Identifiability of Viscoelastic Mechanical Systems

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34 slides

Structural Identifiability

- Finding which unknown parameters of a model can be determined from known data
- “Structural” means we assume “perfect” data
- Biological models
  - Systems Biology
  - Chemical Reaction Networks
- Viscoelastic Mechanical Models

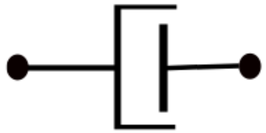
Elastic Material

- Spring 
- Obeys relationship (Hooke's Law):

$$\sigma = E\epsilon$$

- Stress: $\sigma = \sigma(t)$
- Strain: $\epsilon = \epsilon(t)$
- Spring constant: E
- If you know σ and ϵ, can determine E

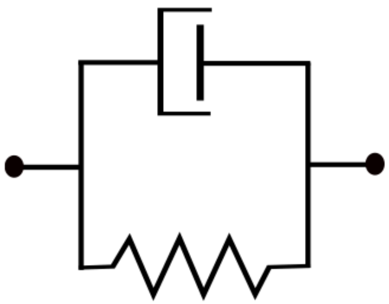
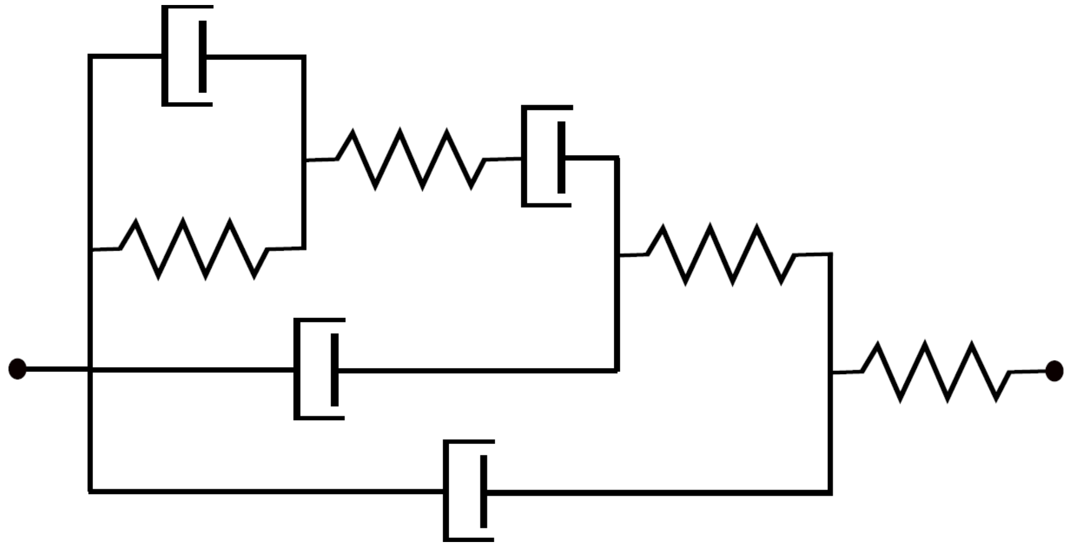
Viscous Material

- Dashpot (or piston) 
- Obeys relationship:

