SDP-based Cheeger inequalities for vertex (and hypergraph) expansion

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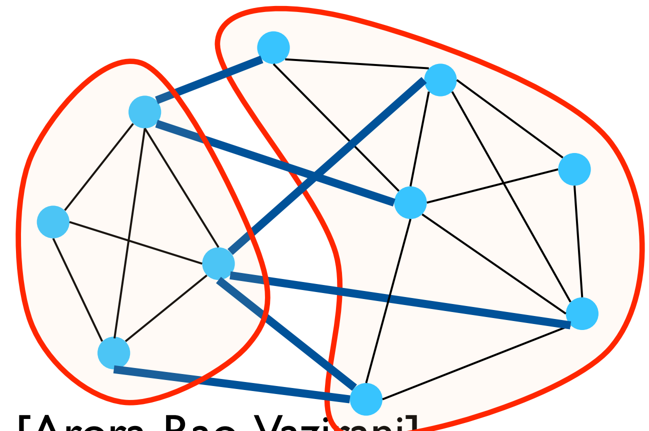
Graph expansion

- $G=(V,E)$, edge weights w
- $S \subset V$

$$\phi(S) = w(S,S)/\min w(S),\ \ w(S)$$

- $\phi(G) = \min_{\top} S\ \phi(S)$



- NP-hard to compute exactly
- Admits polytime $O(\sqrt{\log n})$ approximation [Arora-Rao-Vazirani]
- Improving on earlier $O(\log n)$ approximation

  [Leighton-Rao'88, Linial-London-Rabinovich, Aumann-Rabani]

Graph eigenvalues

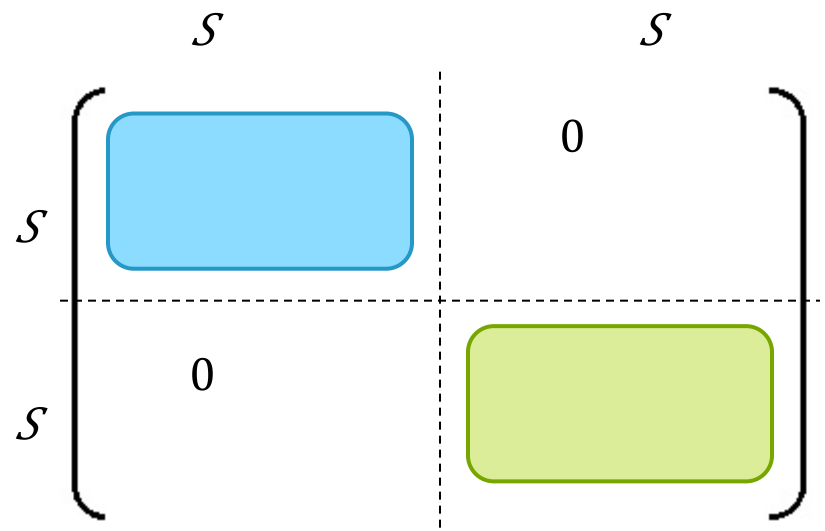
- $A_G = D^{-1/2} A D^{-1/2}$ with $D_{ii} = d_i = \sum_j w_{ij}$
- $A_G = 1/d\, A$ for d-regular graphs
- $L_G = I - A_G$ is positive semidefinite
- $\lambda_1(L_G) = 0$; $L_G D^{1/2} \mathbf{1} = 0$.

$$\lambda_2(L_G) = \min_{x \in R^n,\ x \perp D^{1/2} \mathbf{1}} x^T L_G x / x^T x = \min_{x \in R^n,\ x \cdot d = 0} \sum_{ij \in E} w_{ij} (x_i - x_j)^2 \Big/ \sum_i d_i x_i^2 \geq 0$$

Perron-Frobenius

- $\lambda_2 = 0$ if and only if graph is disconnected.

- If $\lambda_2 \approx 0$, then is graph close to disconnected ?



