$$

and let $\pi_n : \mathbf{S}^{N_{n,4}-1} \times \mathbf{S}^n \to \mathbf{S}^{N_{n,4}-1} \hookrightarrow \mathbb{R}^{N_{n,4}}$ be the projection to the first factor composed with the natural inclusion.

## Proposition

- $(V_n)_{n>0} \in \mathbf{P}^c_{\mathbb{R}}$.
- $(\pi_n(V_n))_{n>0} \in \mathbf{NP}^c_{\mathbb{R}}$.
- $(j_{n,*}\mathbb{Q}_{V_n})_{n>0} \in \mathcal{P}_{\mathbb{R}}$.
- $\left(R\pi_{n,*}(j_{n,*}\mathbb{Q}_{V_n}) \in \mathcal{CS}(\mathbb{R}^{N_{n,4}})\right)_{n>0} \in \mathbf{\Lambda}_{\mathbb{R}}$.

# Another example

Let $V_n \subset \mathbf{S}^{N_{n,4}-1} \times \mathbf{S}^n$ defined by

$$V_n = \{(P, \mathbf{x}) \mid \mathbf{x} \in \mathbf{S}^n, P \in \mathbf{S}^{N_{n,4}-1}, P(\mathbf{x}) = 0\},$$

and let $\pi_n : \mathbf{S}^{N_{n,4}-1} \times \mathbf{S}^n \to \mathbf{S}^{N_{n,4}-1} \hookrightarrow \mathbb{R}^{N_{n,4}}$ be the projection to the first factor composed with the natural inclusion.

## Proposition

- $(V_n)_{n>0} \in \mathbf{P}^c_{\mathbb{R}}$.
- $(\pi_n(V_n))_{n>0} \in \mathbf{NP}^c_{\mathbb{R}}$.
- $(j_{n,*}\mathbb{Q}_{V_n})_{n>0} \in \boldsymbol{\mathcal{P}}_{\mathbb{R}}$.
- $\left(R\pi_{n,*}(j_{n,*}\mathbb{Q}_{V_n}) \in \mathcal{CS}(\mathbb{R}^{N_{n,4}})\right)_{n>0} \in \mathbf{\Lambda}_{\mathbb{R}}$.

# Another example

Let $V_n \subset \mathbf{S}^{N_{n,4}-1} \times \mathbf{S}^n$ defined by

$$V_n = \{(P, \mathbf{x}) \mid \mathbf{x} \in \mathbf{S}^n, P \in \mathbf{S}^{N_{n,4}-1}, P(\mathbf{x}) = 0\},$$

and let $\pi_n : \mathbf{S}^{N_{n,4}-1} \times \mathbf{S}^n \to \mathbf{S}^{N_{n,4}-1} \hookrightarrow \mathbb{R}^{N_{n,4}}$ be the projection to the first factor composed with the natural inclusion.

Proposition

- $(V_n)_{n>0} \in \mathbf{P}^c_{\mathbb{R}}$.
- $(\pi_n(V_n))_{n>0} \in \mathbf{NP}^c_{\mathbb{R}}$.
- $(j_{n,*}\mathbb{Q}_{V_n})_{n>0} \in \boldsymbol{\mathcal{P}}_{\mathbb{R}}$.
- $\left(R\pi_{n,*}(j_{n,*}\mathbb{Q}_{V_n}) \in \mathcal{CS}(\mathbb{R}^{N_{n,4}})\right)_{n>0} \in \mathbf{\Lambda}_{\mathbb{R}}$.

# Conjecture and relation with the classical questions

## Conjecture

$$\mathcal{P}_{\mathbb{R}} \neq \mathbf{\Lambda}_{\mathbb{R}}.$$

## Theorem (B., 2014)

$$\mathbf{P}_{\mathbb{R}}^{c} \neq \mathbf{NP}_{\mathbb{R}}^{c} \Rightarrow \mathcal{P}_{\mathbb{R}} \neq \mathbf{\Lambda}_{\mathbb{R}}.$$

Possibly – using the real analog of Toda's theorem (B.-Zell (2010)) – there is even the stronger implication:

