Recovery of algebraic-exponential data from moments

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$\star$ Part of this work is joint with M. Putinar

- Motivation
- An important property of Positively Homogeneous Functions (PHF)
- Some properties (convexity, polarity)
- Sub-level sets of minimum volume containing **K**
- Exact reconstruction from moments
- Recovery of the defining function of a semi-algebraic set

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Exact reconstruction

Reconstruction of a shape $\mathbf{K} \subset \mathbb{R}^n$ (convex or not)

from knowledge of finitely many moments

$$y_\alpha = \int_{\mathbf{K}} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \, dx, \qquad \alpha \in \mathbb{N}_d^n,$$

for some integer d, is a difficult and challenging problem!

EXACT recovery of $\mathbf{K}$

from $y = (y_\alpha)$, $\alpha \in \mathbb{N}_d^n$, is even more difficult and challenging!

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Exact recovery (continued)

Examples of exact recovery:

- Quadrature (planar) Domains in ($\mathbb{R}^2$) (Gustafsson, He, Milanfar and Putinar (Inverse Problems, 2000))
  • via an exponential transform
- Convex Polytopes (in $\mathbb{R}^n$) (Gravin, Lasserre, Pasechnik and Robins (Discrete & Comput. Geometry (2012))
  • Use Brion-Barvinok-Khovanski-Lawrence-Pukhlikov moment formula for projections $\int_P \langle c, x\rangle^j \, dx$ combined with a Prony-type method to recover the vertices of P.
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Approximate recovery can de done in multi-dimensions

(Cuyt, Golub, Milanfar and Verdonk, 2005) via :

- (multi-dimensional versions of) homogeneous Padé approximants applied to the Stieltjes transform.
- cubature formula at each point of grid
- solving a linear system of equations to retrieve the indicator function of $\mathbf{K}$

This talk: I

- Exact recovery.
- $\mathbf{K} = \{ x \in \mathbb{R}^n : \; g(\mathbf{x}) \; \leq \; 1 \}$ compact.
- g is a nonnegative homogeneous polynomial
- Data are finitely many moments:

$$y_\alpha \;=\; \int_{\mathbf{K}} \mathbf{x}^\alpha \, d\mathbf{x}, \qquad \alpha \in \mathbb{N}^n_d.$$

• Also works for Quasi-homogeneous polynomials, i.e., when

$$g(\lambda^{u_1} x_1, \ldots, \lambda^{u_n} x_n) \;=\; \lambda \, g(x), \qquad x \in \mathbb{R}^n, \, \lambda > 0$$

for some vector $u \in \mathbb{Q}^n$.

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A little detour

Positively Homogeneous functions (PHF) form a wide class of functions encountered in many applications. As a consequence of homogeneity, they enjoy very particular properties, and among them the celebrated and very useful Euler's identity which allows to deduce additional properties of PHFs in various contexts.

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So we are now concerned with PHFs, their sublevel sets and in particular, the integral

$$y \mapsto I_{g,h}(y) \;:=\; \int_{\{x\,:\,g(\mathbf{x})\le y\}} h(\mathbf{x})\,d\mathbf{x},$$

as a function $I_{g,h} : \mathbb{R}_+ \to \mathbb{R}$ when g, h are PHFs.

With y fixed, we are also interested in

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now as a function of g, especially when g is a nonnegative homogeneous polynomial.

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Some motivation

Interestingly, the latter integral is related in a simple and remarkable manner to the non-Gaussian integral

$$\int_{\mathbb{R}^n} h \exp(-g) dx.$$

Functional integrals appear frequently in quantum Physics

. . . ... where a challenging issue is to provide

exact formulas for $\int \exp(-g)\,dx$, the most well-known being when $\deg g = 2$, i.e., $g(\mathbf{x}) = x^T Q x$, with $Q \succ 0$,

$$d = 2 \Rightarrow \int \exp(-g)\,dx \;=\; \frac{\mathrm{Cte}}{\sqrt{\det(Q)}}.$$

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An integral

$$J(g) \;:=\; \int \exp(-g)\,dx$$

is called a discriminant integral.

Next if one write

$$\mathbf{x} \mapsto g(\mathbf{x}) \,=\, \sum_{a\in\mathbb{N}^n} g_a\,\mathbf{x}^a \quad (= \sum_{a\in\mathbb{N}^n} g_a\,\mathbf{x}_1^{a_1}\cdots\mathbf{x}_n^{a_n}).$$

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Integral discriminants satisfy WARD Identities

$$\left(\frac{\partial}{\partial g_{a_1\cdots a_n}}\frac{\partial}{\partial g_{b_1\cdots b_n}} - \frac{\partial}{\partial g_{c_1\cdots c_n}}\frac{\partial}{\partial g_{d_1\cdots d_n}}\right)\cdot J(g) \,=\, 0,$$

where $a_i + b_i = c_i + d_i$ for all i.

which in some (few) low-dimensional cases, permits to obtain exact formulas in terms of algebraic invariants of g. See e.g. Morosov and Shakirov[1]

[1] New and old results in Resultant theory, arXiv.0911.5278v1.

In particular, as a by-product in the important particular case when $h = 1$, they have proved that for all *forms* g of degree d,

$$\begin{aligned}\mathrm{Vol}\left(\{x \,:\, g(x) \leq 1\}\right) &= \int_{\{x\,:\,g(x)\leq 1\}} dx \\ &= \mathrm{cte}(d) \cdot \int_{\mathbb{R}^n} \exp(-g) d\mathbf{x},\end{aligned}$$

where the constant depends only on d and n.

In fact, a formula of exactly the same flavor was already known for convex sets, and was the initial motivation of our work. Namely, if $C \subset \mathbb{R}^n$ is convex, its support function

$$x \mapsto \sigma_C(x) := \sup \{x^T y \,:\, y \in C\},$$

is a PHF of degree 1, and the polar $C^\circ \subset \mathbb{R}^n$ of C is the convex set $\{x \,:\, \sigma_C(x) \leq 1\}$.

Then . . .

$$\mathrm{vol}\,(C^\circ) = \frac{1}{n!} \int_{\mathbb{R}^n} \exp(-\sigma_C(x))\, dx, \qquad \forall C.$$

I. An important property of PHF's

Let $\phi_1, \phi_2 : \mathbb{R}_+ \to \mathbb{R}$ be measurable mappings, and let $g \geq 0$ and h be PHFs of respective degree $0 \neq d, p \in \mathbb{Z}$.

