Positive Semidefinite Rank

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Joint work with:

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Definition: Given $M \in \mathbf{R}_+^{p\times q}$

$$\operatorname{rk}_{\mathrm{psd}}(M) = \min \left\{ k \; : \; \exists \begin{array}{l} A_1, \ldots, A_p \in \mathcal{S}_+^k \\ B_1, \ldots, B_q \in (\mathcal{S}_+^k)^* \\ \text{s.t. } M_{ij} = \langle A_i, B_j \rangle \end{array} \right\}$$

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$$M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$\mathrm{rk}(M) = 3$

$\mathrm{rk}_{\mathrm{psd}}(M) = 2$

$\mathcal{S}_+^2$-factorization of M :

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

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$\mathrm{rk}_{\mathrm{psd}}(I_3) = 3$

Factorizations of matrices/operators:

(i) factoring through subspaces:

$$\mathrm{rk}(M) = \min \{k \,:\, M = AB, \;\; A \in \mathbf{R}^{p \times k}, \;\; B \in \mathbf{R}^{k \times q}\}$$

$$\text{i.e., } \exists \; a_1, \ldots, a_p \in \mathbf{R}^k, \; b_1, \ldots, b_q \in (\mathbf{R}^k)^* \text{ s.t. } \; M_{ij} = \langle a_i, b_j \rangle$$

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(ii) factoring through cones: cone rank of M GPT 2013

$\mathcal{C} = \{C_i\}$ "closed" family of closed convex cones

$$\mathrm{rk}_{\mathcal{C}}(M) = \min\left\{k \,:\, \exists \begin{array}{c} a_1, \ldots, a_p \in C_k,\ \ b_1, \ldots, b_q \in (C_k)^* \\ \text{s.t. } M_{ij} = \langle a_i, b_j \rangle \end{array}\right\}$$

$\mathcal{C} = \{\mathbf{R}^k_+\}$ *nonnegative rank* $\mathrm{rk}_+(M)$

$\mathcal{C} = \{\mathcal{S}^k_+\}$ *psd rank* $\mathrm{rk}_{\mathrm{psd}}(M)$

$$\frac{1}{2}\sqrt{1+8\ \mathrm{rk}(M)}-\frac{1}{2}\ \le\ \mathrm{rk}_{\mathrm{psd}}(M)\ \le\ \mathrm{rk}_{+}(M)\ \le\ \min(p,q)$$

$$\mathrm{rk}(M)=1\ \Leftrightarrow\ \mathrm{rk}_{\mathrm{psd}}(M)=1 \qquad \mathrm{rk}(M)=2\ \Leftrightarrow\ \mathrm{rk}_{\mathrm{psd}}(M)=2$$

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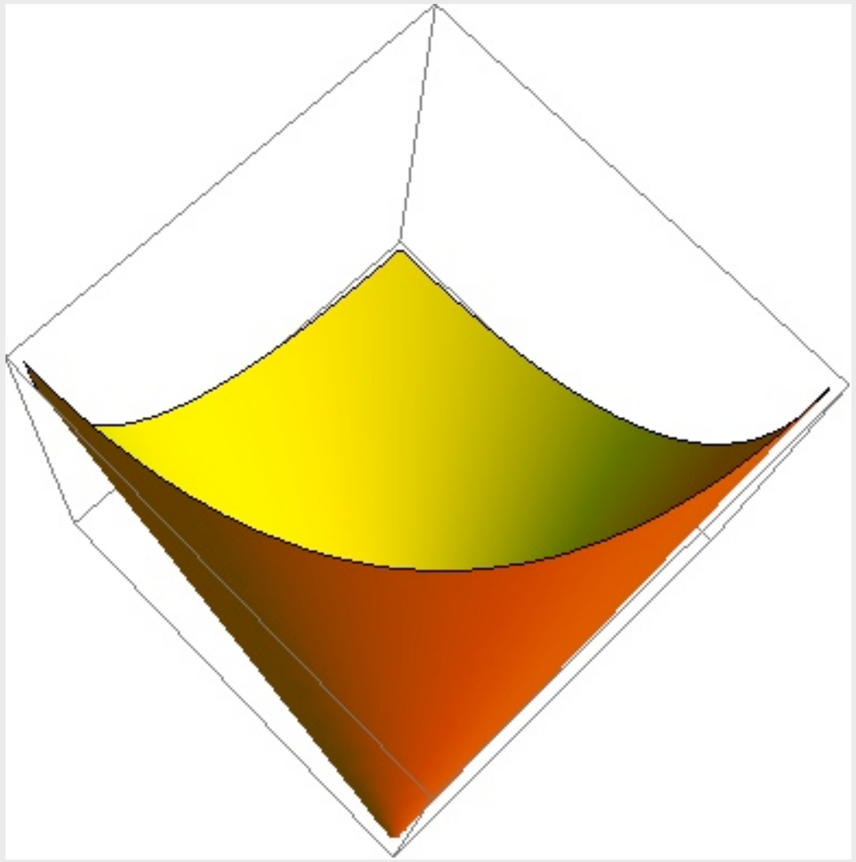
$$M_{abc}=\begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix},\quad a,b,c\geq 0$$

