Certified Symmetry and Dominance Breaking for Combinatorial Optimisation

Bart Bogaerts, **Stephan Gocht**, Ciaran McCreesh, Jakob Nordström

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Symmetry Breaking — Example

- airline scheduling: have 4 aircraft
  goal: assign which aircraft fly
- only need 2 (due to corona)
- aircraft are interchangeable
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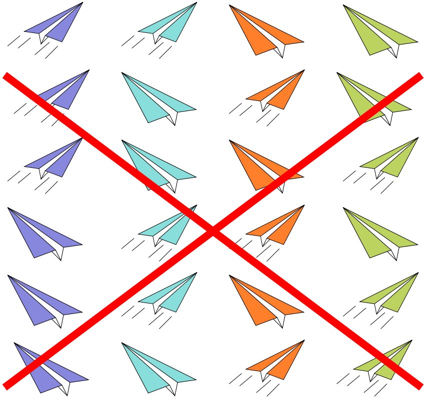
- aircraft are interchangeable
  (encoding will be symmetric)
- idea: make scheduling easier
  by removing symmetric assignments



Symmetry Breaking

manual and automatic techniques used across combinatorial optimisation paradigms

- constraint programming (CP) [GSVW14]
- Boolean satisfiability (SAT) [BHvMW21]
- mixed-integer programming (MIP) [AW13]

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How can we know we didn't remove too many assignments?

Software Verification — How to Ensure Software Behaves as Intended?

- Software testing
  - run collection of test cases to check if software behaves as intended
    - depends on quality of test cases, likely to miss non-trivial defects
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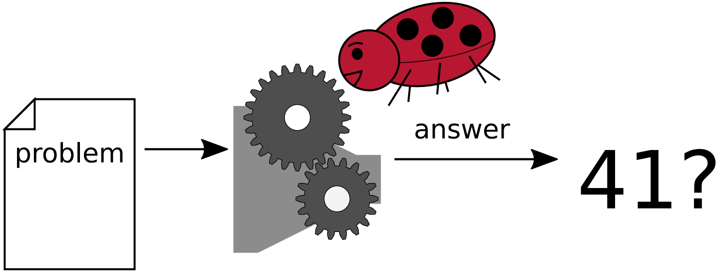
    - out of reach for complex, performance-critical software
- Certifying algorithms, also known as proof logging (this talk)
  - let algorithm output answer and *proof* that answer is correct
  - proof: sequence of simple, efficiently machine-verifiable steps

Detecting Bugs with Certifying Algorithms



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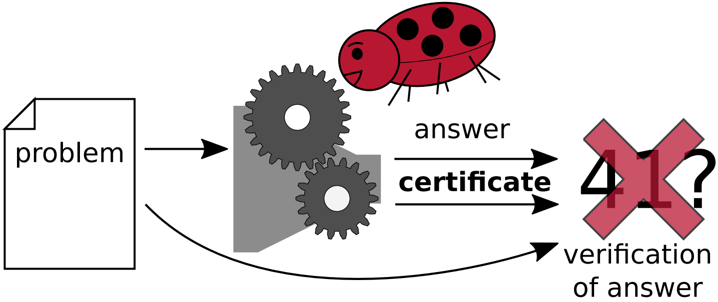
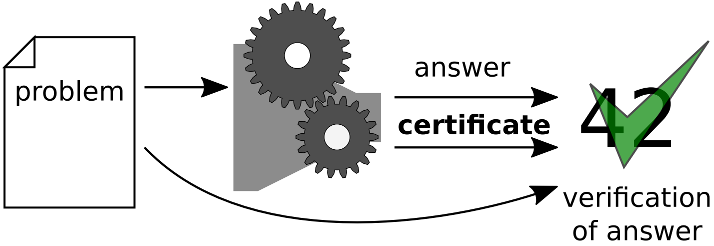


- verification of answer with external tool can detect bugs

Guaranteeing Correctness with Certifying Algorithms



- successful verification of answer with external tool guarantees correct answer

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  - increase trust in solution
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  - analyse certificate to understand and improve solving process
  - could use certificate to audit solution afterwards
- during development
  - simplifies testing: not necessary to know correct answer a priory
  - find bugs even if result is correct
  - locate first unsound step

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- certificate verification
  - should be efficiently machine-verifiable
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But how?

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- Can we use SAT technology?

CNF formula → Symmetry Breaking (BreakID) → SAT solver (Kissat) → result: SAT / UNSAT

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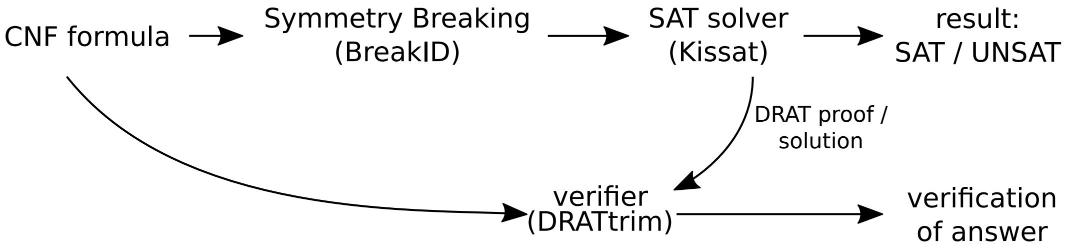