We next show that

$$\frac{\displaystyle\int \phi_1(g)\, h\, d\mathbf{x}}{\displaystyle\int \phi_2(g)\, h\, d\mathbf{x}} = \mathrm{C}(\phi_1, \phi_2, d, p),$$

that is

The ratio DOES DEPENDS ONLY on ϕ_1, ϕ_2 and the degree of homogeneity of g and h!

$$\text{With } t \mapsto \phi_1(t) = 1_{[0,1]}(t) : \to \int_{\{g(\mathbf{x}) \leq 1\}} h(\mathbf{x})\, d\mathbf{x}.$$

$$\text{With } t \mapsto \phi_2(t) = \exp(-t) : \to \int_{\mathbb{R}^n} h(\mathbf{x})\, \exp(-g(\mathbf{x}))\, d\mathbf{x}$$

Theorem

Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ *be a measurable mapping, and let* $g \geq 0$ *and* h *be PHFs of respective degree* $0 \neq d, p \in \mathbb{Z}$ *and such that* $\int |h| \exp(-g) dx$ *is finite,*

$$\int_{\mathbb{R}^n} \phi(g(x))\, h(x)\, dx \;=\; C(\phi, d, p) \cdot \int_{\mathbb{R}^n} h \exp(-g)\, dx,$$

where the constant $C(\phi, d, p)$ *depends only on* ϕ, d, p.
In particular, if the sublevel set $\{x \,:\, g(x) \leq 1\}$ *is bounded, then*

$$\int_{\{x\,:\,g(x)\leq y\}} h\, dx \;=\; \frac{y^{(n+p)/d}}{\Gamma(1+(n+p)/d)} \int_{\mathbb{R}^n} h \exp(-g)\, dx,$$

with Γ *being the standard Gamma function*

Proof for nonnegative h

For simplicity assume that $g(x) > 0$ if $x \neq 0$. With $z = (z_1, \ldots, z_{n-1})$, do the change of variable $x_1 = t$, $x_2 = t\, z_1, \ldots, x_n = t\, z_{n-1}$ so that one may decompose $\int_{\mathbb{R}^n} \phi(g(x))\, h(x) dx$ into the sum

$$\begin{aligned}
& \int_{\mathbb{R}_+ \times \mathbb{R}^{n-1}} t^{n+p-1} \phi(t^d g(1,z))\, h(1,z)\, dt\, dz \\
+ & \int_{\mathbb{R}_+ \times \mathbb{R}^{n-1}} t^{n+p-1} \phi(t^d g(-1,-z))\, h(-1,z)\, dt\, dz, \\
= & \int_{\mathbb{R}^{n-1}} \left(\int_0^\infty t^{n+p-1} \phi(t^d g(1,z))\, dt \right) h(1,z)\, dz \\
+ & \int_{\mathbb{R}^{n-1}} \left(\int_0^\infty t^{n+p-1} \phi(t^d g(-1,-z))\, dt \right) h(-1,-z)\, dz,
\end{aligned}$$

where the last two integrals are obtained from the sum of the previous two by using Tonelli's Theorem.

Next, with the change of variable $u = t\,g(1,z)^{1/d}$ and $u = t\,g(-1,-z)^{1/d}$

$$\int_{\mathbb{R}^n} \phi(g(x))\,h(x)\,dx \;=\; \underbrace{\left(\int_{\mathbb{R}_+} u^{n+p-1}\,\phi(u^d)\,du\right)}_{\mathrm{Cte}(\phi,p,d)} \cdot A(g,h),$$

with

$$A(g,h) = \int_{\mathbb{R}^{n-1}} \left(\frac{h(1,z)}{g(1,z)^{(n+p)/d}} + \frac{h(-1,-z)}{g(-1,-z)^{(n+p)/d}} \right) dz.$$

□

Choosing $\phi(t) = \exp(-t)$ on $[0, +\infty)$ yields:

$$\int_{\mathbb{R}^n} \exp(-g(x))\, h(x)\, dx \;=\; \frac{\Gamma(1+(n+p)/d)}{n+p} \cdot A(g,h),$$

whereas, choosing $\phi(t) = \mathrm{I}_{[0,1]}(t)$ on $[0, +\infty)$ yields:

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And so in particular, whenever g is nonnegative and $\{x \,:\, g(x) \leq 1\}$ has finite Lebesgue volume:

Theorem

If g, h are PHFs of degree $0 < d$ and p respectively, then:

$$\int_{\{x\,:\,g(x)\leq y\}} h\,dx \;=\; \frac{y^{(n+p)/d}}{\Gamma(1+(n+p)/d)}\int_{\mathbb{R}^n} \exp(-g)\,h\,dx$$

$$\mathrm{vol}\,(\{x \,:\, g(x) \leq y\}) \;=\; \frac{y^{n/d}}{\Gamma(1+n/d)}\int_{\mathbb{R}^n} \exp(-g)\,dx$$

An alternative proof

Let g, h be nonnegative so that $I_{g,h}(y)$ vanishes on $(-\infty, 0]$. For $0 < \lambda \in \mathbb{R}$, its Laplace transform $\lambda \mapsto \mathcal{L}_{I_{g,h}}(\lambda) = \int_0^\infty \exp(-\lambda y) I_{g,h}(y)\, dy$ reads:

$$
\begin{aligned}
\mathcal{L}_{I_{g,h}}(\lambda) &= \int_0^\infty \exp(-\lambda y) \left(\int_{\{x: g(x) \leq y\}} h dx \right) dy \\
&= \int_{\mathbb{R}^n} h(x) \left(\int_{g(x)}^\infty \exp(-\lambda y) dy \right) dx \quad \text{[by Fubini]} \\
&= \frac{1}{\lambda} \int_{\mathbb{R}^n} h(x) \exp(-\lambda g(x))\, dx \\
&= \frac{1}{\lambda^{1+(n+p)/d}} \int_{\mathbb{R}^n} h(z) \exp(-g(z))\, dz \quad \text{[by homog]} \\
&= \frac{\int_{\mathbb{R}^n} h(z) \exp(-g(z))\, dz}{\Gamma(1 + (n+p)/d)} \mathcal{L}_{y^{(n+p)/d}}(\lambda).
\end{aligned}
$$

And so, by analyticity and the Identity theorem of analytical functions

$$I_{g,h}(y) = \frac{y^{(n+p)/d}}{\Gamma(1+(n+p)/d)} \int_{\mathbb{R}^n} h(x) \exp(-g(x))\, dx,$$

II. Approximating a non gaussian integral

Hence computing the non Gaussian integral $\int \exp(-g)\,dx$

reduces to computing the volume of the level set
$G := \{x \,:\, g(x) \,\leq\, 1\}$,

. . . which is the same as solving the optimization problem:

$$\begin{array}{rrl} \max_{\mu} & \mu(G) & \\ \text{s.t.} & \mu + \nu & = \lambda \\ & \mu(\mathbf{B} \setminus G) & = 0 \end{array}$$

where :

- $\mathbf{B}$ is a box $[-a, a]^n$ containing G and
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approximate as closely as desired $\mu(G)$ and any FIXED number of moments of μ, by solving an appropriate hierarchy of semidefinite programs (SDP).