$\mathrm{rk}=\mathrm{rk}_{\mathrm{psd}}=1 \Leftrightarrow a=b=c$

else, $\mathrm{rk}=3 \Rightarrow \mathrm{rk}_{\mathrm{psd}}=2,3$

$\mathrm{rk}_{\mathrm{psd}}=2 \Leftrightarrow$

$a^2+b^2+c^2 \leq 2(ab+bc+ac)$



$$\frac{1}{2}\sqrt{1+8\,\mathrm{rk}(M)}-\frac{1}{2}\;\leq\;\mathrm{rk}_{\mathrm{psd}}(M)\;\leq\;\mathrm{rk}_{+}(M)\;\leq\;\min(p,q)$$

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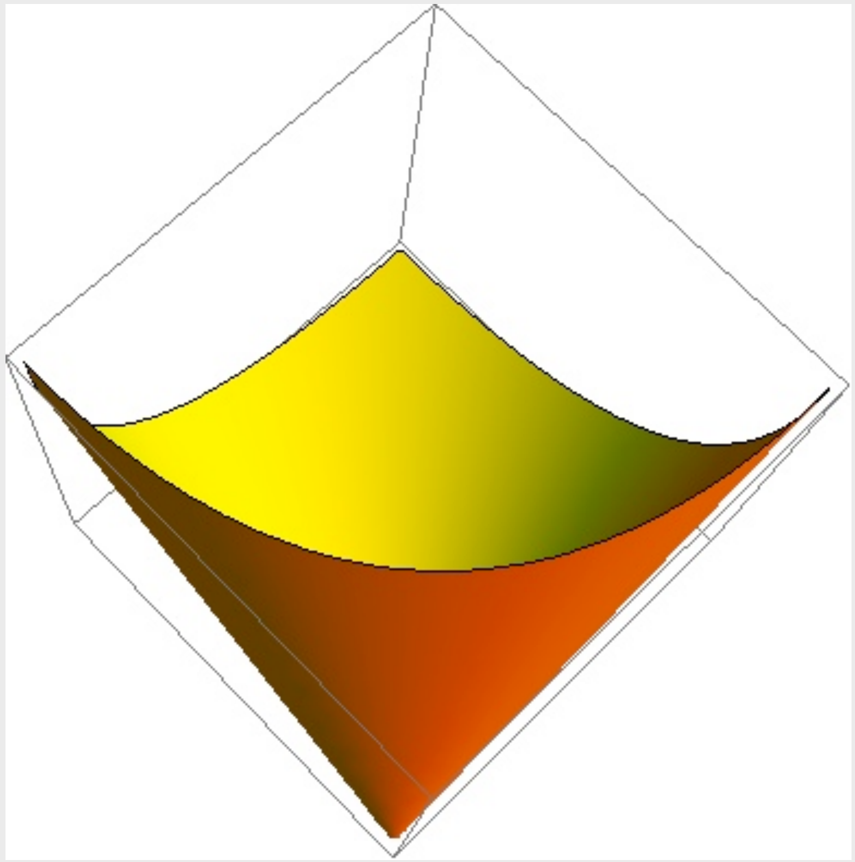