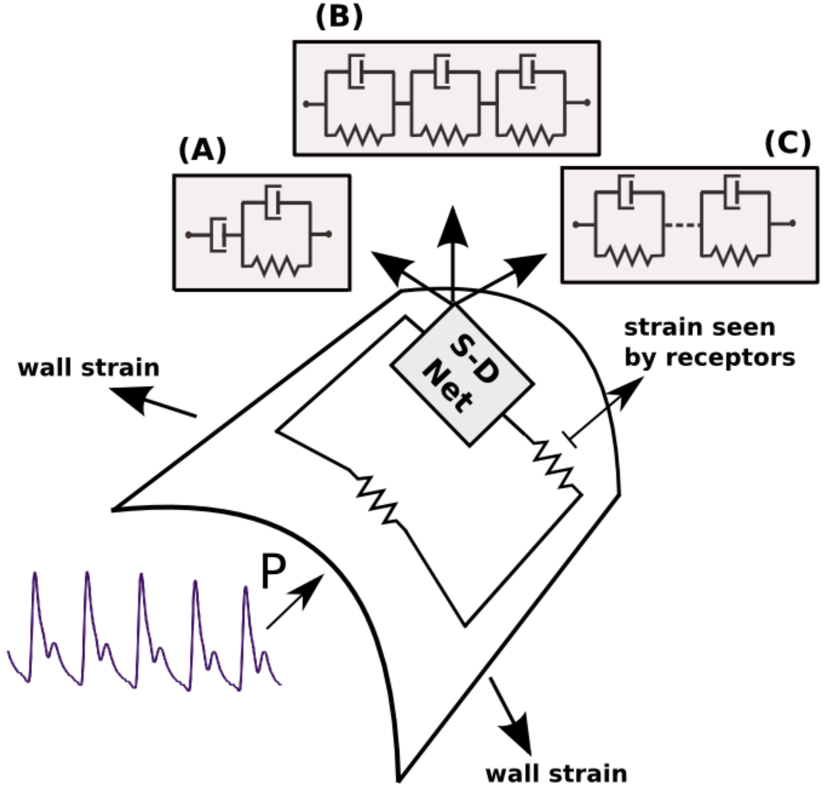
$$\sigma = \eta \epsilon$$

- Stress: $\sigma = \sigma(t)$
- Strain: $\epsilon = \epsilon(t)$
- Dashpot constant: η
- If you know σ and ϵ, can determine η

Viscoelastic material

- Combine springs and dashpots in series or parallel
- Series: 
- Parallel: 
- More complicated: 

Application to cardiovascular modeling

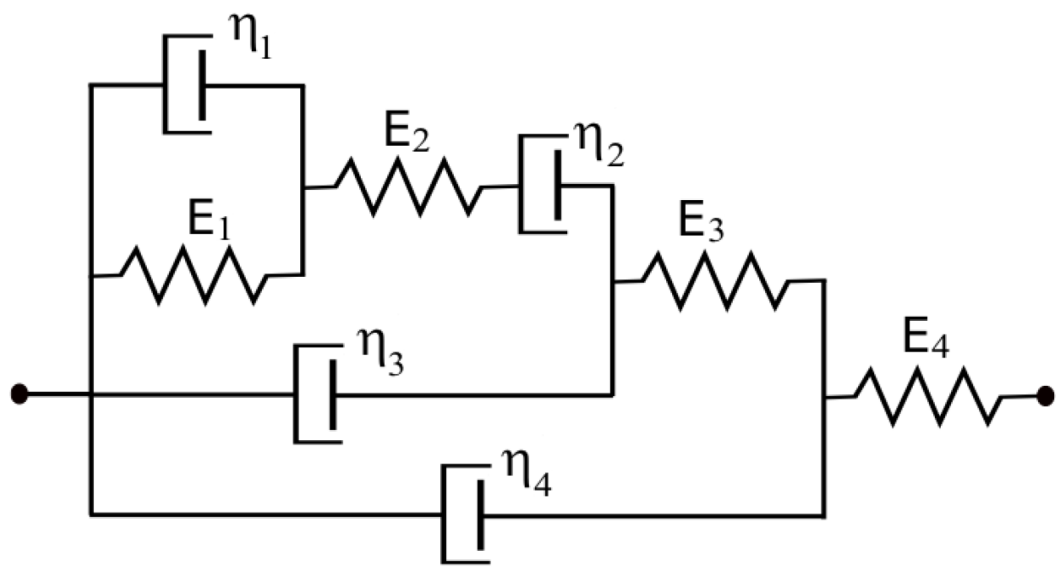


(Madhi 2013)

- Changing blood pressure causes periodic expansion and contraction of arterial walls
- Stress-strain curves of the arterial walls exhibit hysteresis, which means the wall is viscoelastic

Viscoelastic material

- Ex:



- Each individual spring/dashpot has corresponding stress/strain relationship
  – Ex:

$$\sigma_1 = E_1 \epsilon_1 \quad \sigma_3 = E_2 \epsilon_3 \quad \sigma_5 = E_3 \epsilon_5 \quad \sigma_7 = E_4 \epsilon_7$$

$$\sigma_2 = \eta_1 \epsilon_2 \quad \sigma_4 = \eta_2 \epsilon_4 \quad \sigma_6 = \eta_3 \epsilon_6 \quad \sigma_8 = \eta_4 \epsilon_8$$

