Plethysm and Lattice Point Counting

Mateusz Michałek*, Thomas Kahle+

*Polish Academy of Sciences
+Otto-von-Guericke Universität, Germany

Geometric Complexity Theory

arXiv:1408.5708

Representations of GL and SL

$$\{\text{Irreducible representations of } SL(n)\}$$

$$\leftrightarrow$$

$$\{\text{Young diagrams with at most } n-1 \text{ rows}\}$$

Representations of GL and SL

$$\{\text{Irreducible representations of } SL(n)\}$$

$$\leftrightarrow$$

$$\{\text{Young diagrams with at most } n-1 \text{ rows}\}$$

$$\{\text{Irreducible representations of } GL(n)\}$$

$$\leftrightarrow$$

$$\{\text{Young diagrams with at most } n-1 \text{ rows and an integer}\}$$

Representations of GL and SL

$$\{\text{Irreducible representations of } SL(n)\}$$

$$\leftrightarrow$$

$$\{\text{Young diagrams with at most } n-1 \text{ rows}\}$$

$$\{\text{Irreducible representations of } GL(n)\}$$

$$\leftrightarrow$$

$$\{\text{Young diagrams with at most } n-1 \text{ rows and an integer}\}$$

The correspondence is very classical using Schur functors.

Representations of GL and SL

$$\{\text{Irreducible representations of } SL(n)\}$$

$$\leftrightarrow$$

$$\{\text{Young diagrams with at most } n-1 \text{ rows}\}$$

$$\{\text{Irreducible representations of } GL(n)\}$$

$$\leftrightarrow$$

$$\{\text{Young diagrams with at most } n-1 \text{ rows and an integer}\}$$

The correspondence is very classical using Schur functors. To any Young diagram λ one associates a Schur functor S^λ. The irreducible representation of $SL(V)$ is $S^\lambda(V)$.

Representations of GL and SL

$$\{\text{Irreducible representations of } SL(n)\}$$

$$\leftrightarrow$$

$$\{\text{Young diagrams with at most } n-1 \text{ rows}\}$$

$$\{\text{Irreducible representations of } GL(n)\}$$

$$\leftrightarrow$$

$$\{\text{Young diagrams with at most } n-1 \text{ rows and an integer}\}$$

The correspondence is very classical using Schur functors. To any Young diagram λ one associates a Schur functor S^λ. The irreducible representation of $SL(V)$ is $S^\lambda(V)$.

What is it? We take the tensor power of V (to the number of boxes in λ) and we act with Young symmetrizer on it (that is we permute with some coefficients the tensors).

Representations of GL and SL - examples

Of course we can apply S^{λ} to any other representation (not only the trivial one) using the same construction. In this case we will usually obtain a *reducible* representation.

Representations of GL and SL - examples

Of course we can apply S^λ to any other representation (not only the trivial one) using the same construction. In this case we will usually obtain a *reducible* representation.

Example

We can consider the symmetric power of a wedge power $S^d(\bigwedge^k(V))$. This is an example of *plethysm*. In general, the composition of two Schur functors $S^\mu(S^\lambda)$ is called the plethysm.

Representations of GL and SL - examples

Of course we can apply S^λ to any other representation (not only the trivial one) using the same construction. In this case we will usually obtain a *reducible* representation.

Example

We can consider the symmetric power of a wedge power $S^d(\bigwedge^k(V))$. This is an example of *plethysm*. In general, the composition of two Schur functors $S^\mu(S^\lambda)$ is called the plethysm.

$$S^4(S^3(\mathbb{C}^3)) = S^{4,4,4} + \dots$$

Aims

What is the combinatorial formula for plethysm?

Aims

What is the combinatorial formula for plethysm?

This is the hard part. There is no good, combinatorial formula. The problem is (very) open even for $S^d(S^k(V))$.

Aims

What is the combinatorial formula for plethysm?
This is the hard part. There is no good, combinatorial formula. The problem is (very) open even for $S^d(S^k(V))$.
Aim: Understand polynomials in coefficients of polynomials

Aims

What is the combinatorial formula for plethysm?
This is the hard part. There is no good, combinatorial formula. The problem is (very) open even for $S^d(S^k(V))$.
Aim: Understand polynomials in coefficients of polynomials
Modest, reasonable aim: Understand low degree polynomials in coefficients of arbitrary polynomials

Aims

What is the combinatorial formula for plethysm?
This is the hard part. There is no good, combinatorial formula. The problem is (very) open even for $S^d(S^k(V))$.
Aim: Understand polynomials in coefficients of polynomials
Modest, reasonable aim: Understand low degree polynomials in coefficients of arbitrary polynomials
Not much is known...

