On the critical point method and deciding connectivity queries in real algebraic sets: from theory to practice

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Polynomial system solving over the reals: what can exact computation do nowadays?

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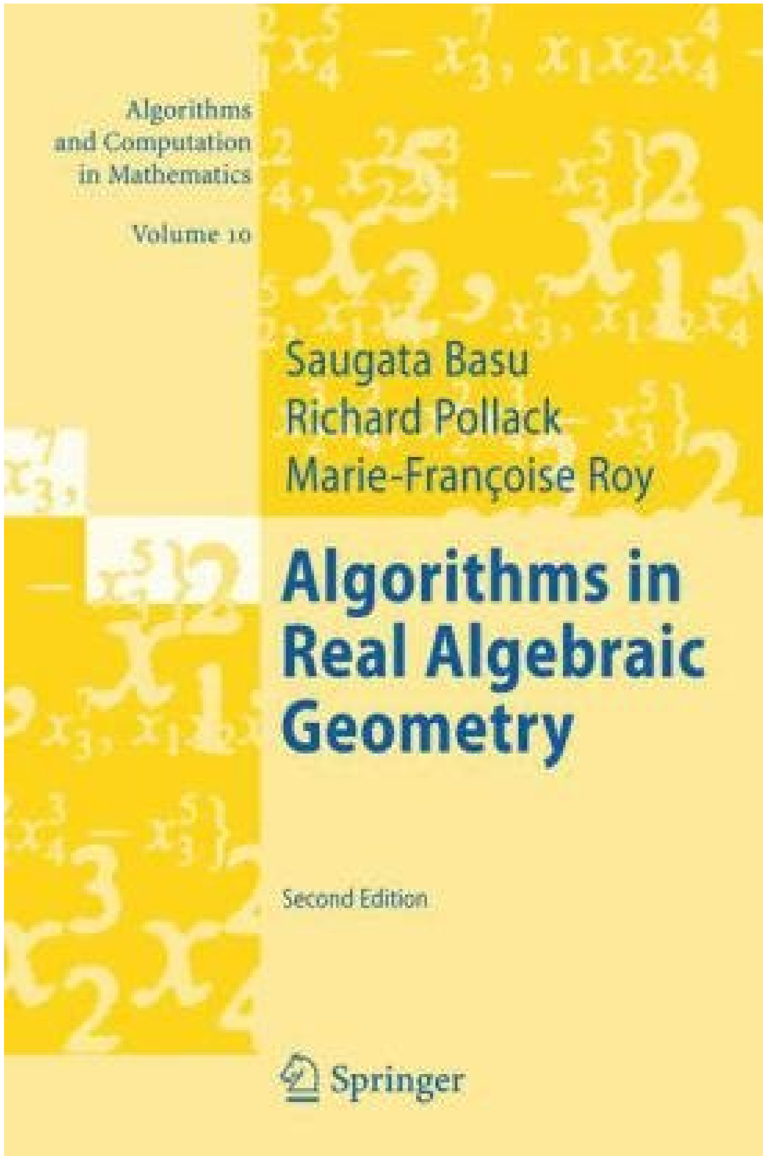


NOT in this talk

- ▷ Univariate problems / zero-dimensional problems.
- ▷ Cylindrical Algebraic Decomposition of real solution sets.
- ▷ Certificates of positivity. Peyrl/Parillo, Kaltofen/Yang/Zhi

We will focus on Chap. 13 $\rightarrow$ 16.

Real Algebraic Geometry and some Applications

Basic objects.

$$F_1 = \cdots = F_p = 0, \quad G_1 > 0, \ldots, G_s > 0$$

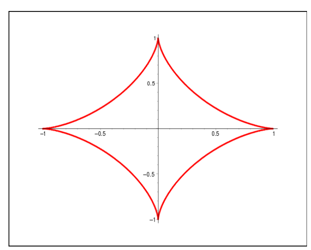
in $\mathbb{Q}[X_1, \ldots, X_n]$ of degree $\leq D$ $\rightarrow$ semi-algebraic set S in $\mathbb{R}^n$

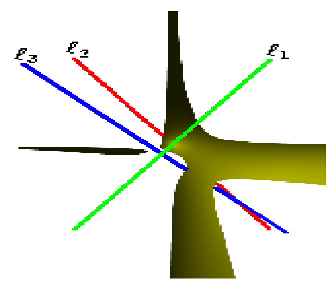
Existence of real solutions?

Projection of S on $X_1, \ldots, X_r$

Topological informations (connectivity, dimension, etc.)

