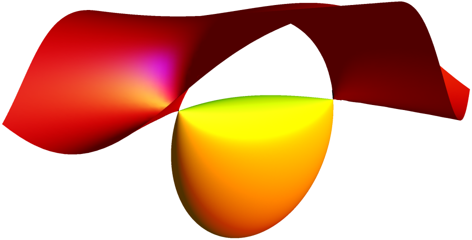
SPECTRAHEDRA AND THEIR SHADOWS

Bernd Sturmfels
UC Berkeley



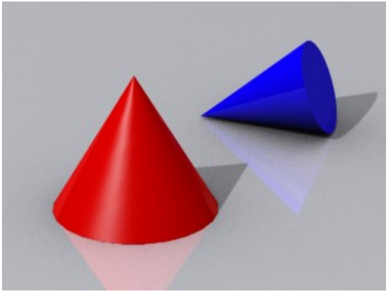
Simons Institute Workshop on
Semidefinite Optimization, Approximation and Applications
Wednesday, September 24, 2014

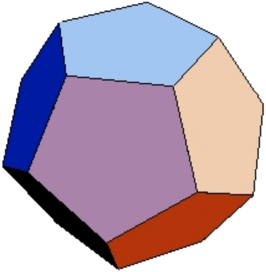
Spectrahedra

A *spectrahedron* of *degree n* in $\mathbb{R}^d$ is a convex body of the form

$$\mathcal{S} = \left\{(x_1, \ldots, x_d) \in \mathbb{R}^d : A_0 + x_1 A_1 + \cdots + x_d A_d \text{ is positive semidefinite}\right\}$$

where $A_0, A_1, \ldots, A_n$ are real symmetric matrices of format $n \times n$.



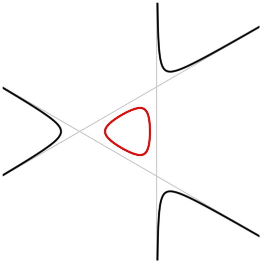


- $n = 1$: $\mathcal{S}$ is a closed half space.
- $n = 2$: $\mathcal{S}$ is a quadric cone.
- Finite intersections of spectrahedra are spectrahedra.

Semidefinite Optimization

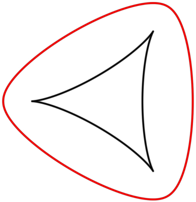
.... is the computational problem of minimizing

a linear function over a spectrahedron.



$(n = 3, d = 2)$

Duality is important in both optimization and projective geometry:



A Familiar Picture

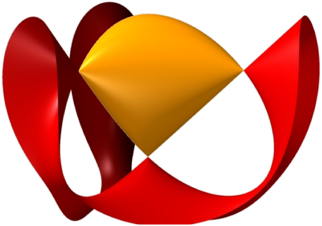
The set of 3×3-correlation matrices is the *elliptope*

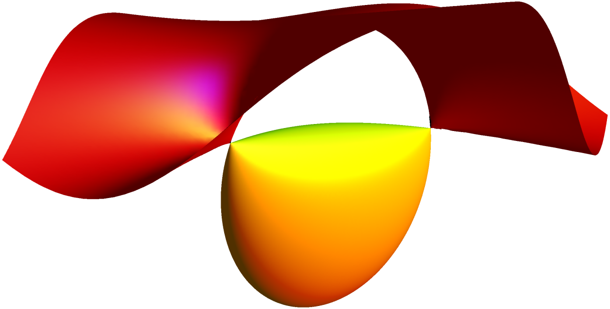
$$\begin{bmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{bmatrix}$$

$(d = n = 3)$

Q: Does every cubic spectrahedron have four nodes?

Cubic Spectrahedra





Two combinatorial types for $n = 3$. How about $n \geq 4$?

Sums of Squares

Let $f(x_1, \ldots, x_m)$ be a polynomial of even degree $2d$.
We wish to compute the global minimum x^* of $f(x)$ on $\mathbb{R}^m$.

This optimization problem is equivalent to

Maximize λ such that $f(x) - \lambda$ is non-negative on $\mathbb{R}^m$.

This problem is very hard.

The optimal value of the following relaxtion gives a lower bound.

Maximize λ such that $f(x) - \lambda$ is a sum of squares of polynomials.

The second problem is much easier. It is a semidefinite program.