Notable exceptions

Proposition (Thrall 1942)

One has $GL(W)$-modules decompositions

$$S^2(S^nW) = \bigoplus S^\lambda W, \qquad \bigwedge{}^2(S^nW) = \bigoplus S^\delta W,$$

where the first sum runs over representations corresponding to λ of weight $2n$ with two rows of even length and the second sum runs over representations corresponding to δ of weight $2n$ with two rows of odd length.

Notable exceptions

Proposition (Thrall 1942)

One has $GL(W)$-modules decompositions

$$S^2(S^nW) = \bigoplus S^\lambda W, \qquad \bigwedge^2(S^nW) = \bigoplus S^\delta W,$$

where the first sum runs over representations corresponding to λ of weight $2n$ with two rows of even length and the second sum runs over representations corresponding to δ of weight $2n$ with two rows of odd length.

There are similar formulas for $S^3(S^k)$ obtained by Agaoka, Chen, Duncan, Foulkes, Garsia, Howe, Plunkett, Remmel, Thrall and others.

Notable exceptions

Proposition (Thrall 1942)

One has $GL(W)$-modules decompositions

$$S^2(S^nW) = \bigoplus S^\lambda W, \qquad \bigwedge{}^2(S^nW) = \bigoplus S^\delta W,$$

where the first sum runs over representations corresponding to λ of weight $2n$ with two rows of even length and the second sum runs over representations corresponding to δ of weight $2n$ with two rows of odd length.

There are similar formulas for $S^3(S^k)$ obtained by Agaoka, Chen, Duncan, Foulkes, Garsia, Howe, Plunkett, Remmel, Thrall and others. Some approaches to the formulas for $S^4(S^k)$ are presented.

Notable exceptions

Proposition (Thrall 1942)

One has $GL(W)$-modules decompositions

$$S^2(S^nW) = \bigoplus S^\lambda W, \qquad \bigwedge^2(S^nW) = \bigoplus S^\delta W,$$

where the first sum runs over representations corresponding to λ of weight $2n$ with two rows of even length and the second sum runs over representations corresponding to δ of weight $2n$ with two rows of odd length.

There are similar formulas for $S^3(S^k)$ obtained by Agaoka, Chen, Duncan, Foulkes, Garsia, Howe, Plunkett, Remmel, Thrall and others. Some approaches to the formulas for $S^4(S^k)$ are presented.
Problem: the formulas are more and more complicated.

Characters

Characters are (symmetric) polynomials that are associated to (and completely determine) a representation.

Characters

Characters are (symmetric) polynomials that are associated to (and completely determine) a representation. The trace of a representation provides a function that is a symmetric polynomial in the eigenvalues.

Characters

Characters are (symmetric) polynomials that are associated to (and completely determine) a representation. The trace of a representation provides a function that is a symmetric polynomial in the eigenvalues. Traces of $S^\lambda(V)$ are called Schur polynomials. They form a basis of the space of symmetric polynomials in $\dim V$ variables.

Characters

Characters are (symmetric) polynomials that are associated to (and completely determine) a representation. The trace of a representation provides a function that is a symmetric polynomial in the eigenvalues. Traces of $S^\lambda(V)$ are called Schur polynomials. They form a basis of the space of symmetric polynomials in $\dim V$ variables.

Example

Consider $V =< e_1, e_2 >$. The trace of the action of the diagonal matrix with eigenvalues x_1, x_2 on $S^2(V)$ equals $x_1^2 + x_1 x_2 + x_2^2$.

Characters

Characters are (symmetric) polynomials that are associated to (and completely determine) a representation. The trace of a representation provides a function that is a symmetric polynomial in the eigenvalues. Traces of $S^\lambda(V)$ are called Schur polynomials. They form a basis of the space of symmetric polynomials in $\dim V$ variables.

Example

Consider $V =< e_1, e_2 >$. The trace of the action of the diagonal matrix with eigenvalues x_1, x_2 on $S^2(V)$ equals $x_1^2 + x_1 x_2 + x_2^2$.

Decomposing a representation is equivalent to decomposing its character. Precisely:

$$W = \sum (S^\lambda V)^{\oplus a_\lambda} \text{if and only if} P_W = \sum a_\lambda P_\lambda,$$

where P_W is the character of W and P_λ are Schur polynomials.

Algebraic approach to plethysm

Plethysm can be defined on the level of symmetric polynomials.

Proposition

For any symmetric polynomial f, the mapping $g \to g \circ f$ is an endomorphism of the ring of symmetric polynomials. For any $n \in \mathbb{N}$, the mapping $g \to \psi_n \circ g$ is an endomorphism of the ring of symmetric polynomials. Moreover,

$$\psi_n \circ g = g \circ \psi_n = g(x_1^n, x_2^n, \dots).$$

This provides an explicit formula for the character of plethysm.