(see: *Approximate volume and integration for basic semi algebraic sets*, Henrion, Lasserre and Savorgnan, SIAM Review 51, 2009.)

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Let $G \subseteq \mathbf{B} := [-1,1]^n$ (possibly after scaling), and let $z = (z_\alpha)$, $\alpha \in \mathbb{N}^n_{2k}$, be the moments of the Lebesgue measure λ on B.

Solve the hierarchy of semidefinite programs:

$$\begin{array}{rl} \rho_k = \max & y_0 \\ \text{s.t.} & \mathbf{M}_k(y), \mathbf{M}_k(v) \succeq 0, \\ & \mathbf{M}_{k-\lceil (d)/2 \rceil}(g\,y) \succeq 0 \\ & \mathbf{M}_{k-1}((1-x_i^2)\,v) \succeq 0, \quad i = 1,\ldots,n \\ & y_\alpha + v_\alpha = z_\alpha, \quad \alpha \in \mathbb{N}^n_{2k} \end{array}$$

for some moment and localizing matrices $\mathbf{M}_k(y)$ and $\mathbf{M}_k(g,y)$.

- The linear constraints $y_\alpha + v_\alpha = z_\alpha$ for all $\alpha \in \mathbb{N}^n_{2k}$ "ensure" $\mu + \nu = \lambda$, while the "$\succeq 0$" constraints "ensure" $\mathrm{supp}\,\mu = G$ and $\mathrm{supp}\,\nu = \mathbf{B}$.

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Another identity

Corollary

If g has degree d and G has finite volume then

$$\frac{\int_{\{x\,:\,g(x)\leq y\}} \exp(-g)\,dx}{\int_{\mathbb{R}^n} \exp(-g)\,dx} = \frac{\int_0^y t^{n/d-1}\exp(-t)dt}{\int_0^\infty t^{n/d-1}\exp(-t)dt} = \frac{\int_0^y t^{n/d-1}\exp(-t)dt}{\Gamma(n/d)}$$

expresses how fast $\mu(\{x\,:\,g(x)\leq y\})$ goes to $\mu(\mathbb{R}^n)$ as $y\to\infty$, for the Borel measure $d\mu=\exp(-g)\,dx$.

It is like for the Gamma function $\Gamma(n/d)$ when approximated by $\int_0^y t^{n/d-1}\exp(-t)dt$.

An interesting issue is to analyze how the Lebesgue volume $\mathrm{vol}\,\{x \in \mathbb{R}^n \,:\, g(x) \leq 1\}$, (i.e. $\mathrm{vol}\,(G)$) changes with g.

Corollary

Let h be a PHF of degree p and let $C_d \subset \mathbb{R}[x]_d$ be the convex cone of homogeneous polynomials of degree at most d such that $\mathrm{vol}(G)$ is finite. Then the function $f_h : C_d \to \mathbb{R}$,

$$g \mapsto f_h(g) := \int_G h\, dx, \qquad g \in C_d,$$

- *is a PHF of degree $-(n+p)/d$,*
- *convex whenever h is nonnegative and strictly convex if $h > 0$ on $\mathbb{R}^n \setminus \{0\}$*

Convexity (continued)

Corollary (continued)

Moreover, if h *is continuous and* $\int |h| \exp(-g)\, dx < \infty$ *then:*

$$\frac{\partial f_h(g)}{\partial g_\alpha} = \frac{-1}{\Gamma(1+(n+p)/d)} \int_{\mathbb{R}^n} x^\alpha\, h \exp(-g)\, dx$$

$$= \frac{-\Gamma(2+(n+p)/d)}{\Gamma(1+(n+p)/d)} \int_G x^\alpha\, h\, dx$$

$$\frac{\partial^2 f_h(g)}{\partial g_\alpha \partial g_\beta} = \frac{-1}{\Gamma(1+(n+p)/d)} \int_{\mathbb{R}^n} x^{\alpha+\beta}\, h \exp(-g)\, dx$$

PROOF: Just use

$$\int_{\{x\,:\,g(x)\leq 1\}} h\,dx \;=\; \frac{1}{\Gamma(1+(n+p)/d)}\int_{\mathbb{R}^n} h\,\exp(-g)\,dx$$

Notice that proving convexity directly would be non trivial but becomes easy when using the previous lemma!

PROOF: Just use

$$\int_{\{x\,:\,g(x)\leq 1\}} h\,dx \;=\; \frac{1}{\Gamma(1+(n+p)/d)} \int_{\mathbb{R}^n} h\, \exp(-g)\, dx$$

Notice that proving convexity directly would be non trivial but becomes easy when using the previous lemma!

III. Polarity

For a set $C \subset \mathbb{R}^n$, recall:

- The support function $x \mapsto \sigma_C(x) := \sup_y \{x^T y \,:\, y \in C\}$
- The POLAR $C^\circ := \{x \in \mathbb{R}^n \,:\, \sigma_C(x) \leq 1\}$
- and for a PHF g of degree d, its Legendre-Fenchel conjugate $g^*(x) = \sup_y \{x^T y - g(y)\}$ is a PHF of degree q with $\frac{1}{d} + \frac{1}{q} = 1$.

Polarity (continued)

Lemma

Let g be a closed proper convex PHF of degree $1 < d$ and let $G = \{x : g(x) \leq 1/d\}$. Then:

$$\begin{aligned} G^\circ &= \{x \in \mathbb{R}^n : g^*(x) \leq 1/q\} \\ \mathrm{vol}\,(G) &= \frac{p^{-n/p}}{\Gamma(1+n/p)} \int \exp(-g)\,dx \\ \mathrm{vol}\,(G^\circ) &= \frac{q^{-n/q}}{\Gamma(1+n/q)} \int \exp(-g^*)\,dx \end{aligned}$$

$\rightarrow$ yields completely symmetric formulas for g and its conjugate g^*.