Need of fast and reliable software – complexity estimates





State-of-the-art and what we want to do

Collins $\sim$ 70's Cylindrical algebraic decomposition – doubly exponential in n

Hong, McCallum, Arnon, Brown, Strzebonski, Anai, Sturm, Weispfenning

Software: QEPCAD, Redlog, SyNRAC, Mathematica, Maple, ...

- Quest for algorithms **singly exponential** in the number of variables

  Grigoriev/Vorobjov, Canny, Renegar, Heintz/Roy/Solerno, Basu/Pollack/Roy

Existence $D^{O(n)}$

Dimension $D^{O(n\,\mathrm{dim})}$

Connectivity $D^{O(n^2)}$

- **Primary goal: obtain fast and reliable software**
- Better understanding of the complexity $\rightarrow$ **constant in the exponent?**

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Existence	$D^{O(n)}$	$\sim O\left(\delta^3\right)$ (regular systems) else $\sim O\left(\delta^4\right)$ with $\delta = D^p(D-1)^{n-p}\binom{n-1}{p-1}$
Dimension	$D^{O(n\,\dim)}$	$\sim O\left(D^{4n\,\dim}\right) \rightsquigarrow$ hypersurfaces Bannwarth/S.
Connectivity	$D^{O(n^2)}$	

Software: RAGlib

- **Primary goal: obtain fast and reliable software**
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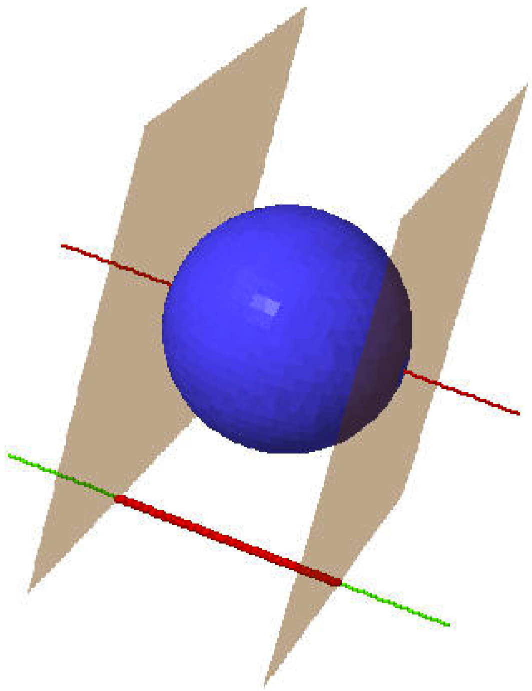


Reduction of the dimension through Global Optimization

Properties of Critical Points

Vorobjov, Renegar, Gournay/Risler, Heintz/Roy/Solerno, Basu/Pollack/Roy 96

▷ **Existence: from n-variate to univariate problems.**

$$q(T) = 0, \qquad X_i = q_i(T)/q_0(T), \qquad (1 \leq i \leq n)$$

▷ **One-block quantifier elimination.**

$$q(Y_1, \ldots, Y_r, T) = 0, \qquad X_i = q_i(Y_1, \ldots, Y_r, T)/q_0(Y_1, \ldots, Y_r, T)$$

▷ **Connectivity queries: reduction to the curve case.**

$$q(U, T) = 0, \qquad X_i = q_i(U, T)/q_0(U, T)$$

Polar varieties – definition

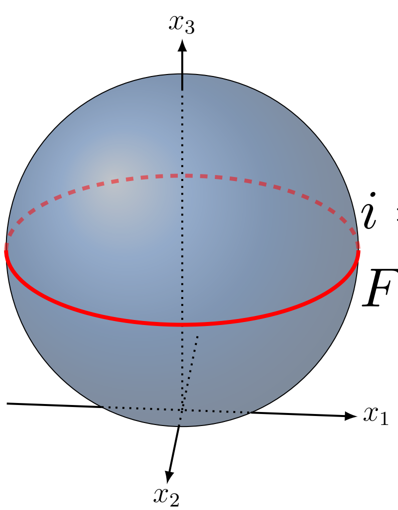
Todd/Severi ~ 30's – Piene/Teissier ~ 75's

For $1 \leq i \leq n$, let $\pi_i : (\mathbf{x}_1, \ldots, \mathbf{x}_n) \rightarrow (\mathbf{x}_1, \ldots, \mathbf{x}_i)$