Character of plethysm

Proposition

For a partition μ of d the character of the representation $S^\mu(S^k W)$ equals

$$P_{S^\mu(S^k W)} = \sum \chi_\mu(\alpha) \frac{D_\alpha}{d!} \psi_\alpha \circ h_k,$$

where the sum is taken over all partitions α of d and D_α is the number of permutations of cycle type α in the group S_d.

Character of plethysm

Proposition

For a partition μ of d the character of the representation $S^\mu(S^k W)$ equals

$$P_{S^\mu(S^k W)} = \sum \chi_\mu(\alpha) \frac{D_\alpha}{d!} \psi_\alpha \circ h_k,$$

Proof.

Instead of presenting Schur polynomials in monomial basis express them in power series:

$$P_\mu = \sum_{\alpha \vdash d} \frac{D_\alpha}{d!} \chi_\mu(\alpha) \psi_\alpha.$$

Substitute in $P_\mu \circ h$ and apply previous remarks. □

Character of plethysm

Proposition

For a partition μ of d the character of the representation $S^\mu(S^kW)$ equals

$$P_{S^\mu(S^kW)} = \sum \chi_\mu(\alpha)\frac{D_\alpha}{d!}\psi_\alpha \circ h_k,$$

Proof.

Instead of presenting Schur polynomials in monomial basis express them in power series:

$$P_\mu = \sum_{\alpha \vdash d} \frac{D_\alpha}{d!}\chi_\mu(\alpha)\psi_\alpha.$$

Substitute in $P_\mu \circ h$ and apply previous remarks. □

Here, $\psi_\alpha \circ h_k$ is just a product of complete symmetric polynomials in powers of variables.

Linear algebra?

Decomposing plethysm is equivalent to presenting $\psi_\alpha \circ h_k$ as a linear combination of Schur polynomials.

Linear algebra?

Decomposing plethysm is equivalent to presenting $\psi_\alpha \circ h_k$ as a linear combination of Schur polynomials.
Why this problem is not completely solved?

Linear algebra?

Decomposing plethysm is equivalent to presenting $\psi_\alpha \circ h_k$ as a linear combination of Schur polynomials.
Why this problem is not completely solved?
The combinatorial formulas become more and more complicated.

Linear algebra?

Decomposing plethysm is equivalent to presenting $\psi_\alpha \circ h_k$ as a linear combination of Schur polynomials.

Why this problem is not completely solved?

The combinatorial formulas become more and more complicated.

What can we do?

Linear algebra?

Decomposing plethysm is equivalent to presenting $\psi_\alpha \circ h_k$ as a linear combination of Schur polynomials.

Why this problem is not completely solved?

The combinatorial formulas become more and more complicated.

What can we do?

There are two possibilities to approach such problems:

Linear algebra?

Decomposing plethysm is equivalent to presenting $\psi_\alpha \circ h_k$ as a linear combination of Schur polynomials.

Why this problem is not completely solved?

The combinatorial formulas become more and more complicated.

What can we do?

There are two possibilities to approach such problems:

1. provide explicit answers using methods, that mathematicians did not use before,
2. hope for weaker qualitative results.

Computational approach

We reduce computation of plethysm to computations of the number of fibers in projections of (many) convex polyhedral cones.

Computational approach

We reduce computation of plethysm to computations of the number of fibers in projections of (many) convex polyhedral cones.
More precisely, we have to compute the number of 'magic' rectangular tables, with nonnegative integral entries, where row and column sums are parameters.

Definition ((α, λ)-matrix)

Fix partitions α, λ and suppose that α has a parts. An $a \times (d-1)$ matrix M with nonnegative integral entries is an (α, λ)-matrix if

1. *each row sums up to k, i.e. $\sum_{j=1}^{d-1} M_{i,j} = k$ for each $1 \leq i \leq a$, and*
2. *the α-weighted entries of the j-th column sum up to λ_j, i.e. $\sum_{i=1}^{a} \alpha_i M_{i,j} = \lambda_j$ for each $1 \leq j \leq d-1$.*

Computational approach

This is a perfect job for a computer! (We use Barvinok program through isl library)

Computational approach

This is a perfect job for a computer! (We use Barvinok program through isl library)
We obtain completely explicit formulas for $S^{\mu}(S^k)$ where μ is any partition of 4 or 5.
(Hence also for $S^{\mu}(\bigwedge^k)$)

Computational approach

This is a perfect job for a computer! (We use Barvinok program through isl library)

We obtain completely explicit formulas for $S^{\mu}(S^{k})$ where μ is any partition of 4 or 5.