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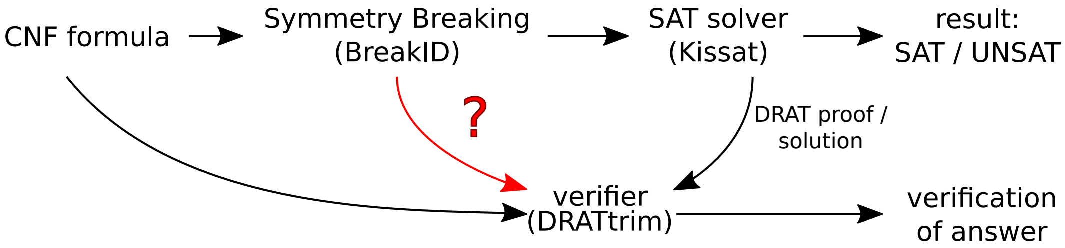

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- without symmetry breaking $\Rightarrow$ exponential loss in reasoning power / performance
- DRAT cannot support symmetry breaking $\Rightarrow$ need to investigate other methods
- not the only reason to look for other methods, what about
  - MaxSAT solving
  - constraint programming (CP)
  - mixed integer programming (MIP)
  - algebraic reasoning / Gröbner basis computations
  - pseudo-Boolean satisfiability and optimization

New Proof Systems are Being Developed

many new proof systems

- delete symmetry reverse unit propagation (DSRUP) [TD20]
- propagation redundancy (PR) [HKB17]
- branch and bound in integer programming [CGS17, EG21]
- practical polynomial calculus (PAC) [RBK18, KFB20, KFBK22]
- extensible RAT (FRAT) [BCH21]
- propagation redundancy for BDDs [BB21]
- **pseudo-Boolean proofs** [EGMN20, GN21, BGMN22]

High Level Idea of Pseudo-Boolean Proofs

- use pseudo-Boolean constraints (0-1 linear inequalities) to describe problem
  - e.g., $x_1 + x_2 + x_3 \geq 1$ or $2z + x_1 + x_2 + x_3 \geq 2$
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- proof constructs sequence of constraints $D_1, D_2, D_3, \ldots, D_L$
  - each constraint is derived by rule in proof system
  - annotation can contain additional information necessary for efficient verification
  - proves there is no solution if D_L is $0 \geq 1$
  - proves optimality if D_L is bound on objective matching known solution

Our Approach

- use pseudo-Boolean proofs (PBP)
- reference implementation of verifier: VeriPB[1]
- **multi-purpose** format: proof logging for wide range of problems / algorithms
  - reasoning with 0-1 linear inequalities (by design)

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This Work

- proof logging for symmetry and dominance breaking
- applied to SAT, constraint programming and max clique solving
- support for optimization

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Running Example

$$\begin{aligned} \text{min: } & 4x_1 + 2x_2 + x_3 \\ \text{s.t. } & x_1 + x_2 + x_3 \geq 2 \end{aligned}$$

- boolean variable x is 0 (false) or 1 (true) (e.g. $x_1 = 1$ means green aircraft flies)
- pseudo-Boolean constraint: linear inequality over variables
- formula F: set of constraints
- objective function f to be minimized

Goal: find assignment minimizing objective and satisfying all constraints

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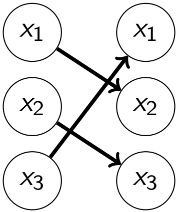

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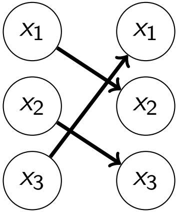

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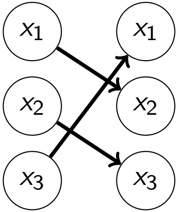

  - constraint is same as before
    $\Rightarrow$ formula is symmetric (ignoring objective)

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$$4x_1 + 2x_2 + x_3 \leq 4x_2 + 2x_3 + x_1$$

- can be simplified to $3x_1 - 2x_2 - x_3 \leq 0$

Output from Symmetry Breaking

- give formula $x_1 + x_2 + x_3 \geq 2$ to symmetry breaker (falsified by red assignments)

Truth Table

objective value	x_1	x_2	x_3
0	0	0	0
1	0	0	1
2	0	1	0
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How prove adding constraint is OK?