Cheeger's inequality

[Cheeger-Alon-Milman]

$$\lambda_2/2 \le \phi(G) \le \sqrt{2\lambda_2}$$

$$\lambda_2 = \min_{x\in R^n,\ x\cdot d=0} \frac{\sum_{ij\in E} w_{ij}(x_i - x_j)^2}{\sum_i d_i x_i^2} = \min_{x\in R^n} \frac{\sum_{ij\in E} w_{ij}(x_i - x_j)^2}{\sum_i d_i x_i^2 - \left(\sum_i d_i x_i\right)^2 / \sum_i d_i}$$

$$\le \min_{x\in\{0,1\}^n} \frac{\sum_{ij\in E} w_{ij}(x_i - x_j)^2}{\sum_i d_i x_i^2 - \left(\sum_i d_i x_i\right)^2 / \sum_i d_i} = \min_S \frac{w(S,\bar S)\, w(V)}{w(S)\, w(\bar S)}$$

$$\le 2\phi(G)$$

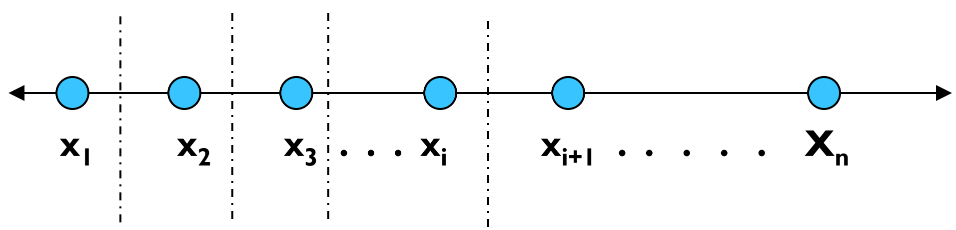
Cheeger's Algorithm

$$\frac{1}{2}\lambda_2 \leq \phi(G) \leq \sqrt{2\lambda_2}$$

x: eigenvector of L_G for λ_2

1. Sort x: $x_1 \leq x_2 \leq \ldots \leq x_n$
2. Consider subsets $S_i = \{x_1, \ldots, x_i\}$
3. Take S: $\operatorname{argmin} \phi(S_i)$

2nd eigenvector of L_G



$\min_i \phi(S_i) \leq \sqrt{2\lambda_2}$, proof via Cauchy-Schwarz

Gives method to certify constant edge expansion

Soo useful and central

Image segmentation

data clustering

network routing and design

VLSI layout

Parallel/distributed computing

...

certificate for constant edge expansion

mixing of Markov chains

graph partitioning

Pseudorandomness

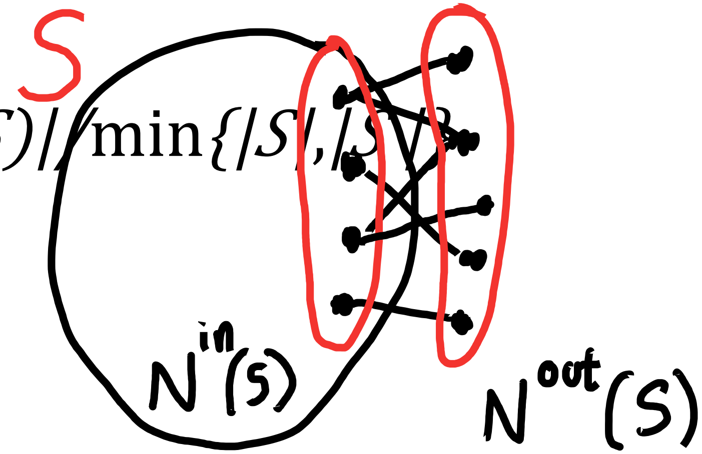
…

Talk outline

- [Vertex expansion] Is there a Cheeger-type inequality for vertex expansion? Can we efficiently verify whether a graph is a vertex expander?
- [Hypergraphs] How to extend expansion and Cheeger inequalities to hypergraphs?
- [Lower bounds] Are these the best possible algorithmic bounds?

Vertex Expansion

$$\phi^V(S) = |N^{in}(S)| + |N^{out}(S)| / \min\{|S|, |\bar{S}|\}$$



$$\phi^V(G) = \min_S \phi^V(S)$$

- Fundamental parameter, with many applications.
- Admits $O(\sqrt{\log n})$ approximation [Feige-Hajiaghayi-Lee'08]
- Cheeger gives $\sqrt{d\, OPT}$, where d is max degree [Alon'85]

- Can constant vertex expansion be certified in polytime?

Vertex expansion

- Max formulation

$$\phi^V(G) = \min_{x\in\{0,1\}^n} \frac{\sum_i \max_{j:ij\in E} (x_i - x_j)^2}{\left(\sum_i x_i^2 - \frac{1}{n}\left(\sum_i x_i\right)^2\right)} = \min_S \frac{|N(S)\cup N(\bar{S})|}{|S||\bar{S}|/n} \leq 2\phi^V(G)$$

Cheeger inequality for vertex expansion

- Relaxing the 0,1 constraint:

$$\lambda_\infty = \min_{x \in R^n,\, \mathbf{x} \perp \mathbf{1}} \sum_i \max_{j: ij \in E} (x_i - x_j)^2 \Big/ \sum_i x_i^2$$

- Theorem [Bobkov-Houdre-Tetali '00]

$$\lambda_\infty / 2 \leq \phi^V(G) \leq c\sqrt{\lambda_\infty}$$

- But how to compute λ_∞ ?