$\mathcal{M}^{p\times q}_{\mathrm{rk}_{\mathrm{psd}}=2}\;\subset\;\mathcal{M}^{p\times q}_{\mathrm{rk}=3}$

Kubjas-Robeva-Robinson 2014

Square root rank

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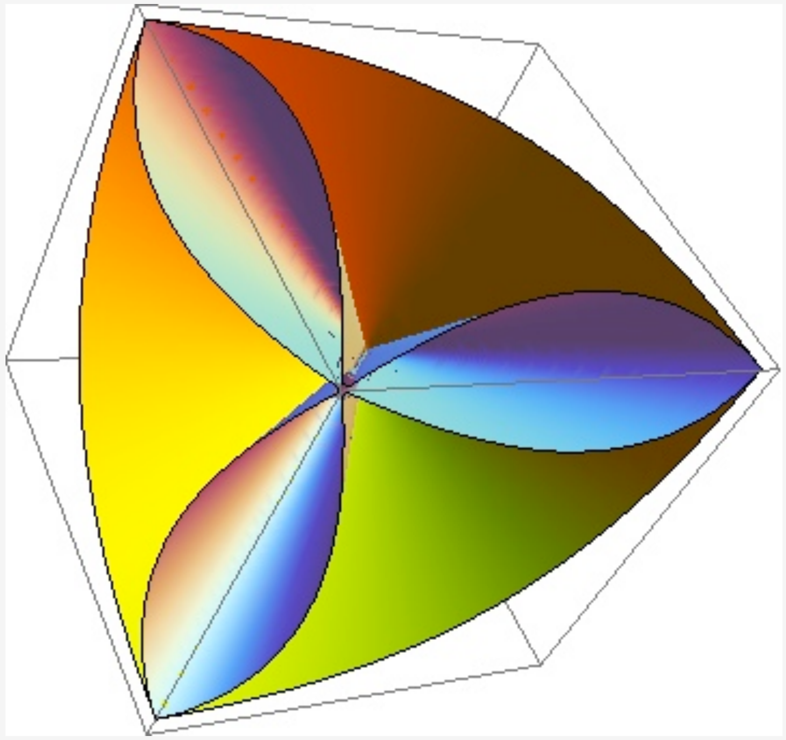
$$M = \begin{bmatrix} 1 & 4 & 9 \\ 9 & 1 & 4 \\ 4 & 9 & 1 \end{bmatrix} \qquad \mathrm{rk}_{\sqrt{}}(M) = 2$$

$$\sqrt{M} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -1 & 2 \\ 2 & -3 & -1 \end{bmatrix}, \quad \mathrm{rk}\sqrt{M} = 2$$