## Theorem (?)

$$\mathbf{P}_{\mathbb{R}}^{c} \neq \mathbf{PH}_{\mathbb{R}}^{c} \Rightarrow \mathcal{P}_{\mathbb{R}} \neq \mathbf{\Lambda}_{\mathbb{R}}.$$

# Conjecture and relation with the classical questions

**Conjecture**

$$\mathcal{P}_{\mathbb{R}} \neq \mathbf{\Lambda}_{\mathbb{R}}.$$

**Theorem (B., 2014)**

$$\mathbf{P}_{\mathbb{R}}^{c} \neq \mathbf{NP}_{\mathbb{R}}^{c} \Rightarrow \mathcal{P}_{\mathbb{R}} \neq \mathbf{\Lambda}_{\mathbb{R}}.$$

Possibly – using the real analog of Toda's theorem (B.-Zell (2010)) – there is even the stronger implication:

**Theorem (?)**

$$\mathbf{P}_{\mathbb{R}}^{c} \neq \mathbf{PH}_{\mathbb{R}}^{c} \Rightarrow \mathcal{P}_{\mathbb{R}} \neq \mathbf{\Lambda}_{\mathbb{R}}.$$

# Conjecture and relation with the classical questions

**Conjecture**

$$\mathcal{P}_{\mathbb{R}} \neq \mathbf{\Lambda}_{\mathbb{R}}.$$

**Theorem (B., 2014)**

$$\mathbf{P}^{c}_{\mathbb{R}} \neq \mathbf{NP}^{c}_{\mathbb{R}} \Rightarrow \mathcal{P}_{\mathbb{R}} \neq \mathbf{\Lambda}_{\mathbb{R}}.$$

Possibly – using the real analog of Toda's theorem (B.-Zell (2010)) – there is even the stronger implication:

**Theorem (?)**

$$\mathbf{P}^{c}_{\mathbb{R}} \neq \mathbf{PH}^{c}_{\mathbb{R}} \Rightarrow \mathcal{P}_{\mathbb{R}} \neq \mathbf{\Lambda}_{\mathbb{R}}.$$

# Conjecture and relation with the classical questions

**Conjecture**

$$\mathcal{P}_{\mathbb{R}} \neq \mathbf{\Lambda}_{\mathbb{R}}.$$

**Theorem (B., 2014)**

$$\mathbf{P}^{c}_{\mathbb{R}} \neq \mathbf{NP}^{c}_{\mathbb{R}} \Rightarrow \mathcal{P}_{\mathbb{R}} \neq \mathbf{\Lambda}_{\mathbb{R}}.$$

Possibly – using the real analog of Toda's theorem (B.-Zell (2010)) – there is even the stronger implication:

**Theorem (?)**

$$\mathbf{P}^{c}_{\mathbb{R}} \neq \mathbf{PH}^{c}_{\mathbb{R}} \Rightarrow \mathcal{P}_{\mathbb{R}} \neq \mathbf{\Lambda}_{\mathbb{R}}.$$

# Conjecture and relation with the classical questions

**Conjecture**

$$\mathcal{P}_{\mathbb{R}} \neq \mathbf{\Lambda}_{\mathbb{R}}.$$

**Theorem (B., 2014)**

$$\mathbf{P}_{\mathbb{R}}^{c} \neq \mathbf{NP}_{\mathbb{R}}^{c} \Rightarrow \mathcal{P}_{\mathbb{R}} \neq \mathbf{\Lambda}_{\mathbb{R}}.$$

Possibly – using the real analog of Toda's theorem (B.-Zell (2010)) – there is even the stronger implication:

**Theorem (?)**

$$\mathbf{P}_{\mathbb{R}}^{c} \neq \mathbf{PH}_{\mathbb{R}}^{c} \Rightarrow \mathcal{P}_{\mathbb{R}} \neq \mathbf{\Lambda}_{\mathbb{R}}.$$

# Conjecture and relation with the classical questions

**Conjecture**

$$\mathcal{P}_{\mathbb{R}} \neq \mathbf{\Lambda}_{\mathbb{R}}.$$

**Theorem (B., 2014)**

$$\mathbf{P}_{\mathbb{R}}^{c} \neq \mathbf{NP}_{\mathbb{R}}^{c} \Rightarrow \mathcal{P}_{\mathbb{R}} \neq \mathbf{\Lambda}_{\mathbb{R}}.$$