Examples

- $g(x) = |x|^3$ so that $g^*(x) = \frac{2}{3\sqrt{3}}|x|^{3/2}$. And so

$$G = [-3^{-1/3}, 3^{-1/3}]; \quad G^\circ = [-3^{1/3}, 3^{1/3}].$$

- TV screen: $g(x) = x_1^4 + x_2^4$ so that $g^*(x) = 4^{-4/3}3(x_1^{4/3} + x_2^{4/3})$. And,

$$G = \{x : x_1^2 + x_2^4 \leq \frac{1}{4}\}; \quad G^\circ = \{x : x_1^{4/3} + x_2^{4/3} \leq 4^{1/3}\}.$$

- $g(x) = |x|$ so that $d \not> 1$, and $g^*(x) = 0$ if $x \in [-1, 1]$, and $+\infty$ otherwise. Hence $G = \{x : |x| \leq 1\} = [-1, 1]$ and with $q = +\infty$,

$$G^\circ = [-1, 1] = \{x : g^*(x) \leq \frac{1}{q} = 0\}.$$

Let $\mathbf{v}_d(x)$ be the vector of monomials (x^α) of degree d, i.e., such that $\alpha_1 + \cdots + \alpha_n = d$. (And so $\mathbf{v}_1(x) = x$.)

If $g \in \mathbb{R}[x]_{2d}$ is homogeneous and SOS then

$$g(x) \,=\, \frac{1}{2}\mathbf{v}_d(x)^T\,\Sigma\,\mathbf{v}_d(x),$$

for some real symmetric positive semidefinite matrix $\Sigma \succeq 0$.

And if $d = 1$ one has the Gaussian property

$$\int_{\mathbb{R}^n} \exp(-g)\,dx \,=\, \frac{(2\pi)^{n/2}}{\sqrt{\det\Sigma}},$$

$$\frac{\displaystyle\int_{\mathbb{R}^n} \mathbf{v}_d(x)\,\mathbf{v}_d(x)^T\,\exp(-g)\,dx}{\displaystyle\int_{\mathbb{R}^n} \exp(-g)\,dx} \,=\, \Sigma^{-1}.$$

Let $\mathbf{v}_d(x)$ be the vector of monomials (x^α) of degree d, i.e., such that $\alpha_1 + \cdots + \alpha_n = d$. (And so $\mathbf{v}_1(x) = x$.)
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In other words, if μ is the Gaussian measure

$$\mu(B) := \frac{\displaystyle\int_B \exp\left(-\frac{1}{2}x^T\Sigma\, x\right)\, dx}{\displaystyle\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}x^T\Sigma\, x\right)\, dx}, \quad \forall B,$$

then its (covariance) matrix of moments of order 2 satisfies:

$$\mathbf{M}_1(\Sigma) := \int_{\mathbb{R}^n} x\, x^T\, d\mu(x) = \Sigma^{-1},$$

and the function

$$\theta_1(\Sigma) := (\det \Sigma)^{1/2} \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\mathbf{v}_1(x)^T\Sigma\, \mathbf{v}_1(x)\right)\, dx.$$

is constant!

... not true anymore for $d > 1$!

However, let $\ell(d) = \binom{n+d-1}{d}$, and $\mathcal{S}_{++}^{\ell(d)}$ be the cone of real positive definite $\ell(d) \times \ell(d)$ matrices. Let $k := n/(2d\ell(d))$.

With $\Sigma \in \mathcal{S}_{++}^{\ell(d)}$, define the probability measure μ

$$\mu(B) := \frac{\int_B \exp\left(-k\mathbf{v}_d(x)^T \Sigma\, \mathbf{v}_d(x)\right)\, dx}{\int_{\mathbb{R}^n} \exp\left(-k\mathbf{v}_d(x)^T \Sigma\, \mathbf{v}_d(x)\right)\, dx}, \quad \forall B,$$

with matrix of moments of order $2d$ given by:

$$\mathbf{M}_d(\Sigma) := \int_{\mathbb{R}^n} \mathbf{v}_d(x)\, \mathbf{v}_d(x)^T \, d\mu(x).$$

Define $\theta_d : \mathcal{S}_{++}^{\ell(d)} \to \mathbb{R}$ to be the function

$$\Sigma \mapsto \theta_d(\Sigma) := (\det \Sigma)^k \int_{\mathbb{R}^n} \exp\left(-k\mathbf{v}_d(x)^T \Sigma\, \mathbf{v}_d(x)\right)\, dx.$$

Theorem

$$\mathbf{M}_d(\Sigma) = \Sigma^{-1} \iff \nabla\theta_d(\Sigma) = 0$$

Hence critical points Σ^* of θ_d have the Gaussian property

$$\frac{\displaystyle\int \mathbf{v}_d(x)\mathbf{v}_d(x)^T \exp\left(-k\mathbf{v}_d(x)^T \Sigma^*\, \mathbf{v}_d(x)\right)\, dx}{\displaystyle\int \exp\left(-k\mathbf{v}_d(x)^T \Sigma^*\, \mathbf{v}_d(x)\right)\, dx} = (\Sigma^*)^{-1}$$

- $\star$ If $d = 1$ then $\theta_d(\cdot)$ is constant and so $\nabla\theta_d(\cdot) = 0$.
- $\star$ If $d > 1$ then $\theta_d(\cdot)$ is constant in each ray $\lambda\Sigma$, $\lambda > 0$.

Proof

$$\begin{aligned}
\nabla\theta_d(\Sigma) &= k\,\frac{\Sigma^{\mathbb{A}}}{\det\Sigma}\,\theta_d(\Sigma) \\
&\quad -k(\det\Sigma)^k\int_{\mathbb{R}^n}\mathbf{v}_d(x)\mathbf{v}_d(x)^T\exp\left(-k\mathbf{v}_d(x)^T\Sigma\,\mathbf{v}_d(x)\right)\,dx \\
&= k\theta_d(\Sigma)\left[\Sigma^{-1}-\mathbf{M}_d(\Sigma)\right]
\end{aligned}$$

and so

$$\mathbf{M}_d(\Sigma)=\Sigma^{-1}\quad\Rightarrow\quad\nabla\theta_d(\Sigma)=0.$$

V. Sublevel sets G of minimum volume

If $\mathbf{K} \subset \mathbb{R}^n$ is compact then computing the ellipsoid ξ of minimum volume containing $\mathbf{K}$ is a classical problem whose optimal solution is called the Löwner-John ellipsoid.
So consider the following problem:

Find an homogeneous polynomial $g \in \mathbb{R}[x]_{2d}$ such that its sub level set $G := \{x : g(x) \leq 1\}$ contains $\mathbf{K}$ and has minimum volume among all such levels sets with this inclusion property.