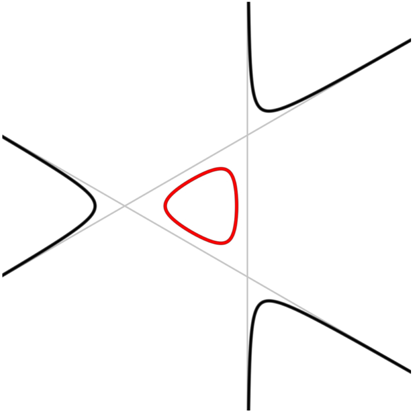
The optimal value of the SDP often agrees with the global minimum. In that case, the optimal matrix of the dual SDP has rank one, and the optimal point x^* can be recovered.

SOS Example

Let $m = 1$, $d = 2$ and $f(x) = 3x^4 + 4x^3 - 12x^2$. Then

$$f(x) - \lambda \quad = \quad \begin{pmatrix} x^2 & x & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & \mu - 6 \\ 2 & -2\mu & 0 \\ \mu - 6 & 0 & -\lambda \end{pmatrix} \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}$$

Our problem is to find (λ, μ) such that the 3×3-matrix is positive semidefinite and λ is maximal.



SOS Example

Let $m = 1$, $d = 2$ and $f(x) = 3x^4 + 4x^3 - 12x^2$. Then

$$f(x) - \lambda \quad = \quad \begin{pmatrix} x^2 & x & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & \mu - 6 \\ 2 & -2\mu & 0 \\ \mu - 6 & 0 & -\lambda \end{pmatrix} \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}$$

Our problem is to find (λ, μ) such that the 3×3-matrix is positive semidefinite and λ is maximal.

The optimal solution is

$$(\lambda^*, \mu^*) \;=\; (-32, -2).$$

Cholesky factorization reveals the SOS representation

$$f(x) - \lambda^* \quad = \quad \left((\sqrt{3}\,x - \frac{4}{\sqrt{3}}) \cdot (x+2)\right)^2 \;+\; \frac{8}{3}(x+2)^2.$$

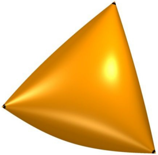
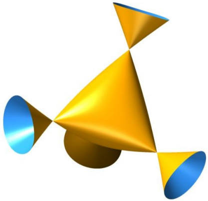
The global minimum is $x^* \;=\; -2$.

This approach works for many polynomial optimization problems.

This Lecture

.... has two objectives:

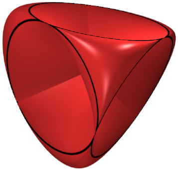
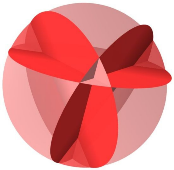
Objective 1: Show many pictures.

Objective 2: Present two recent papers:

Quartic Spectrahedra
(with John Christian Ottem, Kristian Ranestad, Cynthia Vinzant)

Generic Spectrahedral Shadows (with Rainer Sinn)

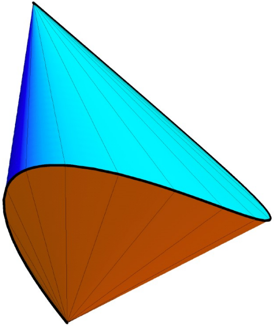



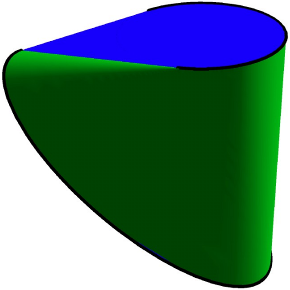
The Quartic Spectrahedron

$$\mathcal{S} \;=\; \left\{ (x,y,z) \in \mathbb{R}^3 \;:\; \begin{bmatrix} 1 & x & y & z \\ x & 1 & x & y \\ y & x & 1 & x \\ z & y & x & 1 \end{bmatrix} \succeq 0 \right\}.$$

is the convex hull of the trigonometric curve

$$\left\{ \left(\cos(\theta), \cos(2\theta), \cos(3\theta) \right) \;:\; \theta \in [0,\pi] \right\}$$





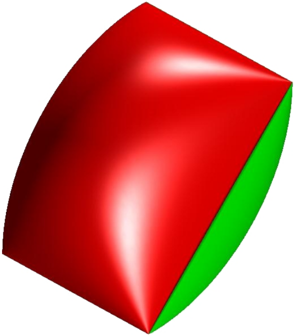
is the intersection of two quadratic cones.

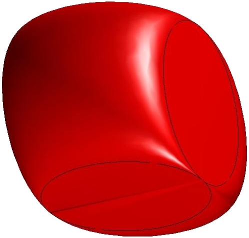
Sweet Dreams

The **pillow** is a quartic spectrahedron:

Q: Is it reducible?

$$\begin{bmatrix} 1 & x & 0 & x \\ x & 1 & y & 0 \\ 0 & y & 1 & z \\ x & 0 & z & 1 \end{bmatrix}$$





Q: Is the *convex dual* of a spectrahedron is a spectrahedron?