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(a, b, c) on this surface

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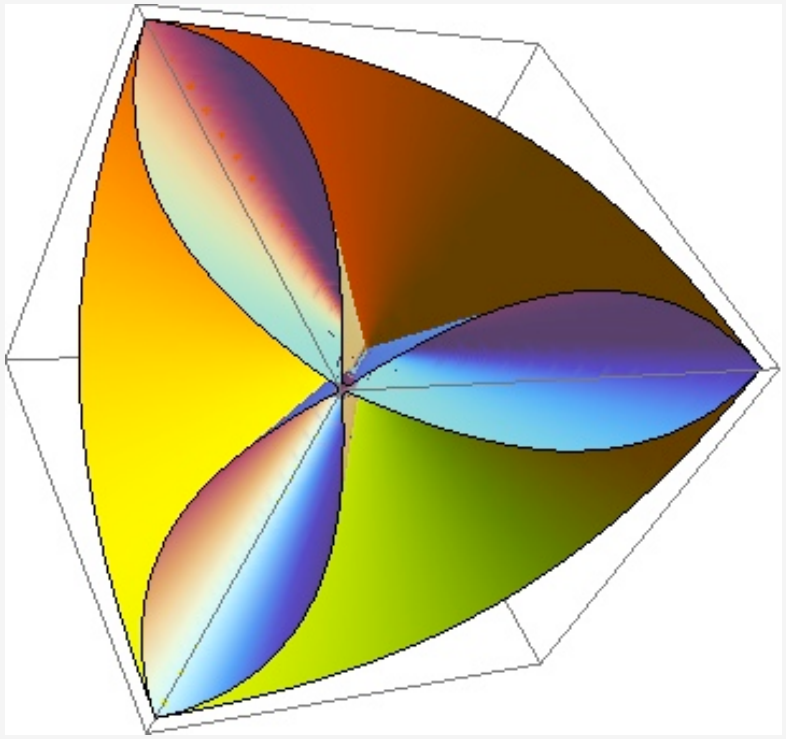


$$\text{rk}_{\sqrt{}}(M) \leq 2$$
$$\Leftrightarrow$$
(a, b, c) on this surface

$$\text{rk}_{\sqrt{}}(M) = \min \{k : \exists\ \mathcal{S}_+^k\text{-factorization of } M \text{ with rank one factors}\}$$

Corollaries:

$$\text{rk}_{\text{psd}}(M) \leq \text{rk}_{\sqrt{}}(M)$$

$$M \in \{0,1\}^{p \times q} \Rightarrow \text{rk}_{\text{psd}}(M) \leq \text{rk}(M)$$

	rk	rk_+	$\mathrm{rk}_{\mathrm{psd}}$	$\mathrm{rk}_{\sqrt{}}$
rk	$=$	$\ll$	$\ll, >$	$\ll, >$
rk_+	$\gg$	$=$	$\gg$	$\ll, \gg$
$\mathrm{rk}_{\mathrm{psd}}$	$<, \gg$	$\ll$	$=$	$\ll$
$\mathrm{rk}_{\sqrt{}}$	$<, \gg$	$\ll, \gg$	$\gg$	$=$

Euclidean distance matrices:

$$M_{ij} = (i-j)^2 \qquad \mathrm{rk}(M) = 3, \ \ \mathrm{rk}_{\mathrm{psd}}(M) = 2$$

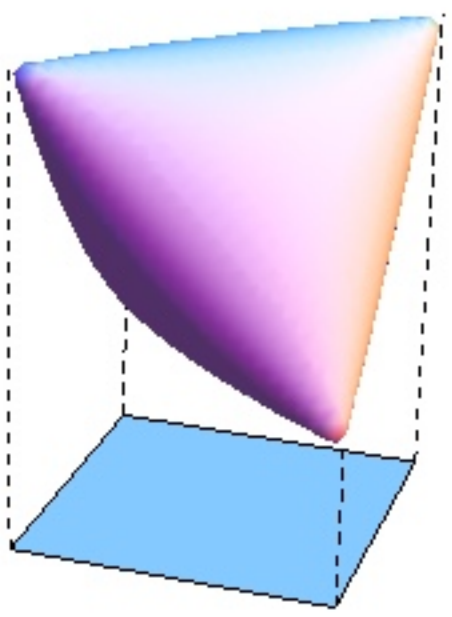
$$\mathrm{rk}_+(M) = \Theta(\log_2 n)$$ Hrubes 2012

Cannot assume all factors have rank one to study psd rank.