Viscoelastic material

- How to determine relationship between *total* stress and *total* strain?
  - Two rules:
    - If combine in **series**,
      - stress is the same for both elements
      - total strain is the sum of individual strains on each element
    - If combine in **parallel**,
      - strain is the same for both elements
      - total stress is the sum of individual stresses on each element

Viscoelastic (VE) System

- Ex 1: Series connection: “Maxwell”

- Equations:

$\sigma_1 = E\epsilon_1$ $\qquad$ $\sigma = \sigma_1 = \sigma_2$

$\sigma_2 = \eta\epsilon_2$ $\qquad$ $\epsilon = \epsilon_1 + \epsilon_2$

- Eliminate individual stress/strain:

$$\sigma_1 / E = \epsilon_1$$

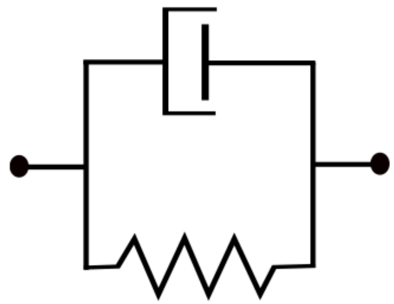
$$+\sigma_2 / \eta = \epsilon_2$$

$$\sigma / E + \sigma/\eta = \epsilon_1 + \epsilon_2 = \epsilon \Rightarrow$$

$$\boxed{\epsilon = \sigma / E + \sigma/\eta}$$

Viscoelastic (VE) System

- Ex 2: Parallel connection: “Voigt”



- Equations:

$\sigma_1 = E\epsilon_1$

$\sigma_2 = \eta\epsilon_2$

$\sigma = \sigma_1 + \sigma_2$

$\epsilon = \epsilon_1 = \epsilon_2$

- Eliminate individual stress/strain:

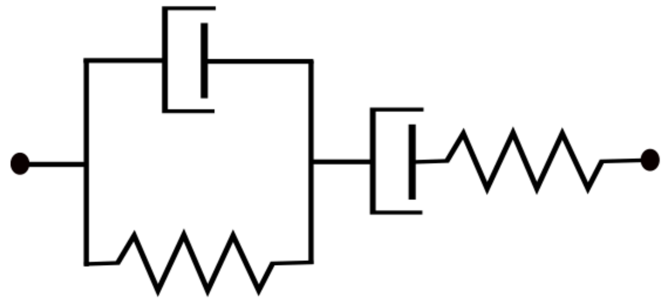
$$\sigma_1 = E\epsilon_1$$

$$+ \ \sigma_2 = \eta\epsilon_2$$

$$\sigma = \sigma_1 + \sigma_2 = E\epsilon + \eta\epsilon \quad \Longrightarrow \quad \boxed{\eta\epsilon + E\epsilon = \sigma}$$

Viscoelastic (VE) System

- Ex 3: Voigt/Maxwell in Series: "Burgers"



- Two equations:

$$\eta_v \epsilon_1 + E_v \epsilon_1 = \sigma_1$$

$$\sigma = \sigma_1 = \sigma_2$$

$$\epsilon_2 = \sigma_2 / E_m + \sigma_2 / \eta_m$$

$$\epsilon = \epsilon_1 + \epsilon_2$$

- Total stress/strain relationship:

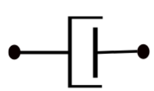
$$E_m \epsilon + E_m E_v / \eta_v \, \epsilon = \sigma + (E_m / \eta_m + E_m / \eta_v + E_v / \eta_v) \sigma + E_m E_v / \eta_m \eta_v \, \sigma$$

Generalize

- Theorem (Mahdi-M-Sullivant): For any configuration of N springs and M dashpots, get total stress-strain relationship of one of 4 types:
  - Type A: $\epsilon^{(n)} + a_{n-1}\epsilon^{(n-1)} + \ldots + a_0 \epsilon = b_n \sigma^{(n)} + \ldots + b_0 \sigma$

    Ex: Spring  $E\epsilon = \sigma$

  - Type B: $\epsilon^{(n+1)} + a_n \epsilon^{(n)} + \ldots + a_1 \epsilon' = b_n \sigma^{(n)} + \ldots + b_0 \sigma$