Symmetroids

Fix projective space $\mathbb{P}^3$ with coordinates $(x : y : z : w)$.
A symmetroid $\mathcal{S}$ of degree n is a surface with equation

$$\det(xA + yB + zC + wD) = 0.$$

Proposition

If A, B, C, D are generic then the singular locus of the symmetroid $\mathcal{S}$ consists of $\binom{n+1}{3}$ nodes. The web $xA+yB+zC+wD$ contains no matrix of rank $\leq n-3$. Those of rank $n-2$ are precisely the nodes.

$\mathcal{S}$ is a *transversal symmetroid* if this holds.

A 3-dimensional spectrahedron is *transversal* if its algebraic boundary is a transversal symmetroid.

Problem

Study the geometry and combinatorics of transversal spectrahedra. For instance, how many of the $\binom{n+1}{3}$ nodes can lie on its boundary?

Degtyarev-Itenberg Theorem

For quartic spectrahedra, this question was answered in a 2010 paper by Alex Degtyarev and Ilia Itenberg:

Theorem

There exists a transversal quartic spectrahedron with β nodes in its boundary and σ real nodes in its symmetroid if and only if

$$0 \leq \beta \leq \sigma, \ 2 \leq \sigma \leq 10, \ \textit{and both } \beta \textit{ and } \sigma \textit{ are even}.$$

Their proof is extremely indirect: it rests on the Global Torelli Theorem for K3 surfaces, and on deep topological results of Kharlamov and Nikhulin for moduli spaces of real K3 surfaces.

It is impossible to use this to construct matrices A, B, C, D.

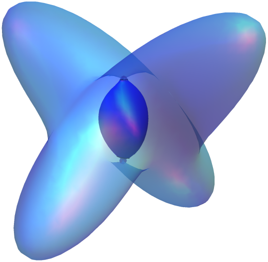
We give a new proof that is direct, computational and geometric.