Let $\mathbf{P}[x]_{2d}$ be the convex cone of homogeneous polynomials of degree $2d$ whose sub-level set $\mathbf{G} = \{x : g(x) \leq 1\}$ has finite Lebesgue volume and with $\mathbf{K} \subset \mathbb{R}^n$, let $C_{2d}(\mathbf{K})$ be the convex cone of polynomials nonnegative on $\mathbf{K}$.

Lemma

Let $\mathbf{K} \subset \mathbb{R}^n$ be compact. The minimum volume of a sublevel set $\mathbf{G} = \{\mathbf{x} : g(\mathbf{x}) \leq 1\}$, $g \in \mathbf{P}[x]_{2d}$, that contains $\mathbf{K} \subset \mathbb{R}^n$ is $\rho/\Gamma(1 + n/2d)$ where:

$$\mathcal{P} : \qquad \rho = \inf_{g \in \mathbf{P}[x]_{2d}} \left\{ \int_{\mathbb{R}^n} \exp(-g)\, dx \ : \ 1 - g \ \in \ C_{2d}(\mathbf{K}) \right\}.$$

a finite-dimensional convex optimization problem!

Let $\mathbf{P}[x]_{2d}$ be the convex cone of homogeneous polynomials of degree $2d$ whose sub-level set $\mathbf{G} = \{x : g(x) \leq 1\}$ has finite Lebesgue volume and with $\mathbf{K} \subset \mathbb{R}^n$, let $C_{2d}(\mathbf{K})$ be the convex cone of polynomials nonnegative on $\mathbf{K}$.

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a finite-dimensional convex optimization problem!

Proof

- We have seen that:

$$\mathrm{vol}\left(\{x\,:\,g(x)\leq 1\}\right)\,=\,\frac{1}{\Gamma(1+n/2d)}\int_{\mathbb{R}^n}\exp(-g)\,dx.$$

Moreover, the sub-level set $\{x\,:\,g(x)\leq 1\}$ contains **K** if and only if $1-g\in C_{2d}(\mathbf{K})$, and so $\rho/\Gamma(1+n/2d)$ is the minimum value of all volumes of sub-levels sets $\{x\,:\,g(x)\leq 1\}$, $g\in\mathbf{P}[\mathbf{x}]_{2d}$, that contain **K**.

- Now since $g\mapsto\int_{\mathbb{R}^n}\exp(-g)dx$ is strictly convex and $C_{2d}(\mathbf{K})$ is a convex cone, problem $\mathcal{P}$ is a finite-dimensional convex optimization problem. $\square$

V (continued). Characterizing an optimal solution

Theorem

(a) $\mathcal{P}$ *has a unique optimal solution* $g^* \in \mathbf{P}[x]_{2d}$ *and there exists a Borel measure* μ^* *supported on* $\mathbf{K}$ *such that:*

$$(*): \quad \left\{ \begin{array}{l} \displaystyle\int_{\mathbb{R}^n} x^\alpha \exp(-g^*)dx \;=\; \int_{\mathbf{K}} x^\alpha \, d\mu^*, \quad \forall |\alpha| = 2d \\ \displaystyle\int_{\mathbf{K}} (1 - g^*) \, d\mu^* = 0 \end{array} \right.$$

In particular, μ^* *is supported on the real variety* $V := \{x \in \mathbf{K} : g^*(\mathbf{x}) = 1\}$ *and in fact,* μ^* *can be substituted with another measure* ν^* *supported on at most* $\binom{n+2d-1}{2d}$ *points of* V.

(b) *Conversely, if* $g^* \in \mathbf{P}[x]_{2d}$ *and* μ^* *satisfy (*) then* g^* *is an optimal solution of* $\mathcal{P}$.

V (continued). Characterizing an optimal solution

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(b) *Conversely, if* $g^* \in \mathbf{P}[x]_{2d}$ *and* μ^* *satisfy (*) then* g^* *is an optimal solution of* $\mathcal{P}$*.*

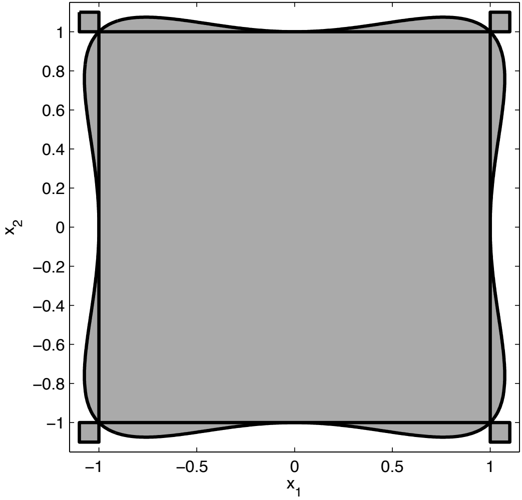
Example

Let $\mathbf{K} \subset \mathbb{R}^2$ be the box $[-1, 1]^2$.

The set $\mathbf{G}_4 := \{\mathbf{x} : g(\mathbf{x}) \leq 1\}$ with g homogeneous of degree 4 which contains $\mathbf{K}$ and has minimum volume is

$$\mathbf{x} \mapsto g_4(\mathbf{x}) := x_1^4 + y_1^4 - x_1^2 x_2^2,$$

with $\mathrm{vol}(\mathbf{G}_4) \approx 4.39$ much better than
- $\pi R^2 = 2\pi \approx 6.28$ for the Löwner-John ellipsoid of minimum volume, and
- the (convex) TV screen $\mathbf{G} := \{\mathbf{x} : (x_1^4 + x_2^4)/2 <= 1\}$ with volume > 5.