    Ex: Dashpot  $\eta\epsilon = \sigma$

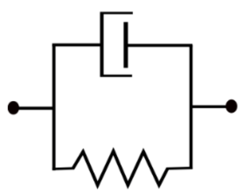
  where: $a_i = a_i(E_1, \ldots, E_N, \eta_1, \ldots, \eta_M)$ $b_j = b_j(E_1, \ldots, E_N, \eta_1, \ldots, \eta_M)$

Generalize

- Theorem (Mahdi-M-Sullivant): For any configuration of N springs and M dashpots, get total stress-strain relationship of one of 4 types:
  - Type C:

$$\epsilon^{(n+1)} + a_n \epsilon^{(n)} + \dots + a_0 \epsilon = b_n \sigma^{(n)} + \dots + b_0 \sigma$$

Ex: Voigt



$$\eta\epsilon + E\epsilon = \sigma$$

  - Type D:

$$\epsilon^{(n)} + a_{n-1} \epsilon^{(n-1)} + \dots + a_1 \epsilon' = b_n \sigma^{(n)} + \dots + b_0 \sigma$$

Ex: Maxwell

$$\epsilon = \sigma/E + \sigma/\eta$$

where: $a_i = a_i(E_1, \dots, E_N, \eta_1, \dots, \eta_M)$ $b_j = b_j(E_1, \dots, E_N, \eta_1, \dots, \eta_M)$

Identifiability problem

- Can recover coefficient values of total stress-strain relationship from known data (Soderstrom 1988)
- Is it possible to recover the unknown parameters of the model from these coefficient values?
- Ex: $E_m \epsilon + E_m E_v / \eta_v \, \epsilon = \sigma + (E_m / \eta_m + E_m / \eta_v + E_v / \eta_v)\sigma + E_m E_v / \eta_m \eta_v \, \sigma$

- Let $c: \mathbb{R}^4 \to \mathbb{R}^4$

$$(E_m, E_v, \eta_m, \eta_v) \longmapsto (E_m, E_m E_v / \eta_v, (E_m / \eta_m + E_m / \eta_v + E_v / \eta_v), E_m E_v / \eta_m \eta_v)$$

Identifiability

- Total stress-strain relationship:

$$\epsilon^{(n+1)} + a_n \epsilon^{(n)} + \ldots + a_0 \epsilon = b_n \sigma^{(n)} + \ldots + b_0 \sigma$$

$$a_i = a_i(E_1, \ldots, E_N, \eta_1, \ldots, \eta_M)$$

$$b_j = b_j(E_1, \ldots, E_N, \eta_1, \ldots, \eta_M)$$

- Let $c: \mathbb{R}^{N+M} \rightarrow \mathbb{R}^{\#coeffs}$

$$(E_1, \ldots, E_N, \eta_1, \ldots, \eta_M) \mapsto (a_n, \ldots, a_0, b_n, \ldots, b_0)$$

Identifiability

- Let $c: \mathbb{R}^{N+M} \rightarrow \mathbb{R}^{\#coeffs}$
- The model is:
  - Globally identifiable iff c is 1-1
  - Locally identifiable iff c is finite-to-one
  - Unidentifiable iff c is infinite-to-one

  (Ljung 1994)
- Consider *generic* identifiability

Main result

Theorem (Mahdi-M-Sullivant): A viscoelastic model is locally ID iff

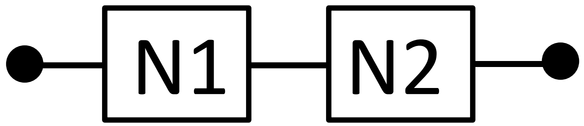
coefficients = # parameters

- Ex 1: $\epsilon = \sigma / E + \sigma / \eta$
- Ex 2: $\eta \epsilon + E \epsilon = \sigma$
- Ex 3: $E_m \epsilon + E_m E_v / \eta_v \, \epsilon = \sigma + (E_m / \eta_m + E_m / \eta_v + E_v / \eta_v) \sigma + E_m E_v / \eta_m \eta_v \, \sigma$

Local Identifiability

- Consider a spring-dashpot system M whose final step connection is a series connection of systems N1 and N2

M =



- Necessary condition for M to be locally id is that N1 and N2 are both locally id
- When is the series connection of two (locally) identifiable models identifiable?