Possibly – using the real analog of Toda's theorem (B.-Zell (2010)) – there is even the stronger implication:

**Theorem (?)**

$$\mathbf{P}_{\mathbb{R}}^{c} \neq \mathbf{PH}_{\mathbb{R}}^{c} \Rightarrow \mathcal{P}_{\mathbb{R}} \neq \mathbf{\Lambda}_{\mathbb{R}}.$$

# Topological complexity of the B-S-S polynomial hierarchy

The *topological complexity* of a semi-algebraic set $S$ is often measured by the sum of the Betti numbers of $S$ with coefficients in $\mathbb{Q}$, which we denote by $b(S)$.

# Topological complexity of the B-S-S polynomial hierarchy

The *topological complexity* of a semi-algebraic set $S$ is often measured by the sum of the Betti numbers of $S$ with coefficients in $\mathbb{Q}$, which we denote by $b(S)$.
It is thus natural to extend this measure to sequences.

# Topological complexity of the B-S-S polynomial hierarchy

The *topological complexity* of a semi-algebraic set $S$ is often measured by the sum of the Betti numbers of $S$ with coefficients in $\mathbb{Q}$, which we denote by $b(S)$.

It is thus natural to extend this measure to sequences.

It follows from bounds of Oleĭnik and Petrovskiĭ (1949), Thom, Milnor, B.-Pollack-Roy, Gabrielov-Vorobjov (2009) that ...

# Topological complexity of the B-S-S polynomial hierarchy

The *topological complexity* of a semi-algebraic set $S$ is often measured by the sum of the Betti numbers of $S$ with coefficients in $\mathbb{Q}$, which we denote by $b(S)$.

It is thus natural to extend this measure to sequences.

It follows from bounds of Oleĭnik and Petrovskiĭ (1949), Thom, Milnor, B.-Pollack-Roy, Gabrielov-Vorobjov (2009) that ...

**Theorem**

*Let* $\mathbf{L} = (S_n \in \mathcal{S}(\mathbb{R}^{m(n)}))_{n>0} \in \mathbf{P}_{\mathbb{R}}$. *Then, there exists a constant* $c_{\mathbf{L}}$, *such that*

$$b(S_n) \leq 2^{n^{c_{\mathbf{L}}}}$$

*for all* $n > 0$.

# Topological complexity of the B-S-S polynomial hierarchy

The *topological complexity* of a semi-algebraic set $S$ is often measured by the sum of the Betti numbers of $S$ with coefficients in $\mathbb{Q}$, which we denote by $b(S)$.
It is thus natural to extend this measure to sequences.
It follows from bounds of Oleĭnik and Petrovskiĭ (1949), Thom, Milnor, B.-Pollack-Roy, Gabrielov-Vorobjov (2009) that ...

**Theorem**

*Let* $\mathbf{L} = (S_n \in \mathcal{S}(\mathbb{R}^{m(n)}))_{n>0} \in \mathbf{P}_{\mathbb{R}}$. *Then, there exists a constant* $c_{\mathbf{L}}$, *such that*

$$b(S_n) \leq 2^{n^{c_{\mathbf{L}}}}$$

*for all* $n > 0$.

One could naively hope to use such a result to distinguish $\mathbf{P}_{\mathbb{R}}$ from $\mathbf{NP}_{\mathbb{R}}$, **co-NP**$_{\mathbb{R}}$ etc., but in fact ....

# Topological complexity of the B-S-S polynomial hierarchy

The *topological complexity* of a semi-algebraic set $S$ is often measured by the sum of the Betti numbers of $S$ with coefficients in $\mathbb{Q}$, which we denote by $b(S)$.

It is thus natural to extend this measure to sequences.

It follows from bounds of Oleĭnik and Petrovskiĭ (1949), Thom, Milnor, B.-Pollack-Roy, Gabrielov-Vorobjov (2009) that ...

One could naively hope to use such a result to distinguish $\mathbf{P}_{\mathbb{R}}$ from $\mathbf{NP}_{\mathbb{R}}$, **co-NP**$_{\mathbb{R}}$ etc., but in fact ....