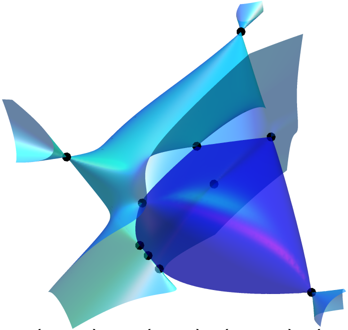
Twenty Types

	A	B	C	D
(2, 2):	$\begin{bmatrix} 3 & 4 & 1 & -4 \\ 4 & 14 & -6 & -10 \\ 1 & -6 & 9 & 2 \\ -4 & -10 & 2 & 8 \end{bmatrix}$	$\begin{bmatrix} 11 & 0 & 2 & 2 \\ 0 & 6 & -1 & 4 \\ 2 & -1 & 6 & 2 \\ 2 & 4 & 2 & 4 \end{bmatrix}$	$\begin{bmatrix} 17 & -3 & 2 & 9 \\ -3 & 6 & -4 & 1 \\ 2 & -4 & 13 & 10 \\ 9 & 1 & 10 & 17 \end{bmatrix}$	$\begin{bmatrix} 9 & -3 & 9 & 3 \\ -3 & 10 & 6 & -7 \\ 9 & 6 & 18 & -3 \\ 3 & -7 & -3 & 5 \end{bmatrix}$
(4, 4):	$\begin{bmatrix} 18 & 3 & 9 & 6 \\ 3 & 5 & -1 & -3 \\ 9 & -1 & 13 & 7 \\ 6 & -3 & 7 & 6 \end{bmatrix}$	$\begin{bmatrix} 17 & -10 & 4 & 3 \\ -10 & 14 & -1 & -3 \\ 4 & -1 & 5 & -4 \\ 3 & -3 & -4 & 6 \end{bmatrix}$	$\begin{bmatrix} 8 & 6 & 10 & 10 \\ 6 & 18 & 6 & 15 \\ 10 & 6 & 14 & 9 \\ 10 & 15 & 9 & 22 \end{bmatrix}$	$\begin{bmatrix} 8 & -4 & 8 & 0 \\ -4 & 10 & -4 & 0 \\ 8 & -4 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
(6, 6):	$\begin{bmatrix} 10 & 8 & 2 & 6 \\ 8 & 14 & 0 & 2 \\ 2 & 0 & 5 & 7 \\ 6 & 2 & 7 & 11 \end{bmatrix}$	$\begin{bmatrix} 11 & -6 & 10 & 9 \\ -6 & 10 & -5 & -5 \\ 10 & -5 & 14 & 11 \\ 9 & -5 & 11 & 9 \end{bmatrix}$	$\begin{bmatrix} 6 & 2 & 6 & -5 \\ 2 & 9 & 2 & 0 \\ 6 & 2 & 6 & -5 \\ -5 & 0 & -5 & 5 \end{bmatrix}$	$\begin{bmatrix} 8 & 6 & 2 & -2 \\ 6 & 9 & 9 & 6 \\ 2 & 9 & 13 & 12 \\ -2 & 6 & 12 & 13 \end{bmatrix}$
(8, 8):	$\begin{bmatrix} 5 & 3 & -3 & -4 \\ 3 & 6 & -3 & -2 \\ -3 & -3 & 6 & 4 \\ -4 & -2 & 4 & 4 \end{bmatrix}$	$\begin{bmatrix} 19 & 10 & 12 & 17 \\ 10 & 14 & 10 & 7 \\ 12 & 10 & 10 & 11 \\ 17 & 7 & 11 & 17 \end{bmatrix}$	$\begin{bmatrix} 5 & 1 & 3 & -3 \\ 1 & 5 & -7 & -1 \\ 3 & -7 & 22 & 7 \\ -3 & -1 & 7 & 10 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 4 & 4 \\ 2 & 2 & 4 & 8 \end{bmatrix}$
(10, 10):	$\begin{bmatrix} 18 & 6 & 6 & -6 \\ 6 & 2 & 2 & -2 \\ 6 & 2 & 2 & -2 \\ -6 & -2 & -2 & 4 \end{bmatrix}$	$\begin{bmatrix} 4 & -6 & 6 & 4 \\ -6 & 13 & -9 & -8 \\ 6 & -9 & 9 & 6 \\ 4 & -8 & 6 & 5 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 4 & 0 & 6 \\ -3 & 0 & 9 & 0 \\ 0 & 6 & 0 & 9 \end{bmatrix}$	$\begin{bmatrix} 9 & -3 & 0 & 0 \\ -3 & 10 & 9 & -6 \\ 0 & 9 & 9 & -6 \\ 0 & -6 & -6 & 4 \end{bmatrix}$
(2, 0):	$\begin{bmatrix} 20 & 6 & -14 & -4 \\ 6 & 18 & 3 & -12 \\ -14 & 3 & 17 & -2 \\ -4 & -12 & -2 & 8 \end{bmatrix}$	$\begin{bmatrix} 54 & -27 & 16 & 12 \\ -27 & 18 & -2 & -15 \\ 16 & -2 & 20 & -10 \\ 12 & -15 & -10 & 21 \end{bmatrix}$	$\begin{bmatrix} 42 & -8 & 9 & -3 \\ -8 & 10 & 5 & -11 \\ 9 & 5 & 29 & 7 \\ -3 & -11 & 7 & 29 \end{bmatrix}$	$\begin{bmatrix} 0 & 9 & 3 & -3 \\ 9 & -9 & -6 & 6 \\ 3 & -6 & -3 & 3 \\ -3 & 6 & 3 & -3 \end{bmatrix}$
(4, 2):	$\begin{bmatrix} 9 & -4 & 1 & 1 \\ -4 & 5 & -3 & -2 \\ 1 & -3 & 3 & 1 \\ 1 & -2 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 6 & 1 & 3 & 4 \\ 1 & 5 & 5 & 2 \\ 3 & 5 & 6 & 2 \\ 4 & 2 & 2 & 8 \end{bmatrix}$	$\begin{bmatrix} 8 & 2 & -6 & 4 \\ 2 & 5 & 1 & 3 \\ -6 & 1 & 6 & -2 \\ 4 & 3 & -2 & 3 \end{bmatrix}$	$\begin{bmatrix} -4 & 4 & -2 & 2 \\ 4 & 0 & 0 & -2 \\ -2 & 0 & 0 & 1 \\ 2 & -2 & 1 & -1 \end{bmatrix}$

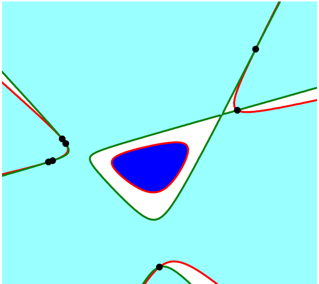
...... etc .... etc ....

Transversal Spectrahedra





$(\beta, \sigma) = (2,2), (8,10), (0,10)$



Back to Cayley (1869)

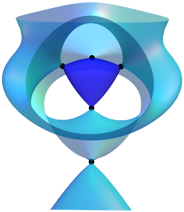
Theorem

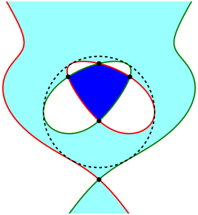
Let p be a node on an irreducible quartic surface $S \subset \mathbb{P}^3$.