Main result

- Theorem (Mahdi-M-Sullivant): Consider two locally id systems N1 and N2. The model resulting in joining N1 and N2 in **series** is either id of Type A, B, C, D, or un-id, denoted by u:

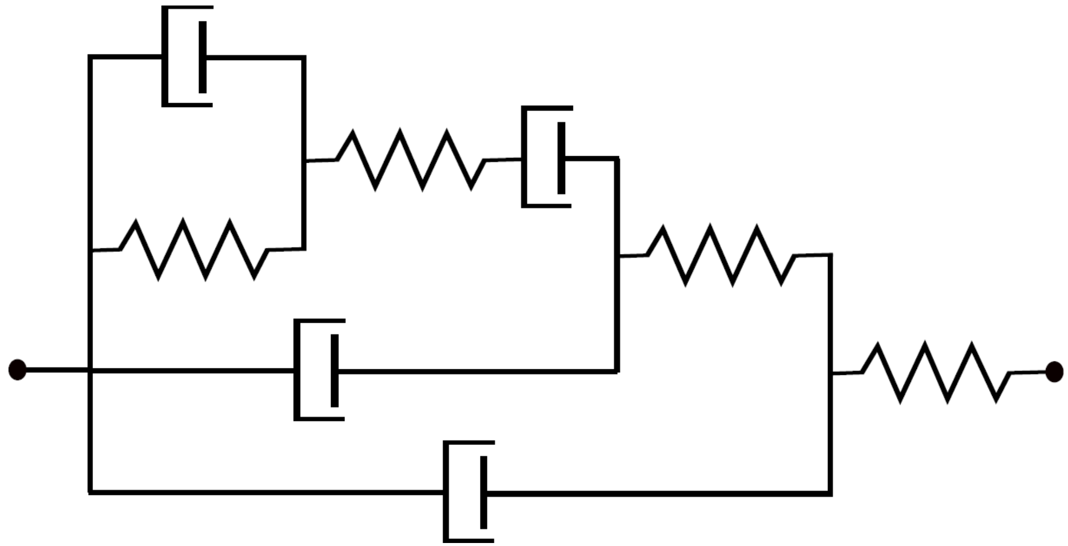
⊙	A	B	C	D	u
A	u	D	A	u	u
B	D	u	B	u	u
C	A	B	C	D	u
D	u	u	D	u	u
u	u	u	u	u	u

Main result

- Theorem (Mahdi-M-Sullivant): Consider two locally id systems N1 and N2. The model resulting in joining N1 and N2 in **parallel** is either id of Type A, B, C, D, or un-id, denoted by u:

⊕	A	B	C	D	u
A	u	C	u	A	u
B	C	u	u	B	u
C	u	u	u	C	u
D	A	B	C	D	u
u	u	u	u	u	u

Real Example (Dietrich 1998)



$$[(((((A \oplus B) \odot A) \odot B) \oplus B) \odot A) \oplus B] \odot A \quad = [((((C \odot A) \odot B) \oplus B) \odot A) \oplus B] \odot A$$

$$= [(((A \odot B) \oplus B) \odot A) \oplus B] \odot A$$

$$= [((D \oplus B) \odot A) \oplus B] \odot A$$

$$= [(B \odot A) \oplus B] \odot A$$

$$= [D \oplus B] \odot A \quad = B \odot A \quad = D$$

Generalize process of combining in Series/Parallel

- Let N1 and N2 be spring-dashpot networks whose stress-strain relationships are:

$$L_1 \epsilon_1 = L_2 \sigma_1$$

$$L_3 \epsilon_2 = L_4 \sigma_2$$

where L_i is a linear differential operator

$$L_1 = a_{n_1} \frac{d^{n_1}}{dt^{n_1}} + \ldots + a_{m_1} \frac{d^{m_1}}{dt^{m_1}}$$

$$L_2 = b_{n_2} \frac{d^{n_2}}{dt^{n_2}} + \ldots + b_{m_2} \frac{d^{m_2}}{dt^{m_2}}$$

$$L_3 = c_{n_3} \frac{d^{n_3}}{dt^{n_3}} + \ldots + c_{m_3} \frac{d^{m_3}}{dt^{m_3}}$$

$$L_4 = e_{n_4} \frac{d^{n_4}}{dt^{n_4}} + \ldots + e_{m_4} \frac{d^{m_4}}{dt^{m_4}}$$