Polar variety W_i associated to π_i and $V(F_1, \ldots, F_p)$

$$F_1 = \cdots = F_p = 0 \quad \text{and} \quad \text{rank}\left(\begin{bmatrix} \frac{\partial F_1}{\partial X_{i+1}} & \cdots & \cdots & \frac{\partial F_1}{\partial X_n} \\ \frac{\partial F_2}{\partial X_{i+1}} & \cdots & \cdots & \frac{\partial F_2}{\partial X_n} \\ \vdots & & & \vdots \\ \frac{\partial F_p}{\partial X_{i+1}} & \cdots & \cdots & \frac{\partial F_p}{\partial X_n} \end{bmatrix}\right) \leq p - 1$$

Example: $F_1 = X_1^2 + X_2^2 + X_3^2 - 1$



$i = 2,$

$F = \frac{\partial F}{\partial X_3} = 0$

Modelings

- ▷ **Minors of the truncated jacobian matrix** $\rightsquigarrow$ Determinantal modeling
- ▷ **Linearly independant vectors in the kernel** $\rightsquigarrow$ Lagrange system

$$\mathbf{F} = 0, \qquad \Lambda \cdot \text{jac}(\mathbf{F}, 1) = \mathbf{0}, \quad \mathbf{u} \cdot \Lambda = 1$$

Geometry of polar varieties

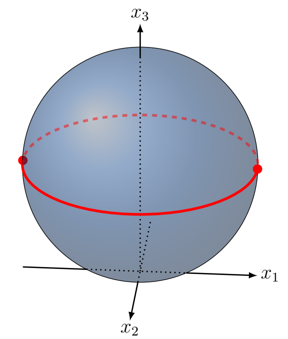
Let $V = \{\mathbf{x} \in \mathbb{C}^n \mid F_1(\mathbf{x}) = \cdots = F_p(\mathbf{x}) = 0\}$ $\rightarrow$ **regularity assumptions**

Transfer of properties of V to polar varieties in generic coordinates.

- **Dimension is well controlled**

  $\dim(W_1) = 0, \dim(W_2) = 1, \ldots, \dim(W_i) = i - 1$

  Bank/Giusti/Heintz/M'bakop/Pardo 97



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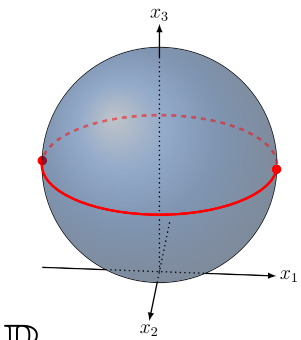
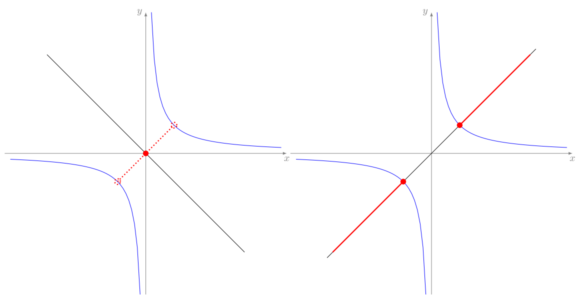


- **Closedness of projections**

  $W_1 \cap (V \cap \mathbb{R}^n) = \emptyset$ and $V \cap \mathbb{R}^n \neq \emptyset \Longrightarrow \pi_1(V \cap \mathbb{R}^n) = \mathbb{R}$



S./Schost 03

Transfer of **Noether position** properties to polar varieties.

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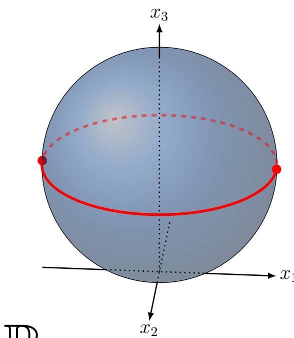

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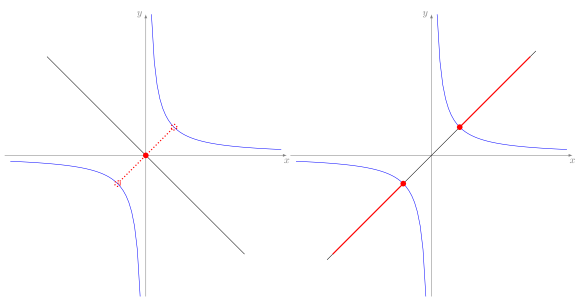