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The Dominance Rule

Idea (From Dominance Breaking)

assignment is dominated if we find strictly better assignment $\Rightarrow$ can remove dominated assignments

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- if for every ρ falsifying C but satisfying F (purple)
- we find ρ' that satisfies F and $f(\rho) > f(\rho')$

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Dominance Rule (simplified)

Can derive constraint C from formula F if a *witnessing* substitution ω is provided such that

$$F \cup \{\neg C\} \models F{\restriction_\omega} \cup \{f > f{\restriction_\omega}\}$$

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The Dominance Rule — Example

- want to add $C: 3x_1 - 2x_2 - x_3 \leq 0$
- choose $\omega = \pi = \{ x_1 \mapsto x_2, x_2 \mapsto x_3, x_3 \mapsto x_1 \}$
- **have to prove:**
  $F \cup \{ \neg C \} \models F\restriction_\omega \cup \{ f > f\restriction_\omega \}$, i.e.,

Dominance Rule (simplified)

Add C if there is *witnessing* substitution ω s.t.

$$F \cup \{ \neg C \} \models F\restriction_\omega \cup \{ f > f\restriction_\omega \}$$

objective value	x_1	x_2	x_3
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1	0	0	1
2	0	1	0
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3	0	1	1
5	1	0	1
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7	1	1	1

ω

ω

The Dominance Rule — Example

- want to add $C: 3x_1 - 2x_2 - x_3 \leq 0$
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- given F: $x_1 + x_2 + x_3 \geq 2$ and $\neg C$: $3x_1 - 2x_2 - x_3 > 0$

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- derive $$F_{\restriction\omega} \equiv (x_1 + x_2 + x_3 \geq 2)_{\restriction\omega} \equiv x_2 + x_3 + x_1 \geq 2$$

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The Dominance Rule — Further Remarks

Dominance Rule (simplified)

Can derive constraint C from formula F if a *witnessing* substitution ω is provided such that

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- in paper: any strict order instead of $f > f{\restriction_\omega}$

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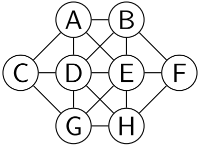
Supported Applications – Symmetry Breaking

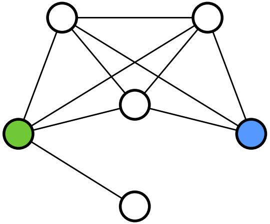
- SAT
  challenge: translate breaking constraint from PB to CNF
- Constraint Programming
  challenge: integer domains instead of 0-1



example: The Crystal Maze puzzle. Place numbers 1 to 8 without repetition, so that adjacent circles do not have consecutive numbers. Puzzle can be mirrored horizontally. Without loss of generality number in A smaller than number in G.

Supported Applications – Dominance Breaking

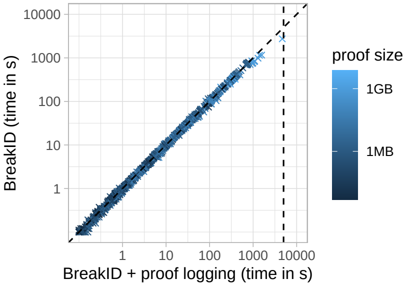
- maximum clique solving (find largest fully connected component)
  challenge: lazy breaking

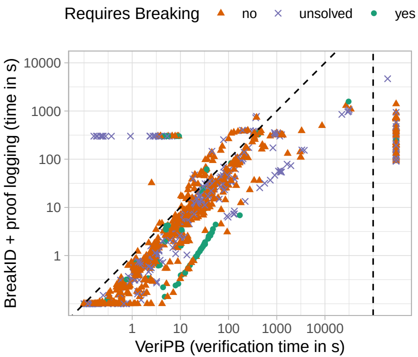


example: consider green but not blue node (every neighbour of blue is also neighbour of green)

Experiments

- evaluated on SAT competition benchmarks
- used BreakID[2] to find and break symmetries





- proof logging overhead negligible
- verification at most 20 times slower than solving for 95 % of instance

[2] https://bitbucket.org/krr/breakid/

Future Work

understand power of dominance rule:

- can we simulate dominance rule with redundance rule or extended cutting planes?

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proof logging for more algorithms and problems:

- symmetric explanation learning [DBB17]
- MaxSAT
- more propagators in constraint programming
- integer programming

Conclusion

- proof logging is well-established standard for SAT solving
- not sufficient for all techniques used in SAT (e.g. symmetry breaking)

3

4

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our work: proof logging for symmetry breaking (BreakID[3]) + verification (VeriPB[4])

- simple to implement + efficient proof checking
- fully evaluated for symmetry breaking on SAT competition benchmarks
- proof of concept for
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  - dynamic dominance breaking for maximum clique

[3] `https://bitbucket.org/krr/breakid/`
[4] `https://gitlab.com/MIAOresearch/VeriPB`

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future work: understand power of dominance rule, improve performance, increase trustworthiness, proof logging for more algorithms and problems

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Strengthening Rules (simplified)

- for formula F, objective f, and sequence of derived constraints $D_1, D_2, \ldots$
- let G_i, be set of constraints added so far ($G_i = F \cup \{ D_1, \ldots, D_{i-1} \}$)
- redundance based strengthening:
  (generalize redundancy from SAT [HKB17, BT19] to PB and optimization)

$$\frac{G_i \cup \{ \neg D_i \} \models (G_i \cup D_i)_{\restriction\omega} \cup \{ f_{\restriction\omega} \leq f \}}{D_i}$$

- dominance based strengthening:

$$\frac{G_i \cup \{ \neg D_i \} \models F_{\restriction\omega} \cup \{ f_{\restriction\omega} < f \}}{D_i}$$

- rules are annotated by:
  - used substitution ω
  - explicit proof for "$\models$"

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