Theorem

*Let* $\mathbf{L} = (S_n \in \mathcal{S}(\mathbb{R}^{m(n)}))_{n>0} \in \mathbf{PH}_{\mathbb{R}}$. *Then, there exists a constant* $c_L$, *such that*

$$b(S_n) \leq 2^{n^{c_{\mathbf{L}}}}$$

*for all* $n > 0$.

# Topological complexity of the B-S-S polynomial hierarchy

The *topological complexity* of a semi-algebraic set $S$ is often measured by the sum of the Betti numbers of $S$ with coefficients in $\mathbb{Q}$, which we denote by $b(S)$.
It is thus natural to extend this measure to sequences.
It follows from bounds of Oleĭnik and Petrovskiĭ (1949), Thom, Milnor, B.-Pollack-Roy, Gabrielov-Vorobjov (2009) that ...

**Theorem**

*Let* $\mathbf{L} = (S_n \in \mathcal{S}(\mathbb{R}^{m(n)}))_{n>0} \in \mathbf{PH}_{\mathbb{R}}$. *Then, there exists a constant* $c_L$, *such that*

$$b(S_n) \leq 2^{n^{c_{\mathbf{L}}}}$$

*for all* $n > 0$.

... But there might be other finer topological/geometric invariants – perhaps, related to complexity of stratification or desingularization ....

# Sheaf polynomial hierarchy and topological complexity

In analogy with the set-theoretic case, it is natural to measure the *topological complexity* of a constructible sheaf $F \in \mathcal{CS}(\mathbf{X})$ by

$$b(F) = \sum_i \dim_{\mathbb{Q}} \mathbb{H}^i(\mathbf{X}, F).$$

# Sheaf polynomial hierarchy and topological complexity

In analogy with the set-theoretic case, it is natural to measure the *topological complexity* of a constructible sheaf $F \in \mathcal{CS}(\mathbf{X})$ by

$$b(F) = \sum_i \dim_{\mathbb{Q}} \mathbb{H}^i(\mathbf{X}, F).$$

Theorem (B., 2014)

*Let* $\mathbf{F} = (F_n \in \mathcal{CS}(\mathbb{R}^{m(n)}))_{n>0} \in \boldsymbol{\mathcal{P}}_{\mathbb{R}}$. *Then, there exists a constant* $c_{\mathbf{F}}$*, such that*

$$b(F_n) \leq 2^{n^{c_{\mathbf{F}}}}$$

*for all* $n > 0$.

# Sheaf polynomial hierarchy and topological complexity

In analogy with the set-theoretic case, it is natural to measure the *topological complexity* of a constructible sheaf $F \in \mathcal{CS}(\mathbf{X})$ by

$$b(F) = \sum_i \dim_{\mathbb{Q}} \mathbb{H}^i(\mathbf{X}, F).$$

Theorem (B., 2014)

*Let* $\mathbf{F} = (F_n \in \mathcal{CS}(\mathbb{R}^{m(n)}))_{n>0} \in \boldsymbol{\mathcal{P}}_{\mathbb{R}}$*. Then, there exists a constant* $c_{\mathbf{F}}$*, such that*

$$b(F_n) \leq 2^{n^{c_{\mathbf{F}}}}$$

*for all* $n > 0$*.*

One could naively hope to use (as before) such a result to distinguish $\boldsymbol{\mathcal{P}}_{\mathbb{R}}$ from $\boldsymbol{\Lambda}_{\mathbb{R}}$, but in fact ....

# Sheaf polynomial hierarchy and topological complexity

In analogy with the set-theoretic case, it is natural to measure the *topological complexity* of a constructible sheaf $F \in \mathcal{CS}(\mathbf{X})$ by

$$b(F) = \sum_i \dim_{\mathbb{Q}} \mathbb{H}^i(\mathbf{X}, F).$$

Theorem (B., 2014)

*Let* $\mathbf{F} = (F_n \in \mathcal{CS}(\mathbb{R}^{m(n)}))_{n>0} \in \boldsymbol{\mathcal{PH}}_{\mathbb{R}}$. *Then, there exists a constant* $c_{\mathbf{F}}$*, such that*

$$b(F_n) \leq 2^{n^{c_{\mathbf{F}}}}$$

*for all* $n > 0$.