The following are equivalent:

- *S is a symmetroid and p corresponds to a rank 2 matrix.*
- *The projection of S from p to $\mathbb{P}^2$ is branched along two cubics C_1 and C_2 that are totally tangent to a common conic Q.*

The pair (p, S) is real if and only if the pair $(C_1 \cup C_2, Q)$ is real. If this holds, then C_1 and C_2 are both real if and only if p is not on the spectrahedron. Equivalently, p is a node on the spectrahedron if and only if the cubic curves C_1 and C_2 are complex conjugates.





A Room with a View

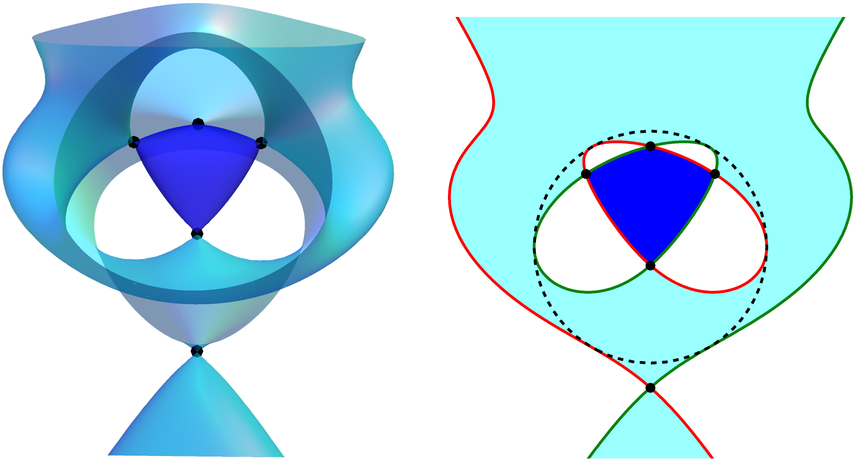


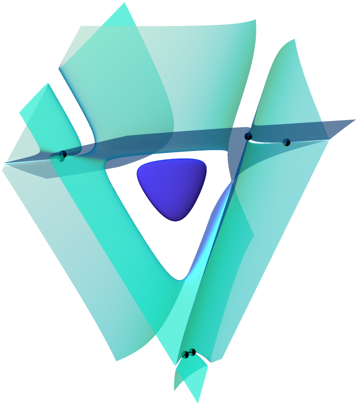
Figure: A quartic spectrahedron and its projection from an outside node. The ramification curve consists of two cubics totally tangent to a conic.

Got Low Rank?

This quartic spectrahedron has $\beta = 0$ and $\sigma = 10$:

$$\begin{bmatrix} 263x-160y-20z-187w & -3x-132y+28z+78w & -114x-30y+4z-76w & 103x+244y+32z-192w \\ -3x-132y+28z+78w & 45x+28y-32z-32w & -35x+40y-32z+24w & 48x+20y-4z+88w \\ -114x-30y+4z-76w & -35x+40y-32z+24w & 275x+25y+96z+80w & -55x-40y-156z-192w \\ 103x+244y+32z-192w & 48x+20y-4z+88w & -55x-40y-156z-192w & 278x-132y+180z-80w \end{bmatrix}$$

Set $x+y+z+w = 1$.



Q: What will semidefinite optimization do for this instance?

What is the rank of the optimal matrix?

Extended Formulations

A *spectrahedral shadow* is a convex set of the form

$$S \;=\; \Big\{ (x_1,\ldots,x_d) \in \mathbb{R}^d \;\Big|\; \exists\, (y_1,\ldots,y_p) \in \mathbb{R}^p : \sum_{i=1}^{d} x_i A_i + \sum_{j=1}^{p} y_j B_j + C \;\succeq\; 0 \Big\}.$$

Here $A_1,\ldots,A_d$, $B_1,\ldots,B_p$ and C are symmetric $n \times n$ matrices. The symbol "$\succeq$" means that the matrix is positive semidefinite.

If $p = 0$ then S is a spectrahedron, i.e. the intersection of the cone of positive definite matrices with an affine-linear space.

Extended Formulations

A *spectrahedral shadow* is a convex set of the form

$$S \;=\; \Big\{ (x_1,\ldots,x_d) \in \mathbb{R}^d \;\Big|\; \exists\,(y_1,\ldots,y_p) \in \mathbb{R}^p : \\ \textstyle\sum_{i=1}^d x_i A_i + \sum_{j=1}^p y_j B_j + C \;\succeq\; 0 \Big\}.$$

Here $A_1,\ldots,A_d$, $B_1,\ldots,B_p$ and C are symmetric $n \times n$ matrices. The symbol "$\succeq$" means that the matrix is positive semidefinite.