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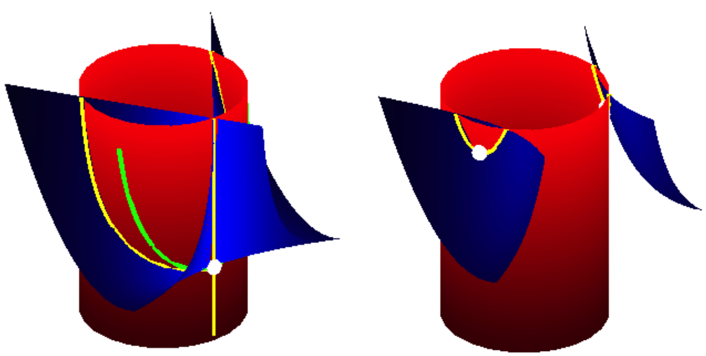
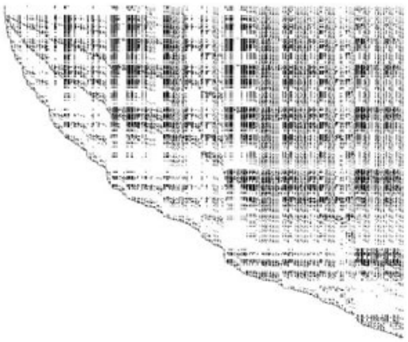


Removal of regularity assumptions Hong/S.

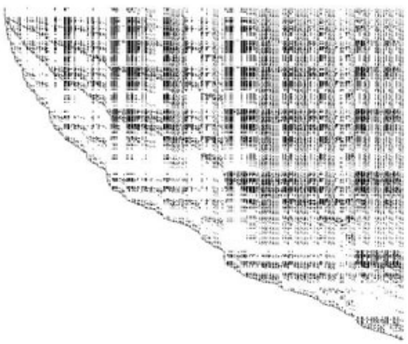
- Deformation techniques without using infinitesimal arithmetic
- Ideal theoretic operations



(Arithmetic) Complexity results

(determinantal modeling, $\deg(F_i) = D$ for $1 \leq i \leq p$)

- $\mathbb{D}_{\mathrm{reg}} \leq D(p-1) + (D-2)n + 2$
- When $D = 2$, $O(n^{2p\omega})$
- $O\left(\frac{1}{\sqrt{n}}((D-1)e)^{n\omega}\right)$ if $D > 2$ and p is fixed.



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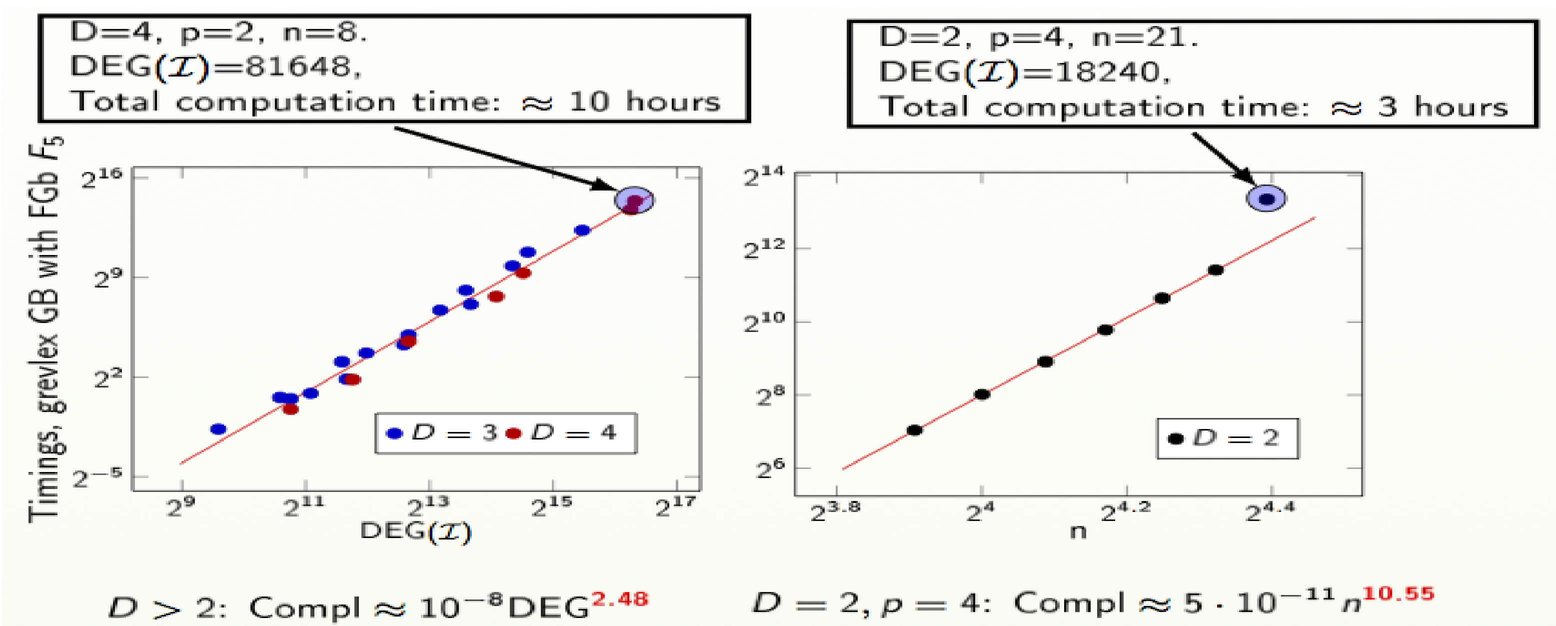