# Complexity theory of constructible functions

Let $\mathbf{X}, \mathbf{Y}$ be compact semi-algebraic sets, and $f : \mathbf{X} \to \mathbf{Y}$ a semi-algebraic continuous map. Then, we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{CS}(\mathbf{X}) & \xrightarrow{Rf_*} & \mathcal{CS}(\mathbf{Y}) \\ \downarrow \mathrm{Eu} & & \downarrow \mathrm{Eu} \\ \mathrm{CF}(\mathbf{X}) & \xrightarrow{\int \cdot \mathbf{d}\chi} & \mathrm{CF}(\mathbf{Y}), \end{array}$$

where we denote by $\mathrm{CF}(\mathbf{X})$ the set of constructible functions $f : \mathbf{X} \to \mathbb{R}$ on a semi-algebraic set $\mathbf{X}$.

The complexity theory of constructible functions is thus a "Euler-Poincaré trace" of that of the category of constructible sheaves.

# Complexity theory of constructible functions

Let $\mathbf{X}, \mathbf{Y}$ be compact semi-algebraic sets, and $f : \mathbf{X} \to \mathbf{Y}$ a semi-algebraic continuous map. Then, we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{CS}(\mathbf{X}) & \xrightarrow{Rf_*} & \mathcal{CS}(\mathbf{Y}) \\ \downarrow \mathrm{Eu} & & \downarrow \mathrm{Eu} \\ \mathrm{CF}(\mathbf{X}) & \xrightarrow{\int \cdot \mathbf{d}\,\chi} & \mathrm{CF}(\mathbf{Y}), \end{array}$$

where we denote by $\mathrm{CF}(\mathbf{X})$ the set of constructible functions $f : \mathbf{X} \to \mathbb{R}$ on a semi-algebraic set $\mathbf{X}$.

The complexity theory of constructible functions is thus a "Euler-Poincaré trace" of that of the category of constructible sheaves.

# Complexity theory of constructible functions

Let $\mathbf{X}, \mathbf{Y}$ be compact semi-algebraic sets, and $f : \mathbf{X} \to \mathbf{Y}$ a semi-algebraic continuous map. Then, we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{CS}(\mathbf{X}) & \xrightarrow{Rf_*} & \mathcal{CS}(\mathbf{Y}) \\ \downarrow{\scriptstyle \mathbf{Eu}} & & \downarrow{\scriptstyle \mathbf{Eu}} \\ \mathrm{CF}(\mathbf{X}) & \xrightarrow{\int \cdot \mathbf{d}\,\chi} & \mathrm{CF}(\mathbf{Y}), \end{array}$$

where we denote by $\mathrm{CF}(\mathbf{X})$ the set of constructible functions $f : \mathbf{X} \to \mathbb{R}$ on a semi-algebraic set $\mathbf{X}$.

The complexity theory of constructible functions is thus a "Euler-Poincaré trace" of that of the category of constructible sheaves.

# Complexity theory of constructible functions

Let $\mathbf{X}, \mathbf{Y}$ be compact semi-algebraic sets, and $f : \mathbf{X} \to \mathbf{Y}$ a semi-algebraic continuous map. Then, we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{CS}(\mathbf{X}) & \xrightarrow{Rf_*} & \mathcal{CS}(\mathbf{Y}) \\ \Big\downarrow \mathbf{Eu} & & \Big\downarrow \mathbf{Eu} \\ \mathrm{CF}(\mathbf{X}) & \xrightarrow{\int \cdot \mathbf{d}\,\chi} & \mathrm{CF}(\mathbf{Y}), \end{array}$$

where we denote by $\mathrm{CF}(\mathbf{X})$ the set of constructible functions $f : \mathbf{X} \to \mathbb{R}$ on a semi-algebraic set $\mathbf{X}$.
The complexity theory of constructible functions is thus a "Euler-Poincaré trace" of that of the category of constructible sheaves.