1
0.8
0.6
0.4
0.2
0
–0.2
–0.4
–0.6
–0.8
–1
x2
–1
–0.5
0
0.5
1
x1

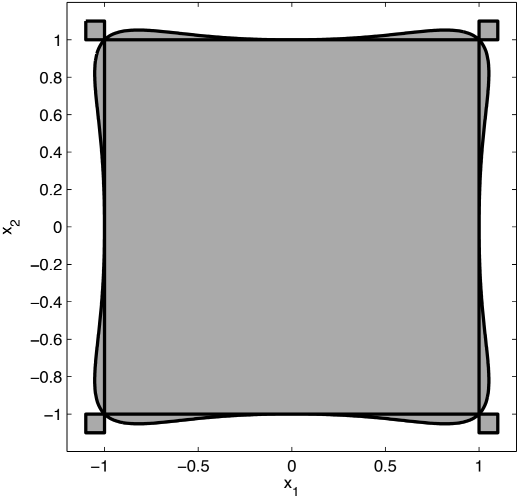
Example (continued)

Let $\mathbf{K} \subset \mathbb{R}^2$ be the box $[-1, 1]^2$.

The set $\mathbf{G}_6 := \{\mathbf{x} : g(\mathbf{x}) \leq 1\}$ with g homogeneous of degree 6 which contains $\mathbf{K}$ and has minimum volume is

$$\mathbf{x} \mapsto g_6(\mathbf{x}) := x_1^6 + y_1^6 - (x_1^4 x_2^2 + x_1^2 x_2^4)/2,$$

with $\mathrm{vol}(\mathbf{G}_6) \approx 4.19$ much better than
- $\pi R^2 = 2\pi \approx 6.28$ for the Löwner-John ellipsoid of minimum volume, and
- better than the set $\mathbf{G}_4$ with volume 4.39.


1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
x2
−1
−0.5
0
0.5
1
x1

VI. Recovering g from moments of G

Write $g(x) = \sum_\beta g_\beta \, x^\beta$.

Lemma

If g is nonnegative and d-homogeneous with G compact then:

$$\underbrace{\int_G x^\alpha \, g(x)\, dx}_{\sum_\beta g_\beta \, y_{\alpha+\beta}}, = \frac{n+|\alpha|}{n+d+|\alpha|} \underbrace{\int_G \mathbf{x}^\alpha \, dx}_{y_\alpha}, \qquad \alpha \in \mathbb{N}^n.$$

and so we see that the moments (y_α) satisfy linear relationships explicit in terms of the coefficients of the polynomial g that describes the boundary of G.

So let us write $\mathbf{g} \in \mathbb{R}^{s(d)}$ the unknown vector of coefficients of the unknown polynomial g.
Let $\mathbf{M}_d(y)$ be the moment matrix of order d whose rows and columns are indexed in the canonical basis of monomials (x^α), $\alpha \in \mathbb{N}^n_d$, and with entries

$$\mathbf{M}_d(y)(\alpha, \beta) = y_{\alpha+\beta}, \qquad \alpha, \beta \in \mathbb{N}^n_d.$$

and let $\mathbf{y}^d$ be the vector (y_α), $\alpha \in \mathbb{N}^n_d$.

Previous Lemma states that

$$\mathbf{M}_d(y)\,\mathbf{g} = \mathbf{y}^d,$$

or, equivalently,

$$\mathbf{g} = \mathbf{M}_d(y)^{-1}\,\mathbf{y}^d,$$

because the moment matrix $\mathbf{M}_d(y)$ is nonsingular whenever G has nonempty interior.

In other words ...

one may recover g EXACTLY from knowledge of moments (y_α) of order d and $2d$!

Non homogeneous polynomials

If g is not quasi-homogeneous then one cannot directly relate

$$\int_{\{\mathbf{x}:g(\mathbf{x})\leq 1\}} d\mathbf{x} \quad \text{and} \quad \int_{\mathbb{R}^n} \exp(-g(\mathbf{x}))\, d\mathbf{x}.$$

But still the Laplace transform $\lambda \mapsto F(\lambda)$ of the function

$$y \mapsto f(y) := \int_{\{\mathbf{x}:\mid g(\mathbf{x})\mid \leq y\}} d\mathbf{x}$$

is the non Gaussian integral

$$\lambda \mapsto F(\lambda) = \frac{1}{\lambda} \int_{\mathbb{R}^n} \exp(-\lambda \mid g(\mathbf{x}) \mid)\, d\mathbf{x}.$$

Nice asymptotic results are available (Vassiliev)

$$f(y) \approx y^a \ln(y)^b, \quad \text{as } y \to \infty$$

for some rationals a, b obtained from the Newton polytope of g.

One even has asymptotic results for

$$y \mapsto \tilde{f}(y) := \#\left(\{\mathbf{x} : |g(\mathbf{x})| \leq y\} \cap \mathbf{Z}^n\right), \quad \text{as } y \to \infty$$

still in the form

$$\tilde{f}(y) \approx y^{a'} \ln(y)^{b'}, \quad \text{as } y \to \infty$$

for some rationals a', b' obtained from the (modified) Newton polytope of g.

Exact recovery

Given a polynomial $g \in \mathbb{R}[\mathbf{x}]_d$ write $g(\mathbf{x}) = \sum_{k=0}^{d} g_k(\mathbf{x})$, where each g_k is homogeneous of degree k.

Lemma

Let $g \in \mathbb{R}[\mathbf{x}]_d$ be such that its level set $\mathbf{G} := \{\mathbf{x} : g(\mathbf{x}) \leq 1\}$ is bounded. Then for every $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$:

$$\int_{\mathbf{G}} \mathbf{x}^\alpha (1 - g(\mathbf{x}))\, d\mathbf{x} = \sum_{k=1}^{d} \frac{k}{n + |\alpha|} \int_{\mathbf{G}} \mathbf{x}^\alpha g_k(\mathbf{x})\, d\mathbf{x}$$

Observe that for each fixed arbitrary $\alpha \in \mathbb{N}^n$ ...

One obtains LINEAR EQUALITIES between MOMENTS of the Lebesgue measure on $\mathbf{G}$!

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Observe that for each fixed arbitrary $\alpha \in \mathbb{N}^n$ ...

One obtains LINEAR EQUALITIES between MOMENTS of the Lebesgue measure on **G**!

Proof:

- Use Stokes' formula

$$\int_{\mathbf{G}} \mathrm{Div}(X)\, f(\mathbf{x})\, d\mathbf{x} + \int_{\mathbf{G}} \langle X, \nabla f(\mathbf{x})\rangle d\mathbf{x} \;=\; \int_{\partial \mathbf{G}} \langle X, \vec{n}_{\mathbf{x}}\rangle\, f\, d\sigma,$$

with vector field $X = \mathbf{x}$ and $f(\mathbf{x}) = \mathbf{x}^{\alpha}(1 - g(\mathbf{x}))$.