If $p = 0$ then S is a spectrahedron, i.e. the intersection of the cone of positive definite matrices with an affine-linear space.

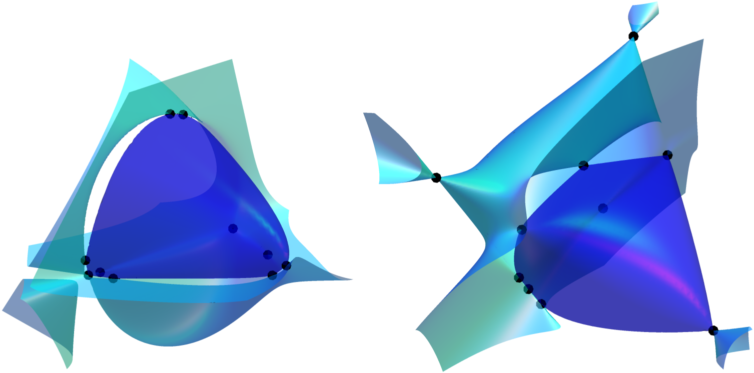
Spectrahedral shadows are projections of spectrahedra.

In Convex Algebraic Geometry, it is conjectured that every closed convex semialgebraic subset of $\mathbb{R}^d$ is a spectrahedral shadow.

For a *generic spectrahedral shadow*, the matrices A_i, B_j, C are generic. They lie outside a certain discriminantal hypersurface.

Ramification

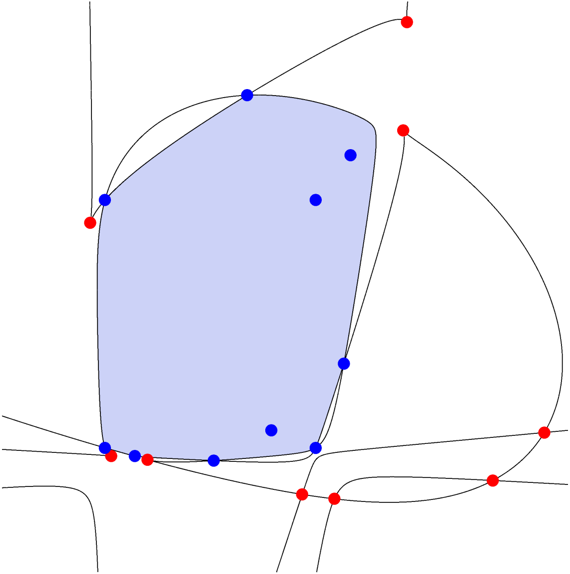
There are twenty generic types for $n = 4$, $d = 3$, $p = 0$.
[Degtyarev-Itenberg 2010], [Ottem-Ranestad-St-Vinzant 2013]



What do you get by projecting these from $\mathbb{R}^3$ into the plane $\mathbb{R}^2$?

What is the degree and number of singular points of the boundary curve of a generic spectrahedral shadow for $n = 4, d = 2, p = 1$?

Shadow

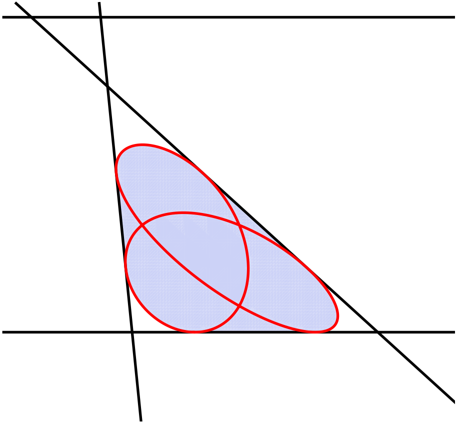


$n = 4, d = 2, p = 1$:

The algebraic boundary is a curve of degree **12** with 46 singular points.

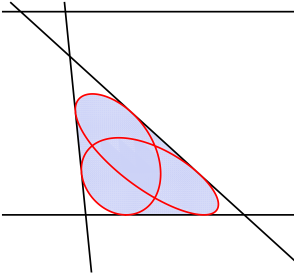
Questions

Let S be a generic spectrahedral shadow of type (n, d, p).



- What ranks occur in the boundary of S?
- How many irreducible components are there in the algebraic boundary of S?
- What are the degrees of these hypersurfaces?