A semidefinite relaxation

- $\lambda_\infty = \min_{x\in R^n,\, \boldsymbol{x}\perp \mathbf{1}} \sum_i \max_{j: ij\in E} (x_i - x_j)^2 / \sum_i x_i^2$

- SDP: $\min_{x_1,\ldots,x_n \in R^n} \sum_i \max_{j: ij\in E} ||x_i - x_j||^2 / \left(\sum_i ||x_i||^2 - 1/n\, ||\sum_i x_i||^2\right) \le \lambda_\infty$

- Theorem. [LRV13; also Steurer-Tetali].

$$\lambda_\infty/2 \le \phi^V(G) \le C\sqrt{SDP\cdot \log d} \le C\sqrt{\lambda_\infty \log d}$$

Cheeger algorithm for vertex expansion

SDP finds vectors $x_1, x_2, \ldots, x_n \in R^n$.

Rounding:

- Pick random Gaussian vector g
- Project and sort, according to $x_i \cdot g$
- Apply a Cheeger sweep to sorted vector

  (or pick a random threshold cut)

Analysis:

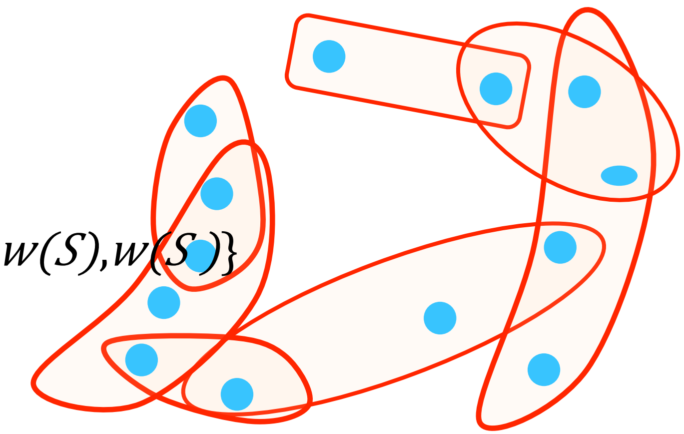
- after projection, vector y satisfies

$$\sum_i \max_{j: ij \in E} (y_i - y_j)^2 / d\left(\sum_i y_i^2 - 1/n \left(\sum_i y_i\right)^2\right) \leq SDP \cdot O(\log d)$$

- Then [BHT] gives a cut of expansion $O(\sqrt{SDP \cdot \log d})$

Hypergraph expansion

$H=(V,E)$, edges are subsets of vertices



$$\phi(H)=\min_{S\subset V} \frac{\sum_{e:\, e\cap S,\ e\cap \bar{S} \neq \phi} w(e)}{\min\{w(S), w(\bar{S})\}}$$

$$\phi(H)\le\min_{x\in\{0,1\}^n} \sum_{e\in E} \max_{i,j\in e} (x_i - x_j)^2 \Big/ d\left(\sum_i x_i^2 - 1/n \left(\sum_i x \ldots\right.\right.$$

Common generalization of vertex and edge expansion

Hypergraph Cheeger

$$\gamma_2 = \min_{x \in \mathbb{R}^n,\, x \perp 1} \sum_{e \in E} \max_{i,j \in e} (x_i - x_j)^2 \Big/ d\left(\sum_i x_i^2 - 1/n \left(\sum_i x_i\right)^2\right)$$

- Theorem.

$$\gamma_2/2 \le \phi(H) \le c\sqrt{\gamma_2}$$

- γ_2 can be approximated by the SDP to within $O(\log r)$.
- Hypergraph expansion to within $O(\sqrt{SDP \cdot \log r})$
- With L_2^2-metric constraints, gives $O(\sqrt{\log n})$ approximation [Louis-Makarychev'14]

Hypergraph dispersion

$$\gamma_2 = \min_{x \in R^n,\, x \perp 1} \frac{\sum_{e \in E} \max_{i,j \in e} (x_i - x_j)^2}{d\left(\sum_i x_i^2 - \frac{1}{n}\left(\sum_i x_i\right)^2\right)}$$

This definition suggests the following dispersion process:

- Start with some distribution x on vertices
- Repeat: each hyperedge finds the two vertices with largest difference $x_i - x_j$ and transfers $\frac{1}{2d}(x_i - x_j)$ from i to j.