# Open problems and future directions

- Study more precisely the complexity of sheaf operations.
- Develop a theory of completeness which generalizes the classical theory.
- Get rid of the compactness/properness restrictions or understand better their significance.
- Role of adjointness in complexity questions ? For example, other pairs of adjoint functors such as the pair $(F \overset{L}{\otimes} \cdot \dashv R\mathcal{H}om(\cdot, F))$ ? More input from abstract category theory ?
- Applications of algorithmic/quantitative sheaf theory in other areas – such as $D$-modules, algebraic theory of PDE's, computational geometry/topology.
- Study the (simpler) complexity theory of constructible functions instead of sheaves (B-S-S analog of Valiant). This has been developed somewhat including a theory of reduction and complete problems (B. (2014).

# Open problems and future directions

- Study more precisely the complexity of sheaf operations.
- Develop a theory of completeness which generalizes the classical theory.
- Get rid of the compactness/properness restrictions or understand better their significance.
- Role of adjointness in complexity questions ? For example, other pairs of adjoint functors such as the pair $(F \overset{L}{\otimes} \cdot \dashv R\mathcal{H}om(\cdot, F))$ ? More input from abstract category theory ?
- Applications of algorithmic/quantitative sheaf theory in other areas – such as $D$-modules, algebraic theory of PDE's, computational geometry/topology.
- Study the (simpler) complexity theory of constructible functions instead of sheaves (B-S-S analog of Valiant). This has been developed somewhat including a theory of reduction and complete problems (B. (2014).

# Open problems and future directions

- Study more precisely the complexity of sheaf operations.
- Develop a theory of completeness which generalizes the classical theory.
- Get rid of the compactness/properness restrictions or understand better their significance.
- Role of adjointness in complexity questions ? For example, other pairs of adjoint functors such as the pair $(F \overset{L}{\otimes} \cdot \dashv R\mathcal{H}om(\cdot, F))$ ? More input from abstract category theory ?
- Applications of algorithmic/quantitative sheaf theory in other areas – such as $D$-modules, algebraic theory of PDE's, computational geometry/topology.
- Study the (simpler) complexity theory of constructible functions instead of sheaves (B-S-S analog of Valiant). This has been developed somewhat including a theory of reduction and complete problems (B. (2014).

# Open problems and future directions

- Study more precisely the complexity of sheaf operations.
- Develop a theory of completeness which generalizes the classical theory.
- Get rid of the compactness/properness restrictions or understand better their significance.
- Role of adjointness in complexity questions ? For example, other pairs of adjoint functors such as the pair $(F \overset{L}{\otimes} \cdot \dashv R\mathcal{H}om(\cdot, F))$ ? More input from abstract category theory ?
- Applications of algorithmic/quantitative sheaf theory in other areas – such as $D$-modules, algebraic theory of PDE's, computational geometry/topology.
- Study the (simpler) complexity theory of constructible functions instead of sheaves (B-S-S analog of Valiant). This has been developed somewhat including a theory of reduction and complete problems (B. (2014).

# Open problems and future directions

- Study more precisely the complexity of sheaf operations.
- Develop a theory of completeness which generalizes the classical theory.
- Get rid of the compactness/properness restrictions or understand better their significance.
- Role of adjointness in complexity questions ? For example, other pairs of adjoint functors such as the pair $(F \overset{L}{\otimes} \cdot \dashv R\mathcal{H}om(\cdot, F))$ ? More input from abstract category theory ?
- Applications of algorithmic/quantitative sheaf theory in other areas – such as $D$-modules, algebraic theory of PDE's, computational geometry/topology.
- Study the (simpler) complexity theory of constructible functions instead of sheaves (B-S-S analog of Valiant). This has been developed somewhat including a theory of reduction and complete problems (B. (2014).

# Open problems and future directions

- Study more precisely the complexity of sheaf operations.
- Develop a theory of completeness which generalizes the classical theory.
- Get rid of the compactness/properness restrictions or understand better their significance.
- Role of adjointness in complexity questions ? For example, other pairs of adjoint functors such as the pair $(F \overset{L}{\otimes} \cdot \dashv R\mathcal{H}om(\cdot, F))$ ? More input from abstract category theory ?
- Applications of algorithmic/quantitative sheaf theory in other areas – such as $D$-modules, algebraic theory of PDE's, computational geometry/topology.
- Study the (simpler) complexity theory of constructible functions instead of sheaves (B-S-S analog of Valiant). This has been developed somewhat including a theory of reduction and complete problems (B. (2014).