Fano Schemes of Determinants and Permanents

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The Space of Singular Matrices

$$D_n = \{M \in \mathrm{Mat}_{\mathbb{C}}(n,n) \mid \det M = 0\} \subset \mathbb{C}^{n^2}$$

Example

$$D_2 = \{x_{11}x_{22} - x_{12}x_{21} = 0\} \subset \mathbb{C}^4$$

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Question

What linear subspaces of $\mathbb{C}^{n^2}$ are contained in D_n?

Examples of Linear Spaces of Singular Matrices

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$$\begin{pmatrix} 0 & \cdots & 0 \\ * & \cdots & * \\ \vdots & & \vdots \\ * & \cdots & * \end{pmatrix} \subset D_n \qquad \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{pmatrix} \subset D_3$$

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Compression Spaces:
Let $V, W \subset \mathbb{C}^n$, $\dim W = s, \dim V = s+1$. Then

$$\{M \in \mathrm{Mat}_{\mathbb{C}}(n,n) \mid MV \subset W\} \subset D_n.$$

Motivation: Geometric Complexity Theory

Geometric approach to Valiant's conjecture (Mulmuley, Sohoni 2001): need to understand closure of $GL_{n^2} \det_n$.

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Maximal linear subspaces of D_n can be used to construct pieces of the boundary of $\overline{GL_{n^2} \det_n}$! (Landsberg, Manivel, Ressayre 2013)

Definition of the Fano Scheme

Definition

The *kth Fano Scheme* of D_n is

$$\mathbf{F}_k(D_n) = \left\{ L \subset \mathbb{C}^{n^2} \;\middle|\; \begin{array}{l} \dim L = k+1 \\ L \subset D_n \end{array} \right\} \subset \mathrm{Gr}(k+1, n^2).$$

Goal: Study the geometry of $\mathbf{F}_k(D_n)$.

Planes of Singular 2×2 Matrices

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$$\begin{pmatrix} \lambda_0 a & \lambda_0 b \\ \lambda_1 a & \lambda_1 b \end{pmatrix}, \begin{pmatrix} \mu_0 c & \mu_1 c \\ \mu_0 d & \mu_1 d \end{pmatrix} \in \mathbf{F}_1(D_2) \qquad \left(\mathbb{C}^2 \setminus \{0\}\right)/\mathbb{C}^* \cong \mathbb{CP}^1$$

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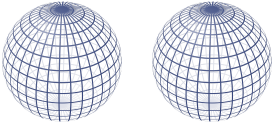
$$\rightsquigarrow \mathbb{CP}^1 \coprod \mathbb{CP}^1 \cong \mathbf{F}_1(D_2).$$



Planes of Singular $n \times n$ Matrices

Kronecker normal form $\rightsquigarrow$ any point of $\mathbf{F}_1(D_n)$ is contained in a compression space.

Theorem (Chan, —)

$\mathbf{F}_1(D_n) \subset \mathrm{Gr}(2, n^2)$ *has exactly* n *irreducible components, each of dimension* $2(n^2 - 2) - (n + 1)$*. This is the expected dimension.*

Higher Dimensional Linear Spaces

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$\mathbf{F}_k(D_n)$ *non-empty* $\iff k < n(n-1)$.

If $1 < k < n(n-1)$, then:

- The dimension of $\mathbf{F}_k(D_n)$ is almost never pure.
- Irreducible components are in general unknown; OK for $k \gg 0$ (Beasley 1987) or $n = 3, 4$ (Atkinson 1983).
- Other bad things ...

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For $0 \leq s \leq n-1$, set $\kappa(s) = n^2 - (n-s)(s+1) - 1$.

Theorem (Chan, —)

Let $1 \leq k < (n-1)n$. Then $\mathbf{F}_k(D_n)$ is disconnected iff

$$n^2 - 2n < k \leq \kappa(0)$$

or if there exists an integer s with $0 < s < n-1$ such that

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n	2	3	4	5	6	7	8
Connected iff $k \leq$ or $k =$	–	3	8	13	21 24	29 35	40 46–48

Compression Space Components

Definition

An s-compression space is the space of matrices compressing a fixed $s+1$-dimensional $V \subset \mathbb{C}^n$ into a fixed s-dimensional $W \subset \mathbb{C}^n$.