Dispersion and Eigenvalues

- Viewing this dispersion process as a (Markov) operator, the process is

$$x^{t+1} = M_{x^t}(x^t)$$

- Does this converge? To what? At what rate?

- Theorem [Louis '14]. Under mild conditions, this process converges to uniform at rate $1/\gamma_2$ and $\exists \mu$: $M_\mu(\mu) = \gamma_2 \mu$.
- Note:

$$\gamma_2 = \min_{x \perp \mathbf{1}} x^T (I - M_x)(x) / x^T x$$

Better algorithmic bounds?

Can we approximate, in polytime,

- edge expansion to better than $\sqrt{OPT}$?
- vertex expansion to better than $\sqrt{OPT \log d}$?

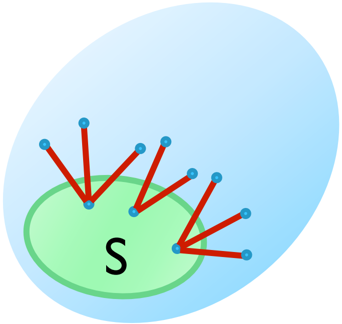
(can we certify that vertex expansion is at least some constant in polynomial time?)

- hypergraph expansion to better than $\sqrt{OPT \log r}$?

Small Set Expansion [Raghavendra-Steurer]

SSE (ε,δ): Given a graph $G=(V,E)$, distinguish between the following two cases :

- $\exists\ S\subset V,\ \mu(S)=\delta$ and $\Phi(S)\leq\varepsilon$
- All subset $S\subset V$ with $\mu(S)=\delta$ have $\Phi(S)\geq 1-\varepsilon$



SSE-Hypothesis: For all $\varepsilon>0, \exists\ \delta>0$ such that SSE(ε,δ) is NP-hard.

SSE-Hardness of approximation

Theorem. [Raghavendra-Steurer-Tulsiani '10]

Assuming the SSE hypothesis, edge expansion is hard to approximate to within $o(\sqrt{OPT})$.

Theorem. [Louis-Raghavendra-V.'13]

Assuming the SSE hypothesis, vertex expansion is hard to approximate to within $o(\sqrt{OPT \log d})$.

Cor. It is SSE-hard to decide if a graph has vertex expansion at most ϵ or at least $\Omega(\sqrt{\epsilon \log d})$.

Similar lower bound for hypergraph expansion.

A series of reductions [LRV'13]

- SSE: ϵ vs $1-\epsilon$ for δ-measure subsets

$\downarrow$

- Balanced analytic expansion

$\downarrow$

- Balanced vertex expansion: ϵ vs $\sqrt{\epsilon \log d}$ for $\min \phi^V(S)$ for $\mu(S) \geq 1/10$

$\downarrow$

- Symmetric vertex expansion: ϵ vs $\sqrt{\epsilon \log d}$ for $\phi^V(G)$

$\downarrow$

- Vertex expansion

Balanced Analytic expansion

- Vertices V and distribution P over d+1 subsets, with marginal P_1 on vertices. For $F: V \to \{0,1\}^n$,

$$\phi(V,P,F) = E_{(X,Y_1,\ldots,Y_d)\sim P}\left(\max_i |F(Y_i) - F(X)|\right) / E_{X,Y\sim P_1}\left(|F(X)-F(Y)|\right)$$

$$\phi(V,P) = \min \phi(V,P,F): \quad E_{X,Y\sim P_1}\left(|F(X)-F(Y)|\right) \geq 1/100 .$$

- Generalizes edge expansion (d=1).
- Reduction to this problem is on the lines of [RST'12]

Analytic expansion

$$E_{(X,Y_1,\dots,Y_d)\sim P}\left(\max_i |F(Y_i)-F(X)|\right)/E_{X,Y\sim P_1}\left(|F(X)-F(Y)|\right)$$

- Sample a vertex X from S
- Sample *d* neighbors of X : $Y_1,\dots,Y_d$
- What is the probability that at least one Y_i lies outside *S*?

- $\phi_1(S)=\phi(S)$, $\phi_\infty(S)=\phi^V(S)$
- Computing $\min_S \phi_d(S)$ ↔ computing vertex expansion in graphs with degree $O(d)$.