Geometric Interpretation

Def: $P \subseteq \mathbf{R}^n$ (convex set) has a psd lift of size k if $P = \pi(\mathcal{S}_+^k \cap L)$

π linear map, L affine space



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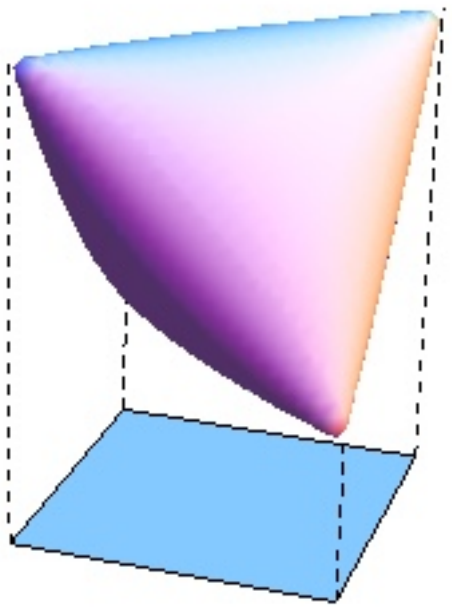

$P \subseteq Q$

$$P = \text{conv}(p_1, \ldots, p_v) \subset \mathbf{R}^n$$

$$Q = \{x \in \mathbf{R}^n \; : \; a_j^T x \leq \beta_j, \;\; j = 1, \ldots, f\}$$

slack matrix: $S_{P,Q} \in \mathbf{R}^{v \times f}, \;\; (S_{P,Q})_{ij} := \beta_j - a_j^\top p_i$

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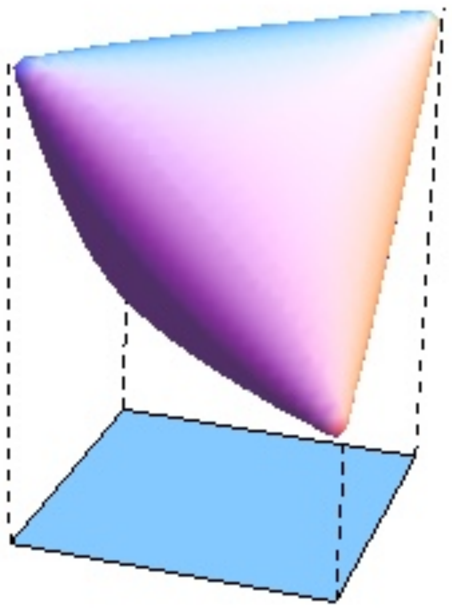

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Thm: $\operatorname{rk}_{\mathrm{psd}}(S_{P,Q}) = \ \min \{k \ : \ \exists \ \pi, L \text{ s.t. } \ P \subseteq \pi(\mathcal{S}_+^k \cap L) \subseteq Q\}$

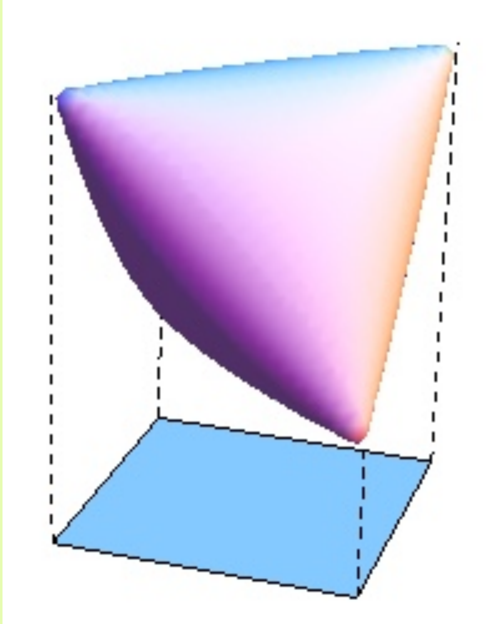
$\operatorname{rk}_{\mathrm{psd}}(S_{P,P}) = \ \min \{k \ : \ \ \text{s.t. } \ P = \pi(\mathcal{S}_+^k \cap L)\}$

Yannakakis 91: polyhedral-lifts, P polytope, $\operatorname{rk}_+(S_{P,P})$

GPT 2013: cone-lifts, P convex set, $\operatorname{rk}_{\mathcal{C}}$(slack operators)

$$P = [-1,1]^2 = \left\{ (x,y,z) \in \mathbf{R}^3 \ : \ \begin{bmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{bmatrix} \succeq 0 \right\}$$

$$S_P = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad \mathrm{rk}_{\mathrm{psd}} = 3$$