Series connection

- $\epsilon = \epsilon_1 + \epsilon_2$
- $\sigma = \sigma_1 = \sigma_2$

$$L_1 \epsilon_1 = L_2 \sigma_1$$

$$L_3 \epsilon_2 = L_4 \sigma_2$$

$$L_3 L_1 \epsilon_1 = L_3 L_2 \sigma_1$$

$$+ \; L_1 L_3 \epsilon_2 = L_1 L_4 \sigma_2$$

$$L_1 L_3 \epsilon = (L_1 L_4 + L_2 L_3) \sigma$$

Parallel connection

- $\epsilon = \epsilon_1 = \epsilon_2$
- $\sigma = \sigma_1 + \sigma_2$

$$L_1 \epsilon_1 = L_2 \sigma_1$$

$$L_3 \epsilon_2 = L_4 \sigma_2$$

$$L_4 L_1 \epsilon_1 = L_4 L_2 \sigma_1$$

$$+ \; L_2 L_3 \epsilon_2 = L_2 L_4 \sigma_2$$

$$(L_1 L_4 + L_2 L_3)\epsilon = L_2 L_4 \sigma$$

Types of viscoelastic models

- The *shape* of a linear operator L_i is a pair of #'s:

$$[n_i, m_i]$$

where n_i = highest differential order

m_i = lowest differential order

Type	Shape in ϵ	Shape in σ	Example	Ex Eqn
A	[n, 0]	[n, 0]	Spring	$E\epsilon=\sigma$
B	[n+1, 1]	[n, 0]	Dashpot	$\eta\dot{\epsilon}=\sigma$
C	[n+1, 0]	[n, 0]	Voigt	$\eta\dot{\epsilon}+E\epsilon=\sigma$
D	[n, 1]	[n, 0]	Maxwell	$\dot{\epsilon}=\dot{\sigma}/E+\sigma/\eta$

Identifiability as "shape factorization problem"

- Let N1 and N2 be spring-dashpot networks whose stress-strain relationships are:

$$L_1 \epsilon_1 = L_2 \sigma_1$$

$$L_3 \epsilon_2 = L_4 \sigma_2$$

- Consider series connection:

$$(L_1 \epsilon_1 = L_2 \sigma_1 , L_3 \epsilon_2 = L_4 \sigma_2) \longmapsto f \epsilon = g \sigma$$

where $f = L_1 L_3$, $g = L_1 L_4 + L_2 L_3$

- Can phrase identifiability problem in terms of shapes $[n_1 , m_1]$, $[n_2 , m_2]$, $[n_3 , m_3]$, $[n_4 , m_4]$

Shape factorization problem for a quadruple of shapes

For a generic pair of polynomials (f, g) with:

- f monic
- $shape(f) = [n_1 + n_3, m_1 + m_3]$,
- $shape(g) = [\max(n_1 + n_4, n_2 + n_3), \min(m_1 + m_4, m_2 + m_3)]$,

does there exist finitely many quadruples of polynomials (L_1, L_2, L_3, L_4) with

- $shape(L_i) = [n_i, m_i]$,
- L_1, L_3 monic,

Example

- Suppose that quadruple is ($[2,0]$, $[2,0]$, $[3,0]$, $[2,0]$), which is a special case of joining A, C in series
- Let (f,g) be a generic pair of polynomials where f and g are degree 5 polynomials with nonzero constant term and f is monic:

$$f = x^5 + f_4 x^4 + \ldots + f_0 \qquad g = g_5 x^5 + g_4 x^4 + g_3 x^3 + \ldots + g_0$$

- Does there exist finitely many polynomials

$$L_1 = x^2 + a_1 x + a_0 \qquad L_2 = b_2 x^2 + b_1 x + b_0$$

$$L_3 = x^3 + c_2 x^2 + c_1 x + c_0 \qquad L_4 = e_2 x^2 + e_1 x + e_0$$

such that $f = L_1 L_3$ and $g = L_1 L_4 + L_2 L_3$?

