A solution

$$f_1 := x_1^d, \quad f_2 := x_1 x_n^{d-1} - x_2^d, \qquad \dots$$

$$\dots, \quad f_{n-1} := x_{n-2} x_n^{d-1} - x_{n-1}^d, \quad f_n := t^h - x_{n-1} x_n^{d-1}$$

$$\begin{aligned} t^{d^{n-1}h} = x_n^{d^n-d} x_1^d \quad & -x_n^{d^n-d^2}\left((x_1 x_n^{d-1})^d - (x_2^d)^d\right) - \cdots \\ \cdots \quad & -x_n^{d^n-d^{n-1}}\left((x_{n-2} x_n^{d-1})^{d^{n-2}} - (x_{n-1}^d)^{d^{n-2}}\right) \\ + \quad & \left((t^h)^{d^{n-1}} - (x_{n-1} x_n^{d-1})^{d^{n-1}}\right) \end{aligned}$$

An arithmetic strong Nullstellensatz

- $f, f_1, \ldots, f_s \in k[\mathbf{t}][x_1, \ldots, x_n]$ $(\mathbb{Z}[x_1, \ldots, x_n])$
  s.t. f vanishes on the common zeros of $f_1, \ldots, f_s$
- $d_0 := \deg(f)$, $d_i := \deg(f_i)$ with $d_1 \geq \cdots \geq d_s$
- $h_0 := \mathrm{h}(f)$, $h :\geq \mathrm{h}(f_i)$, $1 \leq i \leq s$, $n_0 := \min\{s, n+1\}$

Then, there exists $\alpha \in k[\mathbf{t}] - \{0\}$ $(\mathbb{Z} - \{0\})$, $\mu \in \mathbb{N}$ and $g_1, \ldots, g_s \in k[\mathbf{t}][\mathbf{x}]$ $(\mathbb{Z}[\mathbf{x}])$, s.t.

$$\alpha\, f^{\mu} = g_1 f_1 + \cdots + g_s f_s$$

with

- $\mu \leq 2 \prod_{j=1}^{n_0} d_j$, $\deg_{\mathbf{x}}(g_i f_i) \leq 4 d_0 \prod_{j=1}^{n_0} d_j$
- $\mathrm{h}(\alpha), \mathrm{h}(g_i) + \mathrm{h}(f_i) \leq 3h_0 \prod_{j=1}^{n_0} d_j + 2hd_0 \sum_{\ell=1}^{n_0} \prod_{j \neq \ell} d_j \ (+c(n,s) d_0 \prod_{j=1}^{n_0} d_j)$