Generalizations to mixed degrees in Spaenlehauer, SIOPT 2014

Polar varieties and homotopy techniques

$$\boxed{\delta = D^p \ (D-1)^{n-p} \binom{n-1}{p-1}}$$ → generic degree of the 0-dim. polar variety

▷ **Symbolic homotopy (Geometric Resolution Algorithm)**

(Giusti/Lecerf/Salvy, Heintz, Montana, Solerno, Pardo, etc.)

Incremental algorithm → well-suited for complete intersections

Use of **Lagrange system** yields a complexity $\sim O(\delta^3)$ S./Schost

▷ **(Semi-)Numerical homotopy**

Sommese/Wampler Bates, Hauenstein, Leykin, Verschelde, etc.

Path-tracking from a "good" start system ⇝ **Lagrange system**

BUT **far from being optimal for** **mixed** **degrees**.

On-going work (Hauenstein/S.):

▷ Dedicated homotopy in the case of mixed systems.

▷ Possible generalization to determinantal systems.

Summary

Software RAGlib (Real Algebraic Geometry Library)

Scales to $\simeq$ 8–10 variables ($D = 4$, $n = 6$, dense equation $\rightarrow$ 2 hours)

Applications in biology, comput. geometry, numerical analysis, robotics, etc.

- Non-validity of models in bio-informatics
- Discovery fo the stability region of MacCormack's scheme for PDEs
- Computational geometry: Voronoi diagram, Perspective problems, etc.

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Used for an engineering application

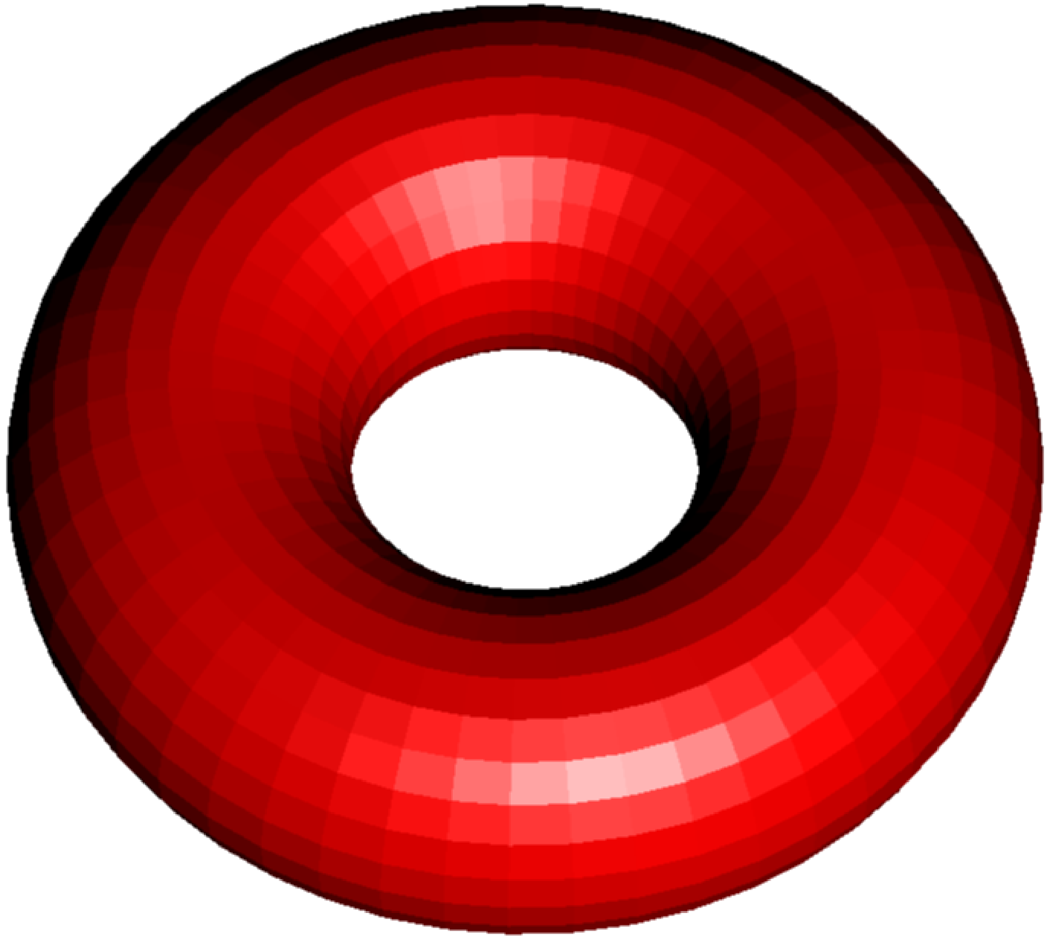
Systems of inequalities with $\simeq$ 6 $\rightarrow$ 8 variables