Example cont'd

- Alternatively, for generic values of $f_4, \ldots, f_0$ and $g_5, \ldots, g_0$, does the system of 11 equations in 11 unknowns:

$$f_4 = a_1 + c_2$$

$$f_3 = a_0 + a_1 c_2 + c_1$$

$$f_2 = a_0 c_2 + a_1 c_1 + c_0$$

$$f_1 = a_0 c_1 + a_1 c_0$$

$$f_0 = a_0 c_0$$

$$g_5 = b_2$$

$$g_4 = b_1 + b_2 c_2 + e_2$$

$$g_3 = b_0 + b_1 c_2 + b_2 c_1 + a_1 e_2 + e_1$$

$$g_2 = b_0 c_2 + b_1 c_1 + b_2 c_0 + a_0 e_2 + a_1 e_1 + e_0$$

$$g_1 = b_0 c_1 + b_1 c_0 + a_0 e_1 + a_1 e_0$$

$$g_0 = b_0 c_0 + a_0 e_0$$

have only finitely many solutions?

Solving shape factorization problem

- For given shapes $[n_1, m_1]$, $[n_3, m_3]$, there are at most finitely many ways to factor $f = L_1 L_3$ into monic factors
- Once we fix L_1 and L_3, then

$$g = L_1 L_4 + L_2 L_3$$

is a *linear* system in the unknowns L_2 and L_4

- One solution iff # coefficients = # parameters

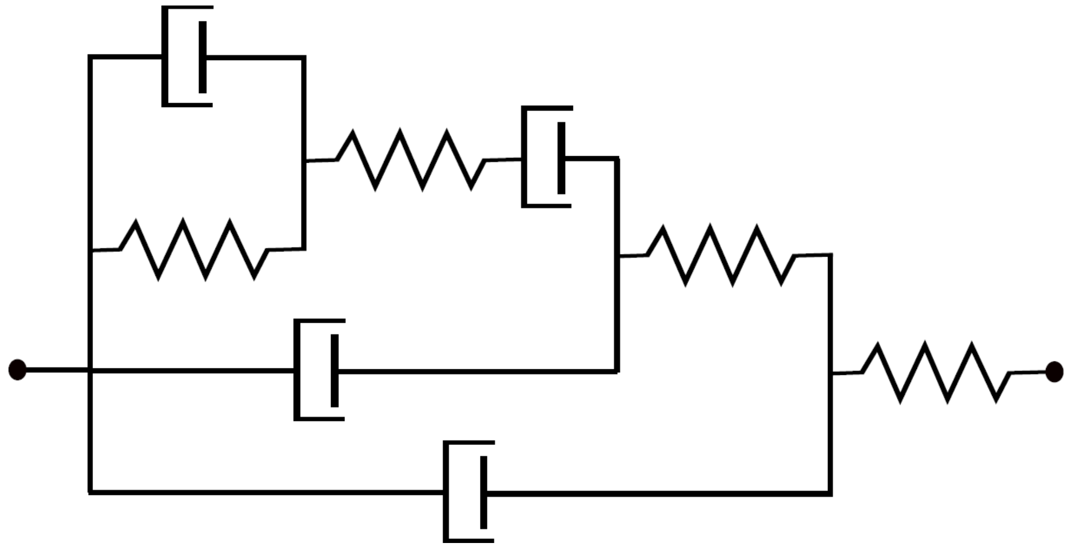
Global identifiability

- When do we have unique soln for shape factorization problem?

  $f = L_1 L_3$ has a unique factorization iff $n_1 = m_1$ or $n_3 = m_3$

- A VE model is globally id iff
  - locally id and
  - network is constructed by adding a spring or dashpot in series or in parallel, **one at a time**

Real Example (Dietrich 1998)



$$[(((((A\oplus B)\odot A)\odot B)\oplus B)\odot A)\oplus B]\odot A \quad =[((((C\odot A)\odot B)\oplus B)\odot A)\oplus B]\odot A$$

$$=[(((A\odot B)\oplus B)\odot A)\oplus B]\odot A$$

$$=[((D\oplus B)\odot A)\oplus B]\odot A$$

$$=[(B\odot A)\oplus B]\odot A$$

$$=[D\oplus B]\odot A \quad =B\odot A \quad =D$$

Globally identifiable!

Summary

- Necessary and sufficient condition for local id:

  # parameters = # coefficients

- Nec. and suff. conditions for global id:

  local id & single spring/dashpot one at a time

- Identifiability tables
- Future work: RLC circuits

References

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