- Then observe that $\mathrm{Div}(X) = n$ and:

$$\langle X, \nabla f(\mathbf{x})\rangle \;=\; |\alpha|\, f - \mathbf{x}^{\alpha} \sum_{k=1}^{d} k\, g_k(\mathbf{x}).$$

$\star$ In the general case, when $\partial \mathbf{G}$ may have singular points, or lower dimensional components, we can invoke Sard's theorem, for the (smooth) sublevel sets

$$G_{\gamma} = \{\, \mathbf{x} : g(\mathbf{x}) < \gamma \,\}$$

and pass to the limit $\gamma \to 1,\ \gamma < 1$. $\square$

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and pass to the limit $\gamma \to 1,\ \gamma < 1$. $\square$

Let $\mathbf{G} \subset \mathbb{R}^n$ be open with $\mathbf{G} = \operatorname{int} \overline{\mathbf{G}}$ and with real algebraic boundary $\partial\mathbf{G}$. A polynomial of degree d vanishes on $\partial\mathbf{G}$.

Define a renormalised moment-type matrix $\mathbf{M}_k^d(y)$ as follows:

- $s(d)$ $(= \binom{n+d}{n})$ columns indexed by $\beta \in \mathbb{N}_d^n$,

- countably many rows indexed by $\alpha \in \mathbb{N}_k^n$,

and with entries:

$$\mathbf{M}_k^d(y)(\alpha,\beta) := \frac{n+|\alpha|+|\beta|}{n+|\alpha|} y_{\alpha+\beta}, \qquad \alpha \in \mathbb{N}_k^n,\ \beta \in \mathbb{N}_d^n.$$

Theorem

Let $\mathbf{G} \subset \mathbb{R}^n$ *be a bounded open set with real algebraic boundary. Assume that* $\mathbf{G} = \mathrm{int}\,\overline{\mathbf{G}}$ *and a polynomial of degree d vanishes on* $\partial\mathbf{G}$ *and not at* 0*. Then the linear system*

$$\mathbf{M}_{2d}^{d}(y)\begin{bmatrix} -1 \\ g \end{bmatrix} = 0,$$

admits a unique solution $\mathbf{g} \in \mathbb{R}^{s(d)-1}$*, and the polynomial* g *with coefficients* $(0, \mathbf{g})$ *satisfies*

$$(\mathbf{x} \in \partial G) \Rightarrow (g(\mathbf{x}) = 1).$$

Sketch of the proof

The identity (obtained from Stokes' theorem)

$$\int_{\mathbf{G}} \mathbf{x}^{\alpha}(1 - g(\mathbf{x}))\, d\mathbf{x} = \sum_{k=1}^{d} \frac{k}{n + |\alpha|} \int_{\mathbf{G}} \mathbf{x}^{\alpha} g_k(\mathbf{x})\, d\mathbf{x}$$

for all $\alpha \in \mathbb{N}^n_k$

in fact reads:

$$\mathbf{M}^d_k(y) \begin{bmatrix} -1 \\ \mathbf{g} \end{bmatrix} = 0,$$

Conversely, if $\mathbf{g}$ solves

$$\mathbf{M}^d_{2d}(y) \begin{bmatrix} -1 \\ \mathbf{g} \end{bmatrix} = 0,$$

then

$$\int_{\partial \mathbf{G}} \langle \mathbf{x}, \vec{n}_{\mathbf{x}} \rangle (1 - g(\mathbf{x}))\, \mathbf{x}^{\alpha}\, d\sigma = 0, \quad \forall \alpha \in \mathbb{N}^n_{2d}.$$

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As $\partial\mathbf{G}$ is algebraic, one may write

$$\vec{n}_{\mathbf{x}} = \frac{\nabla h(\mathbf{x})}{\|\nabla h(\mathbf{x})\|},$$

for some polynomial h. Therefore

$$\begin{aligned} 0 &= \int_{\partial\mathbf{G}} \langle \mathbf{x}, \vec{n}_{\mathbf{x}} \rangle (1 - g(\mathbf{x}))\, \mathbf{x}^\alpha \, d\sigma \quad \forall \alpha \in \mathbb{N}^n_{2d} \\ &= \int_{\partial\mathbf{G}} \underbrace{\langle \mathbf{x}, \nabla h(\mathbf{x}) \rangle}_{\in \mathbb{R}[\mathbf{x}]_d} \underbrace{(1 - g(\mathbf{x}))}_{\in \mathbb{R}[\mathbf{x}]_d} \mathbf{x}^\alpha \underbrace{\frac{1}{\|\nabla h\|} d\sigma}_{d\sigma'} \quad \forall \alpha \in \mathbb{N}^n_{2d} \end{aligned}$$

$$\Rightarrow \int_{\partial\mathbf{G}} \underbrace{\langle \mathbf{x}, \nabla h(\mathbf{x}) \rangle^2}_{\neq 0 \; \sigma - a.e.} (1 - g(\mathbf{x}))^2 d\sigma' = 0 \quad \square$$

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$$\vec{n}_{\mathbf{x}} = \frac{\nabla h(\mathbf{x})}{\|\nabla h(\mathbf{x})\|},$$

for some polynomial h. Therefore

$$\begin{aligned} 0 &= \int_{\partial \mathbf{G}} \langle \mathbf{x}, \vec{n}_{\mathbf{x}} \rangle (1 - g(\mathbf{x}))\, \mathbf{x}^{\alpha}\, d\sigma \quad \forall \alpha \in \mathbb{N}^n_{2d} \\ &= \int_{\partial \mathbf{G}} \underbrace{\langle \mathbf{x}, \nabla h(\mathbf{x}) \rangle}_{\in \mathbb{R}[\mathbf{x}]_d} \underbrace{(1 - g(\mathbf{x}))}_{\in \mathbb{R}[\mathbf{x}]_d}\, \mathbf{x}^{\alpha} \underbrace{\frac{1}{\|\nabla h\|} d\sigma}_{d\sigma'} \quad \forall \alpha \in \mathbb{N}^n_{2d} \end{aligned}$$

$$\Rightarrow \quad \int_{\partial \mathbf{G}} \underbrace{\langle \mathbf{x}, \nabla h(\mathbf{x}) \rangle^2}_{\neq 0\ \sigma - a.e.} (1 - g(\mathbf{x}))^2 d\sigma' = 0 \quad \square$$

For sake of rigor the boundary $\partial\mathbf{G}$ can be written

$$\partial\mathbf{G} = Z_0 \cup Z_1,$$

with Z_0 being a finite union of smooth $n-1$-submanifolds of $\mathbb{R}^n$ leaving $\mathbf{G}$ on one side, Z_1 is a union of the lower dimensional strata, and $\sigma(Z_1) = 0$.