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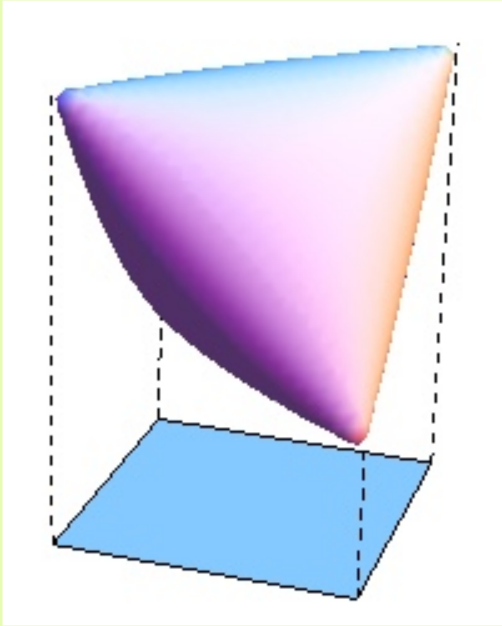

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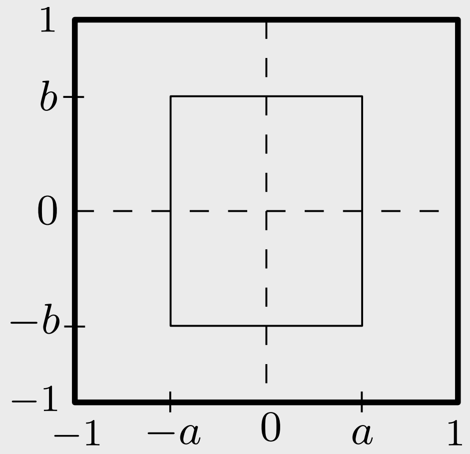
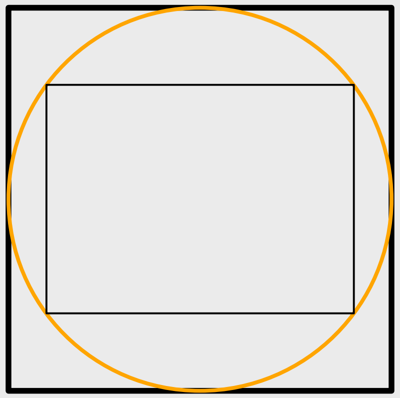
$$Q = [-1,1]^2, \quad P = [-a,a] \times [-b,b]$$





$$S_{P,Q} = \begin{bmatrix} 1+a & 1+b & 1-a & 1-b \\ 1-a & 1+b & 1+a & 1-b \\ 1-a & 1-b & 1+a & 1+b \\ 1+a & 1-b & 1-a & 1+b \end{bmatrix}$$

$$\mathrm{rk}_{\mathrm{psd}} = \begin{cases} 3 & \text{if } a^2+b^2 > 1 \\ 2 & \text{if } 0 < a^2+b^2 \le 1 \\ 1 & \text{if } a = b = 0 \end{cases}$$

PSD ranks of polytopes

Briet-Dadush-Pokutta 2013: not all $\{0,1\}$-polytopes in $\mathbf{R}^n$ have small psd lifts

no concrete 0/1-polytope family so far with large psd rank

favorite candidate: $\mathrm{COR}(n) = \mathrm{conv}(aa^\top : a \in \{0,1\}^n)$

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GPT 13: $\mathrm{rk}_{\mathrm{psd}}(P) = k \Rightarrow P$ has at most $k^{O(k^2 n)}$ facets

eg. psd rank of n-gons grow to infinity with n

GRT 13: $\mathrm{rk}_{\mathrm{psd}}(\text{generic } n\text{-gon}) = \Omega(\sqrt[4]{n})$

no family so far with large gap between psd and nonnegative rank

Equivariant psd lifts of polytopes:

(lifts that respect the symmetries of the polytope -- a la Yannakakis)

Fawzi-Parrilo-Saunderson 2013: $\mathrm{rk}_{\mathrm{psd}}^{\mathrm{eq}}(\mathrm{CUT}(n)) \geq \binom{n}{\lceil n/4 \rceil}$

Lee-Raghavendra-Steurer-Tan 2013

FSP 2014: $\mathrm{rk}_{\mathrm{psd}}^{\mathrm{eq}}(\text{regular } n\text{-gon}) = \Theta(\log n)$

GPT 13: $\mathrm{rk}_{+}^{\mathrm{eq}}(\text{regular } n\text{-gon}) = n \qquad n \text{ prime power}$

provides exponential gap in sizes of LP and PSD equivariant lifts

Lower bounds on psd rank

support not powerful: $\mathrm{rk}_{\mathrm{psd}}(M^2) \leq \mathrm{rk}(M)$ Lee-Theis 12

Barvinok 12:

$$M \text{ has } k \text{ distinct entries } \Rightarrow \mathrm{rk}_{\mathrm{psd}}(M) \leq \binom{k-1+\mathrm{rk}(M)}{k-1}$$

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A lower bounding technique using results of Renegar:

(1) $L \cap \mathcal{S}_+^k$ is bounded by polynomials of degree $\leq k$

(2) bound poly describing algebraic boundary of $\pi(L \cap \mathcal{S}_+^k) = P$

(3) degree of this boundary poly $\geq \#$ facets of P

e.g. GPT 13: $\mathrm{rk}_{\mathrm{psd}}(n\text{-gon}) = \Omega\left(\sqrt{\frac{\log n}{\log\log n}}\right)$ ... deWolf-Lee-Wei 14

Polytopes of min psd rank Gouveia-Robinson-T. 13:

Theorem: $\mathrm{rk}(S_P) = n + 1 \leq \mathrm{rk}_{\mathrm{psd}}(P)$

equality holds $\Leftrightarrow \mathrm{rk}_{\sqrt{}}(S_P) = n + 1$

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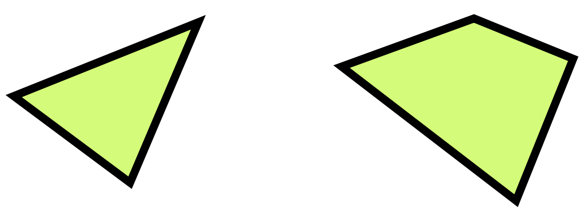
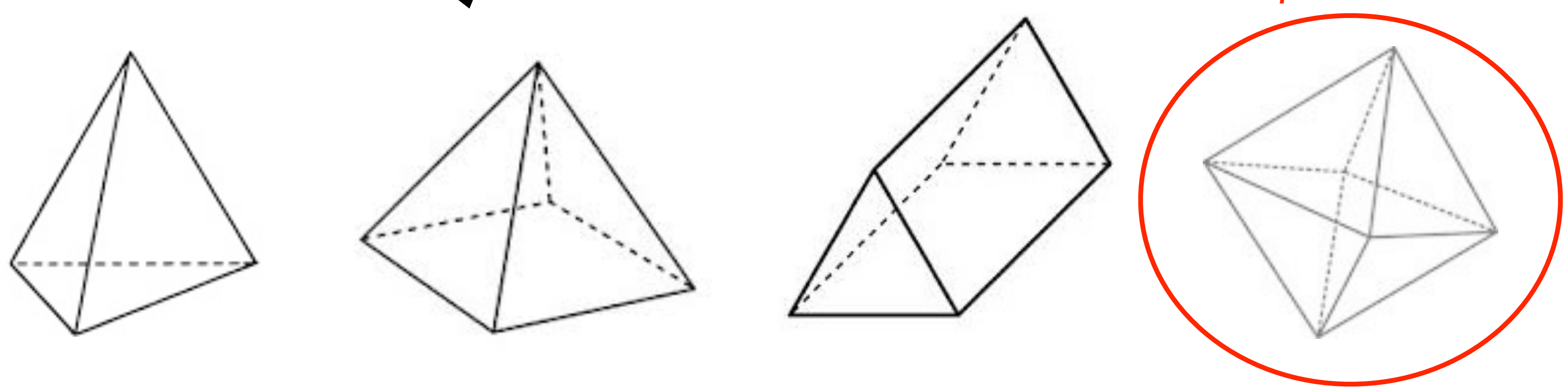
in $\mathbf{R}^2$:



biplanar

in $\mathbf{R}^3$:



and their polars

Theorem: $\mathrm{rk}_{\mathrm{psd}}(\mathrm{STAB}(G)) = n + 1 \Leftrightarrow G$ perfect

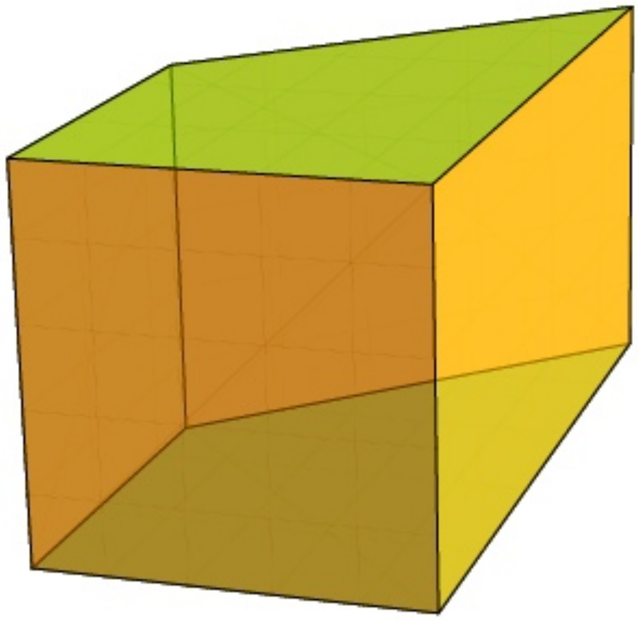
Grande-Sanyal 14: *psd minimal matroid polytopes*

a corollary ....

$\exists\, M \,:\, \mathrm{rk}^{\mathbf{Q}}_{\mathrm{psd}}(M) > \mathrm{rk}^{\mathbf{R}}_{\mathrm{psd}}(M)$ Fawzi-Gouveia-Robinson 14

M is the slack matrix of a biplanar cuboid

$$\begin{bmatrix} 0 & 0 & 2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \end{bmatrix}$$



Open for nonnegative rank

PSD rank of convex sets

$$C \subset \mathbf{R}^n \Leftarrow \mathrm{rk}_{\mathrm{psd}}(C) = \mathrm{rk}_{\mathrm{psd}}(\text{slack operator of } C)$$

Long history of constructing psd lifts of interesting convex sets in optimization/control/real algebraic geometry ....

$$i.e. \text{ show } \mathrm{rk}_{\mathrm{psd}}(C) < \infty$$

Helton-Nie Conjecture:

Every convex semi-algebraic set has finite psd rank.

Scheiderer 12: true for convex hulls of algebraic curves

Complexity issues:

Robinson 14: Square root rank is NP-hard to compute.

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & a_1^2 \\ 0 & 1 & \cdots & 0 & a_2^2 \\ \vdots & & \vdots & & \\ 0 & 0 & \cdots & 1 & a_n^2 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}$$

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Is psd rank NP-hard to compute?

Vavasis 2009: nonnegative rank is NP-hard

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For fixed k, is it NP-hard to decide if $\mathrm{rk}_{\mathrm{psd}}(M) \leq k$?

Arora-Ge-Kannan-Moitra 12, Moitra 13:

poly-time decidable for nonnegative rank

Some of the things I did not talk about ...

- Space of psd factorizations
- Quantum information interpretation of psd rank
- Approximate factorizations and approximations of polytopes
- Symmetric psd factorizations