Unreachable by current CAD implementations

through Erich Kaltofen's consulting activity

New challenges: topological informations such as connectivity queries?

Example: torus



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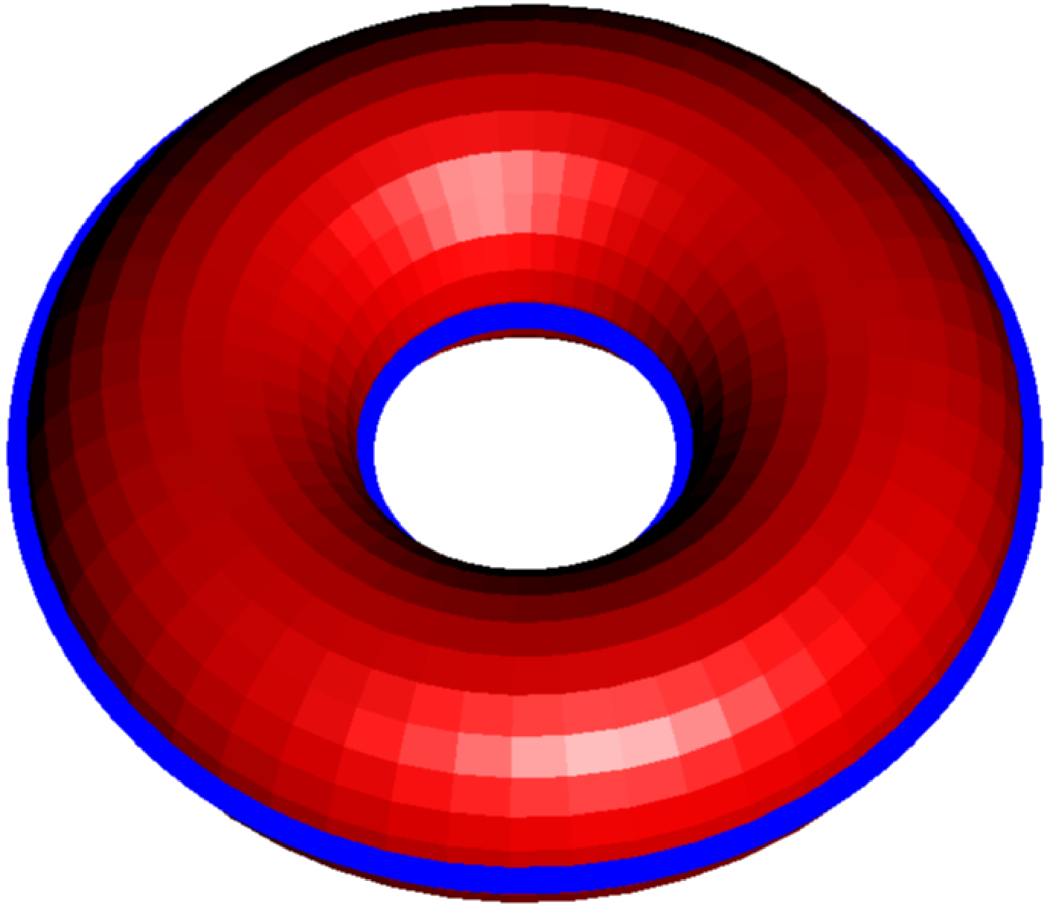