- $\kappa(s) = n^2 - (n-s)(s+1) - 1$ is the projective dimension of any s-compression space.
- $k+1$-dimensional subspaces of s-compression spaces form irreducible components of $\mathbf{F}_k(D_n)$.

Torus Fixed Points

$(\mathbb{C}^*)^n \times (\mathbb{C}^*)^n$ acts on D_n by scaling rows and columns.
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$\rightsquigarrow$ torus action on $\mathbf{F}_k(D_n)$.

- $\mathbf{F}_k(D_n)$ only has finitely many fixed points under this action.
- Each fixed point lies on a compression space component!

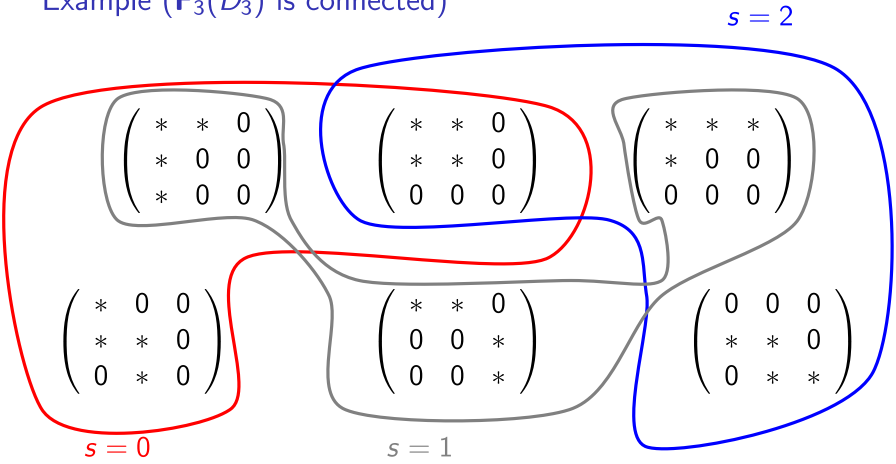
Example

$$\begin{pmatrix} * & * & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix} \in \mathbf{F}_3(D_3)$$

Argument for Connectedness

- Sufficient to connect compression space components! Do this at torus fixed points.

Example ($\mathbf{F}_3(D_3)$ is connected)



Argument for Disconnectedness

To prove disconnectedness:

- Exhibit a compression space component with smooth torus fixed points.

Example ($\mathbf{F}_4(D_3)$ is disconnected)

$s = 1$

$$\begin{pmatrix} * & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}$$

$s = 0$

$$\begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & * & * \\ * & * & * \\ 0 & 0 & 0 \end{pmatrix}$$

$s = 2$

Fano Schemes of Permanents

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$P_n = \{\mathrm{perm}_n = 0\} \subset \mathbb{C}^{n^2} \rightsquigarrow$ Fano scheme $\mathbf{F}_k(P_n)$.

- Partial characterization of connectedness of $\mathbf{F}_k(P_n)$.
- Understanding of some components of $\mathbf{F}_k(P_n)$.
- Component structure of $\mathbf{F}_k(P_n)$ appears "more complicated" than that of $\mathbf{F}_k(D_n)$: e.g. $\mathbf{F}_3(P_3)$ has 21 components, while $\mathbf{F}_3(D_3)$ has 3.

Product Rank of perm_3

Let f be a form of degree d. Its *product rank* $\mathbf{pr}(f)$ is the smallest r such that

$$f = \sum_{i=1}^{r} \prod_{j=1}^{d} l_{ij}$$

for some linear forms l_{ij}.

- $\mathbf{pr}(\mathrm{perm}_3) \leq 4$ (Glynn 2013).
- $\mathbf{pr}(\mathrm{perm}_3) > 3$ using structure of $\mathbf{F}_5(P_3)$:

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- $\mathbf{pr}(\mathrm{perm}_3) \leq 4$ (Glynn 2013).
- $\mathbf{pr}(\mathrm{perm}_3) > 3$ using structure of $\mathbf{F}_5(P_3)$: Suppose

$$\mathrm{perm}_3 = \sum_{i=1}^{3} \prod_{j=1}^{3} l_{ij}.$$

If l_{ij} are linearly dependent, then all maximal linear spaces $L \subset P_3$ contain a common line. ↯ But if l_{ij} are linearly independent, then $\mathbf{F}_5(P_3)$ contains 27 points. ↯