A series of reductions

- SSE: ϵ vs $1-\epsilon$ for δ-measure subsets

$\downarrow$

- Balanced analytic expansion: ϵ vs $\sqrt{\epsilon \log d}$

$\downarrow$

- Balanced vertex expansion: ϵ vs $\sqrt{\epsilon \log d}$ for $\min \phi^V(S)$ for $\mu(S) \geq 1/10$

$\downarrow$

- Symmetric vertex expansion: ϵ vs $\sqrt{\epsilon \log d}$ for $\phi^V(G)$

$\downarrow$

- Vertex expansion

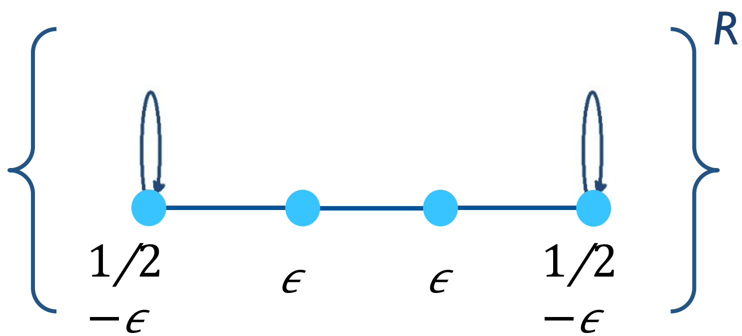
Transforming an SSE instance

- G_0 = SSE instance
- H = "gadget" (small graph)
- $G = G_0 \times H^R$ + "random" edges to smooth

Claim:

1. $\phi_\delta(G_0) \leq \epsilon$ maps to $\phi^V(G) \leq \epsilon'$
2. $\phi_\delta(G_0) \geq 1-\epsilon$ maps to $\phi^V(G) \geq \sqrt{\epsilon'} \log d$

Gadgets for dictators



- Dictator cuts: subset defined by all copies of some vertices from base graph

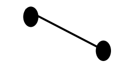
"Completeness": Dictator cuts have analytic expansion $\leq \varepsilon$

"Soundness": Cuts far from dictators have analytic expansion $\geq \sqrt{\varepsilon \log d}$

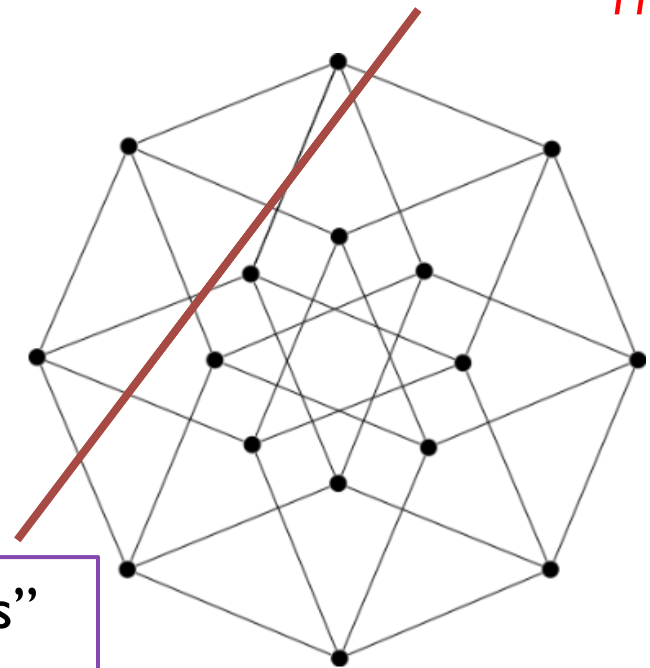
- Reduction from SSE via this gadget gives ε vs $\sqrt{\varepsilon \log d}$ hardness for vertex expansion

Analytic Expansion of Product Graphs

Take any graph H with spectral gap ε.

H =


Fix $G = H^{R}$

G =


Theorem:

For all S which are far from "axis cuts"
$\phi_d(S) \geq \sqrt{\varepsilon} \log d$

Reduce this computation to bounding analytic expansion of the *Gaussian Graph* via Invariance Principles [Issakson,Mossel-12].

Soundness of the gadget

- Need to show that every subset "far" from a dictator has high analytic expansion.
- This is done via an invariance principle for low-degree polynomials [Isaksson-Mossel '12]
- Reduces to showing that the infinite Gaussian graph has large analytic expansion.