Convexity

Theorem

Let $\mathbf{G} \subset \mathbb{R}^n$ *be a bounded convex open set with real algebraic boundary. Assume that* $\mathbf{G} = \operatorname{int} \overline{\mathbf{G}}$, $0 \in \mathbf{G}$, *and a polynomial of degree* d *vanishes on* $\partial\mathbf{G}$ *and not at* 0. *Then the linear system*

$$\mathbf{M}_d^d(y) \begin{bmatrix} -1 \\ g \end{bmatrix} = 0,$$

admits a unique solution $\mathbf{g} \in \mathbb{R}^{s(d)-1}$, *and the polynomial* g *with coefficients* $(0, \mathbf{g})$ *satisfies*

$$(\mathbf{x} \in \partial G) \Rightarrow (g(\mathbf{x}) = 1).$$

$\star$ As in the previous proof, if

$$\mathbf{M}_d^d(y) \begin{bmatrix} -1 \\ g \end{bmatrix} = 0,$$

then

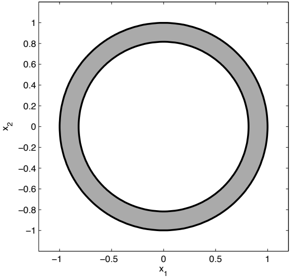
$$\int_{\partial \mathbf{G}} \langle \mathbf{x}, \vec{n}_{\mathbf{x}} \rangle (1 - g(\mathbf{x}))^2 \, d\sigma = 0.$$

But one now uses that if $0 \in \mathbf{G}$ then $\langle \mathbf{x}, \vec{n}_{\mathbf{x}} \rangle \geq 0$.

Example

Let us consider the two-dimensional example of the annulus

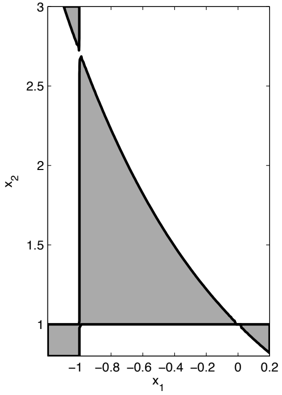
$$\mathbf{G} := \{ \mathbf{x} \in \mathbb{R}^2 : \; 1 - x_1^2 - x_2^2 \geq 0; \; x_1^2 + x_2^2 - 2/3 \geq 0 \}.$$



The polynomial $(1 - x_1^2 - x_2^2)(x_1^2 + x_2^2 - 2/3)$ is the unique solution of $\mathbf{M}_4^4(y)\,[-1, g] = 0$.

Example continued: Non-algebraic boundary

Let $\mathbf{G} = \{\mathbf{x} \in \mathbb{R}^2 : x_1 \geq -1;\ x_2 \geq 1;\ x_2 \leq \exp(-x_1)\,\}$.



We now look as the eigenvector g of the smallest eigenvalue of $\mathbf{M}_3^3(y)$.

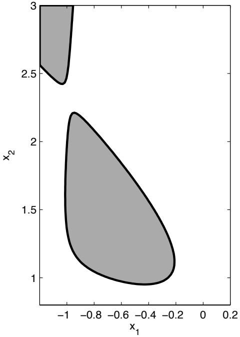


Figure: Shape $\mathbf{G}' = \{\mathbf{x} : g(\mathbf{x}) \leq 0\}$ with $d = 3$

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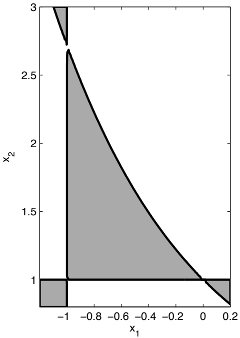

Figure: Shape $\mathbf{G}' = \{\mathbf{x} : g(\mathbf{x}) \leq 0\}$ with $d = 4$

A consequence in Probability

Consider the Probability measure μ

uniformly supported on a set G of the form $\{\mathbf{x} : g(\mathbf{x}) \leq 1\}$, for some polynomial $g \in \mathbb{R}[\mathbf{x}]_d$.

Then :

- ALL moments $y_\alpha := \int_G \mathbf{x}^\alpha \, d\mu$, $\alpha \in \mathbb{N}^n$, are determined from those up to order $3d$ (and $2d$ if G is convex) !

- A similar result holds true if now μ has a density $\exp(h(\mathbf{x}))$ on G (for some $h \in \mathbb{R}[\mathbf{x}]$).

$\rightarrow$ is an extension to such measures of a well-known result for exponential families

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Conclusion

• Compact sub-level sets $G := \{x : g(x) \leq y\}$ of homogeneous polynomials exhibit surprising properties. E.g.:

- convexity of $\mathrm{volume}(G)$ with respect to the coefficients of g
- Integrating a PHF h on G reduce to evaluating the non Gaussian integral $\int h \exp(-g) dx$
- A variational property yields a Gaussian-like property
- exact recovery of G from finitely moments.
  (Also works for quasi-homogeneous polynomials with bounded sublevel sets!)
- exact recovery for sets with algebraic boundary of known degree

Practical and important issues

- COMPUTATION!: Efficient evaluation of $\int_{\mathbb{R}^n} \exp(-g)\,dx$, or equivalently, evaluation of $\mathrm{vol}\,(\{x : g(x) \leq 1\})$!

  • The property

$$\int_G \mathbf{x}^\alpha g(x)\,dx \;=\; \frac{n+|\alpha|}{n+d+|\alpha|}\int_G x^\alpha\,dx, \qquad \forall\alpha,$$

  helps a lot to improve efficiency of the method in Henrion, Lasserre and Savorgnan (SIAM Review)

Some references

- J.B. Lasserre. A Generalization of Löwner-John's ellipsoid Theorem. *Math. Program.*, to appear.
- J.B. Lasserre. Recovering an homogeneous polynomial from moments of its level set. *Discrete & Comput. Geom.* **50**, pp. 673–678, 2013.
- J.B. Lasserre and M. Putinar. Reconstruction of algebraic-exponential data from moments. Submitted
- J.B. Lasserre. Unit balls of constant volume: which one has optimal representation? submitted.

THANK YOU!