Algebra



Let S be the spectrahedral shadow of type $(3, 2, 2)$ defined by

$$\begin{pmatrix} y_1 & x_1 & x_2 \\ x_1 & y_2 & -x_1 - \frac{6}{5}x_2 - \frac{7}{10} \\ x_2 & -x_1 - \frac{6}{5}x_2 - \frac{7}{10} & 2 - y_1 - y_2 \end{pmatrix} \succeq 0.$$

Its algebraic boundary consists of two curves of degree 4:

For rank 1 we get the irreducible quartic

$$100x_1^4 + 240x_1^3x_2 + 344x_1^2x_2^2 + 240x_1x_2^3 + 144x_2^4 + 140x_1^3 \\ +368x_1^2x_2 + 380x_1x_2^2 + 168x_2^3 + 49x_1^2 + 140x_1x_2 + 49x_2^2.$$

For rank 2 we get the reducible quartic

$$(2x_2 - 3)(22x_2 + 17)(20x_1 + 2x_2 + 17)(20x_1 + 22x_2 - 3).$$

Answers

Let $\delta(m, n, r)$ denote the *algebraic degree of semidefinite programming*, as defined in [Nie-Ranestad-Sturmfels 2010], and computed in [von Bothmer-Ranestad 2009].

Theorem

Let S be a generic spectrahedral shadow of type (n, d, p). The rank r of any general point in the boundary of S satisfies

$$\binom{n-r+1}{2} \leq p+1 \text{ and } \binom{r+1}{2} \leq \binom{n+1}{2} - (p+1).$$

The points of rank r form an irreducible component of the algebraic boundary of S. The degree of that hypersurface is independent of d, and it is equal to $\delta(p+1, n, r)$.

Quiz: What does this mean for $d = 1$?

Setting $p = m - 1$ in the inequalities gives the *Pataki range* in SDP.

Numbers

Degrees of the boundary components of generic spectrahedral shadows:

	$n=3$		$n=4$		$n=5$		$n=6$		$n=7$		$n=8$		$n=9$		$n=10$	
p	r	deg	r	deg	r	deg	r	deg	r	deg	r	deg	r	deg	r	deg
1	2	6	3	12	4	20	5	30	6	42	7	56	8	72	9	90
2	2	4	3	16	4	40	5	80	6	140	7	224	8	336	9	480
	1	4	2	10	3	20	4	35	5	56	6	84	7	120	8	165
3	1	6	3	8	4	40	5	120	6	280	7	560	8	1008	9	1680
			2	30	3	90	4	210	5	420	6	756	7	1260	8	1980
4	1	3	2	42	4	16	5	96	6	336	7	896	8	2016	9	4032
					3	207	4	672	5	1722	6	3780	7	7434	8	13464
5			2	30	3	290	5	32	6	224	7	896	8	2688	9	6720
			1	8	2	35	4	1400	5	4760	6	13020	7	30660	8	64680
							3	112	4	294	5	672	6	1386	7	2640
6			2	10	3	260	4	2040	6	64	7	512	8	2304	9	7680
			1	16	2	140	3	672	5	9600	6	33540	7	96120	8	238920
									4	2352	5	6720	6	16632	7	36960
7			1	12	3	140	4	2100	5	14532	7	128	8	1152	9	5760
					2	260	3	1992	4	9576	6	66948	7	238140	8	706860
											5	34800	6	104544	7	273240
8			1	4	3	35	4	1470	5	16485	6	104692	8	256	9	2560
					2	290	3	3812	4	25998	5	122400	7	474145	8	1708630
													6	451638	7	1399860
9					2	207	4	630	5	13650	6	127596	7	761364	9	512
					1	16	3	5184	4	52143	5	324624	6	1490049	8	3401574
							2	126	3	672	4	2772	5	9504	7	5524728
															6	28314

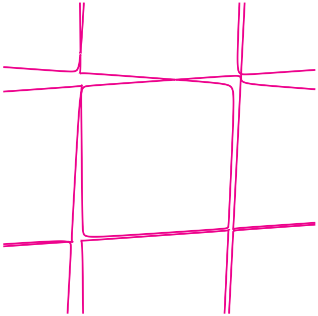
Punchline: these degrees are independent of $d = \dim(S)$

Degeneration

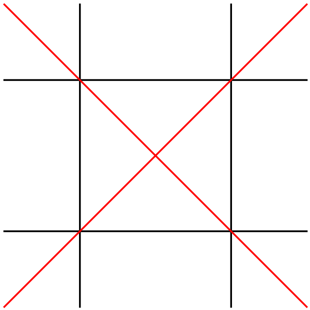
A spectrahedral shadow of type $(4, 2, 1)$ with a parameter ϵ:

$$\begin{pmatrix} 1 & x_1 & x_2 & y \\ x_1 & 1 & y & x_2 \\ x_2 & y & 1 & x_1 \\ y & x_2 & x_1 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 2y & 0 & 0 & 0 \\ 0 & 3x_2 & 0 & 0 \\ 0 & 0 & 5y & 0 \\ 0 & 0 & 0 & -7x_1 \end{pmatrix}.$$