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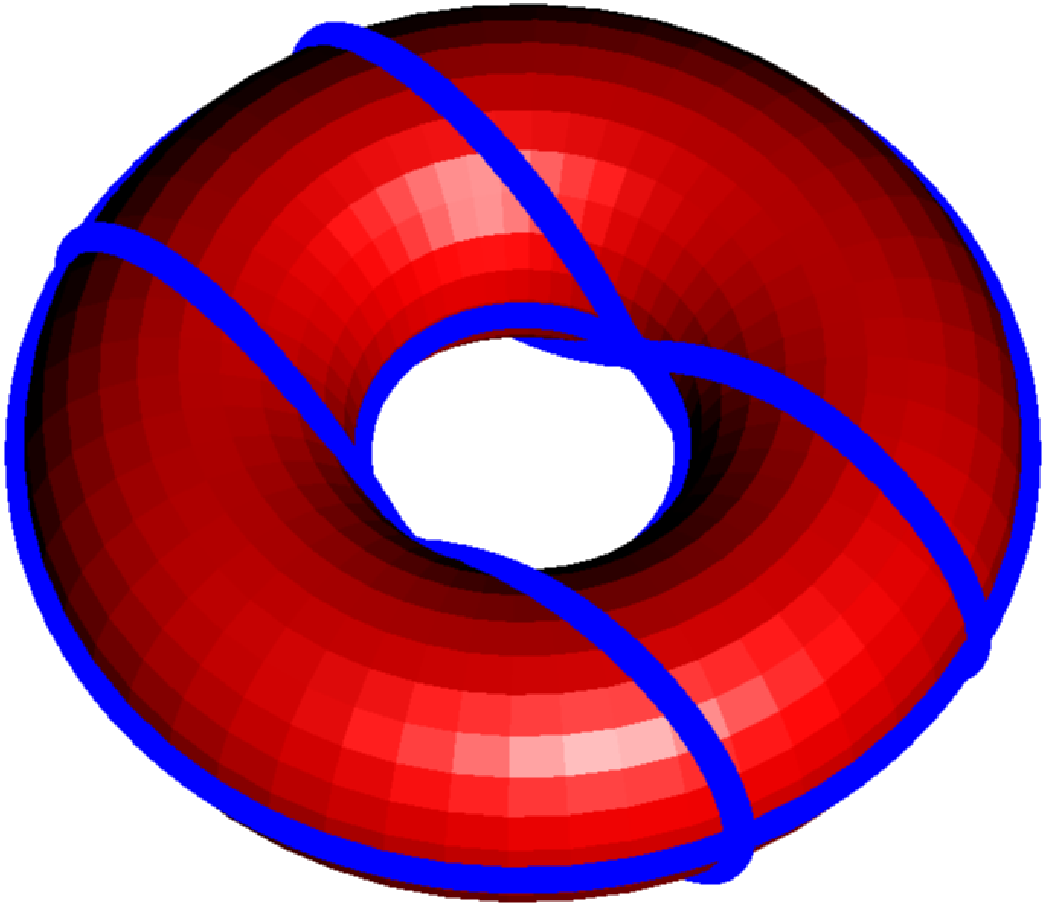

Roadmap

Complexity results for roadmaps

$D^{O(n^2)}$ (probabilistic) / $D^{O(n^4)}$ (deterministic) **Canny** ($\simeq$ 88)

Further improvements (Grigoriev/Vorobjov, Gourney/Risler, Heintz/Roy/Solerno)

$\rightsquigarrow D^{O(n^2)}$ deterministic **Basu/Pollack/Roy**

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A landmark result: Basu/Roy 2014 $(nD)^{\tilde{O}(n)}$ deterministic, no hyp.

BUT $(n^{\log(n)}D)^{O(n\log^2(n))}$ and **Output size**: $(n^{\log_2(n)}D)^{n\log_2(n)}$

$\boxed{n \leq \log_2^2(n) \quad \text{for } 4 \leq n \leq 16}$, $\boxed{\sqrt{n} \leq \log_2^2 n \quad \text{for } 4 \leq n \leq 2^{16} = 65536}$

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Input: smoothness and compactness hyp. $\mathbf{f} = (f_1, \ldots, f_p)$ red. reg. sequence

$$\boxed{O\left(E(4n\lambda D)^{6(\lambda+3)(n+3\lambda)}\right)}$$

with $\lambda = \log_2(n-p)$ and $E = \mathsf{EvalComplexity}(\mathbf{f})$

Output: $(nD)^{2(\lambda+3)(n+3\lambda)}$

A general form of Canny's algorithm

S./Schost 2011

Input / output

- input: a "nice" system $\mathbf{f} = f_1, \ldots, f_p$ defining $V = V(\mathbf{f})$ ($d = \dim(V)$)
- output: a roadmap of $V(\mathbf{f})$

Main idea: for a suitable $i \leq d$

- **recursive call on W_i**

  $\rightarrow$ we need W_i to satisfy the input assumptions.