(Hence also for $S^{\mu}(\bigwedge^{k})$)

The result is a piecewise quasipolynomial. That is we obtain a chamber decomposition of the parameter space and in each chamber the result is a quasipolynomial, that is a polynomial with coefficients that depend on the rest of the parameter modulo a fixed number.

Computational approach

This is a perfect job for a computer! (We use Barvinok program through isl library)

We obtain completely explicit formulas for $S^{\mu}(S^k)$ where μ is any partition of 4 or 5.

(Hence also for $S^{\mu}(\bigwedge^k)$)

The result is a piecewise quasipolynomial. That is we obtain a chamber decomposition of the parameter space and in each chamber the result is a quasipolynomial, that is a polynomial with coefficients that depend on the rest of the parameter modulo a fixed number.

Such chamber decomposition is already necessary for the case of cubics.

Assymptotics

The coefficient of S^λ inside $S^\mu(S^k)$ for fixed μ is a piecewise quasipolynomial in λ_i.

Assymptotics

The coefficient of S^{λ} inside $S^{\mu}(S^{k})$ for fixed μ is a piecewise quasipolynomial in λ_i.
What is the assymptotic bahaviour/the leading term?

Assymptotics

The coefficient of S^λ inside $S^\mu(S^k)$ for fixed μ is a piecewise quasipolynomial in λ_i.
What is the assymptotic bahaviour/the leading term?
There is a very natural candidate

$$\frac{\dim|\mu|}{d!}LR$$

observed already by Howe in '80 and identified by him for $S^4(S^k)$.

Assymptotics

The coefficient of S^λ inside $S^\mu(S^k)$ for fixed μ is a piecewise quasipolynomial in λ_i.
What is the assymptotic bahaviour/the leading term?
There is a very natural candidate

$$\frac{\dim|\mu|}{d!}LR$$

observed already by Howe in '80 and identified by him for $S^4(S^k)$.
We can prove that this is in fact the leading term (it is obvious that the term appears, it is not obvious that the 'error terms' are smaller).

Assymptotics

The coefficient of S^λ inside $S^\mu(S^k)$ for fixed μ is a piecewise quasipolynomial in λ_i.
What is the assymptotic bahaviour/the leading term?
There is a very natural candidate

$$\frac{\dim|\mu|}{d!}LR$$

observed already by Howe in '80 and identified by him for $S^4(S^k)$.
We can prove that this is in fact the leading term (it is obvious that the term appears, it is not obvious that the 'error terms' are smaller).
I will be more than happy to discuss other approaches/remarks after the talk.

Conclusions and future plans

Conclusions and future plans

- When the inner functor is fixed see 'Computing multiplicities of Lie group representations' Christandl, Doran, Walter

Conclusions and future plans

- When the inner functor is fixed see 'Computing multiplicities of Lie group representations' Christandl, Doran, Walter
- The multiplicities $k\lambda$ in $S^{\mu}(S^{kn})$ are a quasipolynomial in k.

Conclusions and future plans

- When the inner functor is fixed see 'Computing multiplicities of Lie group representations' Christandl, Doran, Walter
- The multiplicities $k\lambda$ in $S^{\mu}(S^{kn})$ are a quasipolynomial in k.
- Explicit results are on: http://www.thomas-kahle.de/plethysm.html

Conclusions and future plans

- When the inner functor is fixed see 'Computing multiplicities of Lie group representations' Christandl, Doran, Walter
- The multiplicities $k\lambda$ in $S^{\mu}(S^{kn})$ are a quasipolynomial in k.
- Explicit results are on: http://www.thomas-kahle.de/plethysm.html
- Symbolic evaluation

Conclusions and future plans

Example

The multiplicity of the isotypic component of

$$\lambda =$$

$$(616036908677580244, 1234567812345678, 12345671234567, 123456123456)$$

in $S^5(S^{123456789123456789})$ *equals*

$$240963570406235277976739158010615905293817243845463524159304407436599680700160 51.$$

The evaluation of our formula on this example takes under one second and this time is almost entirely constant overhead for dealing with the data structure. Evaluation on much larger arguments (for instance with a million digits) is almost as quick.

Conclusions and future plans

Example

The multiplicity of the isotypic component of

$$\lambda =$$

$$(616036908677580244, 1234567812345678, 12345671234567, 123456123456)$$

in $S^5(S^{123456789123456789})$ *equals*

$$24096357040623527797673915801061590529381724384546352415930440743659968070016051.$$

The evaluation of our formula on this example takes under one second and this time is almost entirely constant overhead for dealing with the data structure. Evaluation on much larger arguments (for instance with a million digits) is almost as quick.

Well... to be honest...

Thank you!