Our curve of degree 12 degenerates to a square:



$\Longrightarrow$



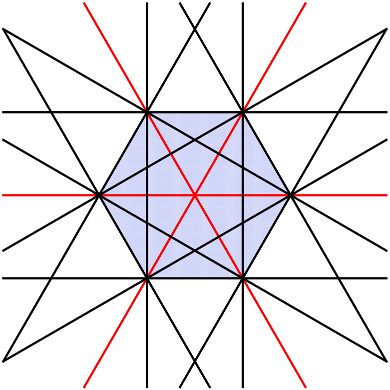
$\epsilon = 1/50$ $\epsilon = 0$

Pablo

The **regular hexagon** as a spectrahedral shadow of type $(4, 2, 3)$:

$$\begin{pmatrix} 1 & x_1 & x_2 & y_3 \\ x_1 & \frac{1}{2}(1+y_1) & \frac{1}{2}y_2 & y_1 \\ x_2 & \frac{1}{2}y_2 & \frac{1}{2}(1-y_1) & -y_2 \\ y_3 & y_1 & -y_2 & 1 \end{pmatrix} \succeq 0.$$

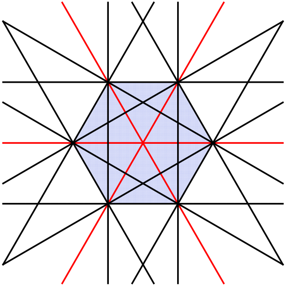
This matrix is due to Hamza Fawzi and James Saunderson.



Deformation

type (4, 2, 3)

$$\begin{pmatrix} 1 & x_1 & x_2 & y_3 \\ x_1 & \frac{1}{2}(1+y_1) & \frac{1}{2}y_2 & y_1 \\ x_2 & \frac{1}{2}y_2 & \frac{1}{2}(1-y_1) & -y_2 \\ y_3 & y_1 & -y_2 & 1 \end{pmatrix}$$

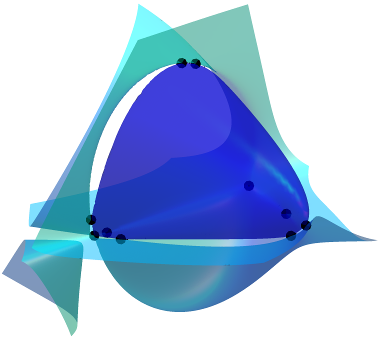


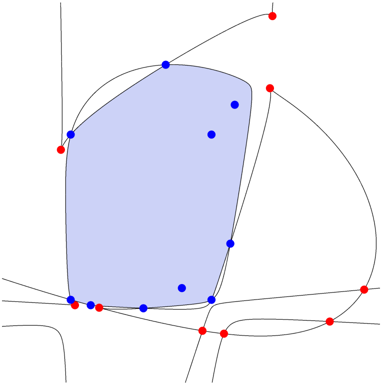
Curve of degree 8 ?
Curve of degree 30 ?

	$n=3$		$n=4$		$n=5$		$n=6$		$n=7$		$n=8$		$n=9$		$n=10$	
p	r	deg	r	deg	r	deg	r	deg	r	deg	r	deg	r	deg	r	deg
1	2	6	3	12	4	20	5	30	6	42	7	56	8	72	9	90
2	2	4	3	16	4	40	5	80	6	140	7	224	8	336	9	480
	1	4	2	10	3	20	4	35	5	56	6	84	7	120	8	165
3	1	6	3	8	4	40	5	120	6	280	7	560	8	1008	9	1680
			2	30	3	90	4	210	5	420	6	756	7	1260	8	1980
4	1	3	2	42	4	16	5	96	6	336	7	896	8	2016	9	4032
					3	207	4	672	5	1722	6	3780	7	7434	8	13464
5			2	30	3	290	5	32	6	224	7	896	8	2688	9	6720
			1	8	2	35	4	1400	5	4760	6	13020	7	30660	8	64680
							3	112	4	294	5	672	6	1386	7	2640

Conclusion of the paper with Rainer Sinn

From the algebra perspective, we now understand the geometry of spectrahedral shadows S when the given matrices are generic.





In applications, the given matrices A_i, B_j, C are very special.
To understand those S, lots and lots of work is still required.