- **recursive call on finitely many fibers of π_{i-1}** (about D^n)

  $\rightarrow$ we need to perform computations over "base points".

  Done by revisiting Lecerf's geometric resolution algorithm

- **merge the results**

Expectedly, running time about $D^{O(\rho n)}$, where ρ is the depth of the recursion.

We want to use this recursive scheme with $i \simeq (n-p)/2$

Smoothness of polar varieties

Bank/Giusti/Heintz/S./Schost

$$W_i = \{\mathbf{x} \in V \mid \dim(\pi_i(T_\mathbf{x} V) \leq i-1\}$$

can be decomposed into **Thom-Boardman strata**.

$$S_{i,i-1} = \{\mathbf{x} \in V \mid \dim(\pi_i(T_\mathbf{x} V)) = i-1\}$$
$$S_{i,i-2} = \{\mathbf{x} \in V \mid \dim(\pi_i(T_\mathbf{x} V)) = i-2\}$$
$$\vdots$$
$$S_{i,j} = \{\mathbf{x} \in V \mid \dim(\pi_i(T_\mathbf{x} V)) = j\}$$
$$\vdots$$
$$S_{i,0} = \{\mathbf{x} \in V \mid \dim(\pi_i(T_\mathbf{x} V)) = 0\}$$

Mather, Alzati/Ottaviani, Alzati/Ottaviani

In **generic** coordinates:

- ▷ T-B strata are locally closed smooth constructible sets.
- ▷ Dimension of T-B is controlled.
- ▷ When $i \leq \frac{\dim(V(\mathbf{f}))+3}{2}$, $S_{i,j}$ has dimension ≤ 0 $1 \leq j \leq p-2$.

Our divide-and-conquer algorithm

Recursive calls: require to manipulate polar varieties of polar varieties

$$
\begin{array}{llllll}
\mathbf{F}_1 & = & \mathbf{f}, & \mathbf{L}_1.\mathrm{jac}(\mathbf{f}, i_1), & \mathbf{L}_1.\mathbf{u}_1 = 1 & \to W_{i_1} \text{ with } i_2 \simeq d/2 \\
\mathbf{F}_2 & = & \mathbf{F}_1, & \mathbf{L}_2.\mathrm{jac}(\mathbf{F}_1, i_2), & \mathbf{L}_2.\mathbf{u}_2 = 1 & \to W_{i_2} \text{ with } i_2 \simeq i_1/2 \\
 & & & \vdots & & \\
\mathbf{F}_r & = & \mathbf{F}_{r-1}, & \mathbf{L}_r.\mathrm{jac}(\mathbf{F}_{r-1}, i_r), & \mathbf{L}_2.\mathbf{u}_2 = 1 & \to W_{i_r} \text{ with } i_r \simeq i_{r-1}/2
\end{array}
$$

Multi-homogeneous systems, with $\simeq n^2$ variables, and multi-degrees

$$
\begin{array}{ll}
(D\ ,\quad 0, \ldots, 0) & \text{for } f_1, \ldots, f_{p_1} \\
(D-1, 0, \ldots, 0) & \text{for } f_{p_1+1}, \ldots, f_{p_1+p_2} \\
\vdots & \vdots \\
(D-1, 1, \ldots, 1) & \text{for } f_{p_1+\cdots+p_{k-1}+1}, \ldots, f_{p_1+\cdots+p_k}.
\end{array}
$$

Good control on the degree bounds $\to \simeq (nD)^n$

First implementation

- ▷ Based on the latest Faugère's FGb library for computing Gröbner bases and rational parametrization.

- ▷ No assumption is checked ; routines for optimizating the choice of linear changes of variables are **not** implemented.

- ▷ Tests on **random dense** systems of quadrics and quartics (worst case).

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- ▷ Tests on **random dense** systems of quadrics and quartics (worst case).
- ▷ scales to problems of dimension 5 (case of quartics) and problems of dimension 9 (quadrics).
- ▷ new ideas behind this implementation and **careful** computation of "critical points of critical points".
- ▷ size of the output is clearly overestimated.

Conclusions and Perspectives

- ▷ Strong interaction between algorithm/software design and complexity
- ▷ Exact methods based on the critical point method can now solve efficiently non-trivial problems
- ▷ You should **NOT** conclude that CAD becomes useless
  - ▷ no other alternative to quantifier elimination
  - ▷ **extremely efficient** for low-dimensional problems
  - ▷ highly non-trivial (and efficient) algorithms for curve of surface topology based on CAD-like techniques
- ▷ **Diversity is good**.