Invariance Principles

$E[\Gamma_{1-\eta} F(X_1 ,\dots,X_n)]$

(average over random boolean inputs)

$\approx$

$E[\Gamma_{1-\eta} F(G_1 ,\dots,G_n)]$

(average over random Gaussian inputs)

Analytic Vertex Expansion of Gadget

Analytic Vertex Expansion of *Gaussian graph*

Gaussian graph

- $\mathcal{G}_\epsilon$: complete (weighted) graph on $V=R^n$
- $w(u,v) \propto \exp(-\|u-v\|^2/2\epsilon)$
- $\mathrm{w}(u,v)=\mathrm{P}[X=u, Y=v]$,

  where X and Y are $(1-\epsilon)$–correlated Gaussians

- Fix $S\subset V$, sample $X\sim\mathcal{N}(0,1)^n$, $Y_1,\ldots,Y_d\sim\mathcal{N}(X,\varepsilon I)$

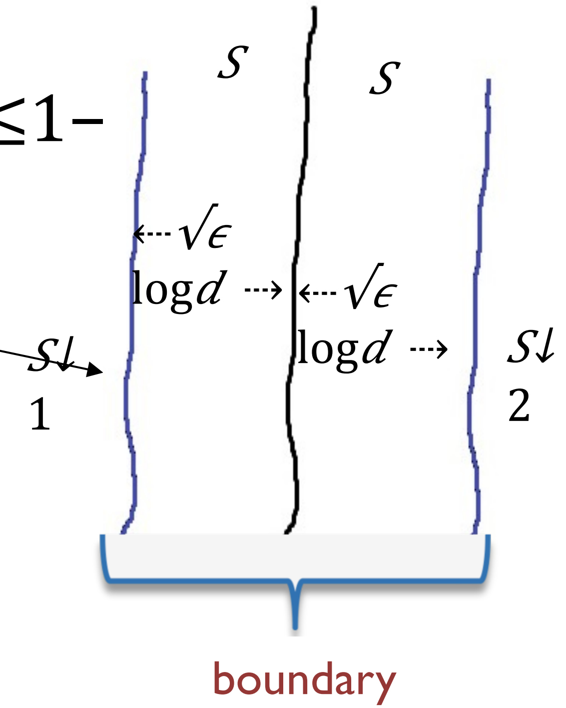
- Theorem. $\phi_d(S)\geq c\sqrt{\varepsilon \log d}$ for all $S\subset V$

Analytic Expansion of the Gaussian Graph

$S_1 = \{X \in S : \mu_X(\Re^n \setminus S) < 1/2d\}$ and $S_2 = \{X \in \Re^n \setminus S : \mu_X(S) < 1/2d\}$

If $\|u - v\| \le \sqrt{\epsilon}\log d$ then $d_{TV}(P_u, P_v) \le 1 -$

Vertices in this region are likely to have one of d neighbors cross the boundary.



$\mu(boundary) \ge \sqrt{\varepsilon}\log d \cdot \mu(S)\mu(\Re^n \setminus S)$
via [Borell-75]

Analytic vertex expansion of S $\ge \sqrt{\epsilon}\log d$.

Conclusion

Given a graph G distinguish between the following cases:

- (Non-expander) G has a set with $\phi^V(S) < \varepsilon_0$
- (Vertex Expander) $\phi_G^V \geq 0.1$

$\sqrt{OPT \cdot \log d}$ upper and lower bounds for approximating vertex expansion.

Edge Expansion	Vertex Expansion
$O(\log n)$ [Leighton,Rao-88]	$O(\log n)$ [Leighton,Rao-88]
$O(\sqrt{\log n})$ [Arora,Rao,Vazirani-04]	$O(\sqrt{\log n})$ [Feige,Hajiaghayi,Lee-05]
$O(\sqrt{OPT})$ [Alon-Milman-86]	$O(\sqrt{d \cdot OPT})$ [Alon-Milman 86]
	$\boldsymbol{O(\sqrt{OPT \log d})}$
No PTAS assuming ETH [Ambuhl-Mastrolilli-Svensson-07]	
$\Omega(\sqrt{OPT})$ under SSE [Raghavendra,Steurer,Tulsiani-12]	
	$\boldsymbol{\Omega(\sqrt{OPT \log d})}$ **under SSE**

Open questions

- Better approximations for edge expansion, vertex expansion, hypergraph expansion?
- Analyze Miller's algorithm
- Show NP-hard to approximate to within some constant factor (1.01)
  [AMS07]: No PTAS unless SAT has subexp algorithms
- Give local (small-space) implementation of a hypergraph dispersion process