Approaches to bounding the exponent of matrix multiplication

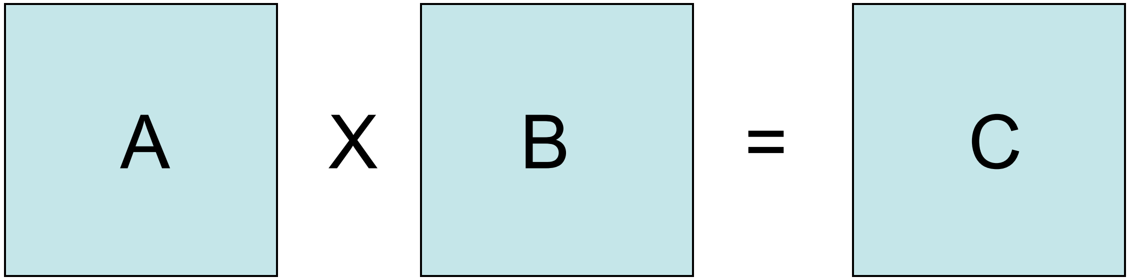
Chris Umans

Caltech

Based on joint work with Noga Alon, Henry Cohn, Bobby Kleinberg, Amir Shpilka, Balazs Szegedy

Simons Institute Sept. 17, 2014

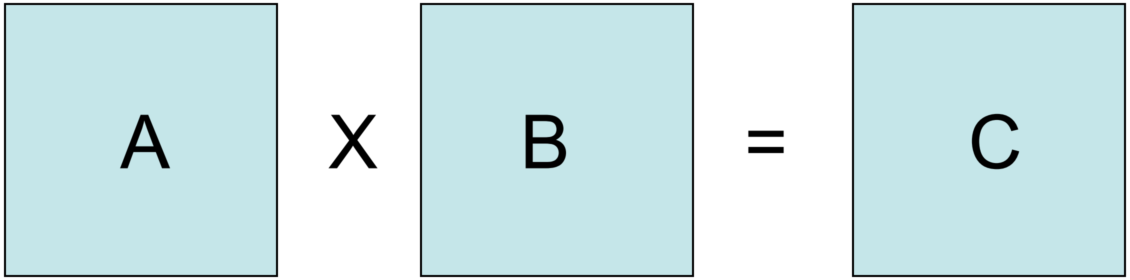
Introduction



A X B = C

- Standard method: $O(n^3)$ operations
- Strassen (1969): $O(n^{2.81})$ operations

Introduction



A X B = C

- Standard method: $O(n^3)$ operations
- Strassen (1969): $O(n^{2.81})$ operations

The exponent of matrix multiplication:
smallest number ω such that for all $\varepsilon>0$
$O(n^{\omega + \varepsilon})$ operations suffice

History

Standard algorithm	$\omega \leq 3$
Strassen (1969)	$\omega < 2.81$
Pan (1978)	$\omega < 2.79$
Bini; Bini et al. (1979)	$\omega < 2.78$
Schönhage (1981)	$\omega < 2.55$
Pan; Romani; Coppersmith + Winograd (1981-1982)	$\omega < 2.50$
Strassen (1987)	$\omega < 2.48$
Coppersmith + Winograd (1987)	$\omega < 2.375$
Stothers (2010)	$\omega < 2.3737$
Williams (2011)	$\omega < 2.3729$
Le Gall (2014)	$\omega < 2.37286$

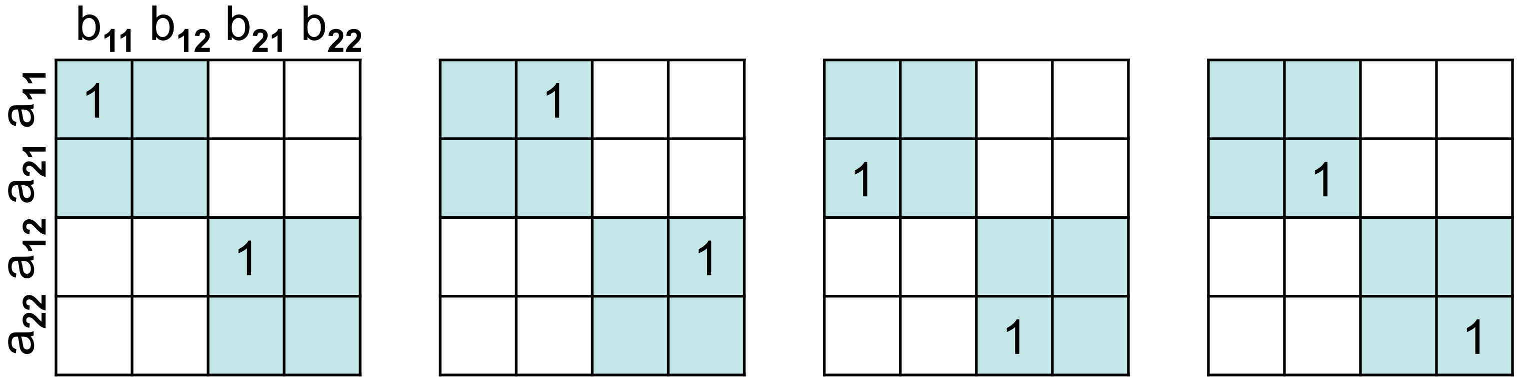
Outline

1. main ideas from Strassen 1969 through Le Gall 2014
2. approach via embedding into semi-simple algebra multiplication
   - groups
   - coherent configurations/association schemes

The matrix multiplication tensor

$\langle n,n,n \rangle$ is a $n^2 \times n^2 \times n^2$ tensor described by trilinear form $\sum_{i,j,k} X_{i,j} Y_{j,k} Z_{k,i}$

$$\begin{array}{|c|c|}\hline a_{11} & a_{12} \\ \hline a_{21} & a_{22} \\ \hline \end{array} \times \begin{array}{|c|c|}\hline b_{11} & b_{12} \\ \hline b_{21} & b_{22} \\ \hline \end{array} = \begin{array}{|c|c|}\hline c_{11} & c_{12} \\ \hline c_{21} & c_{22} \\ \hline \end{array}$$



$$\begin{array}{c|c|c|c|c|}
 & b_{11} & b_{12} & b_{21} & b_{22} \\ \hline
a_{11} & 1 & & & \\ \hline
a_{21} & & & & \\ \hline
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a_{22} & & & & \\ \hline
\end{array}
\quad
\begin{array}{|c|c|c|c|}\hline
 & 1 & & \\ \hline
 & & & \\ \hline
 & & & 1 \\ \hline
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\end{array}
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\begin{array}{|c|c|c|c|}\hline
 & & & \\ \hline
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\end{array}
\quad
\begin{array}{|c|c|c|c|}\hline
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The matrix multiplication tensor

<n,n,n> is a n^2 x n^2 x n^2 tensor described by trilinear form $\sum_{i,j,k} X_{i,j} Y_{j,k} Z_{k,i}$

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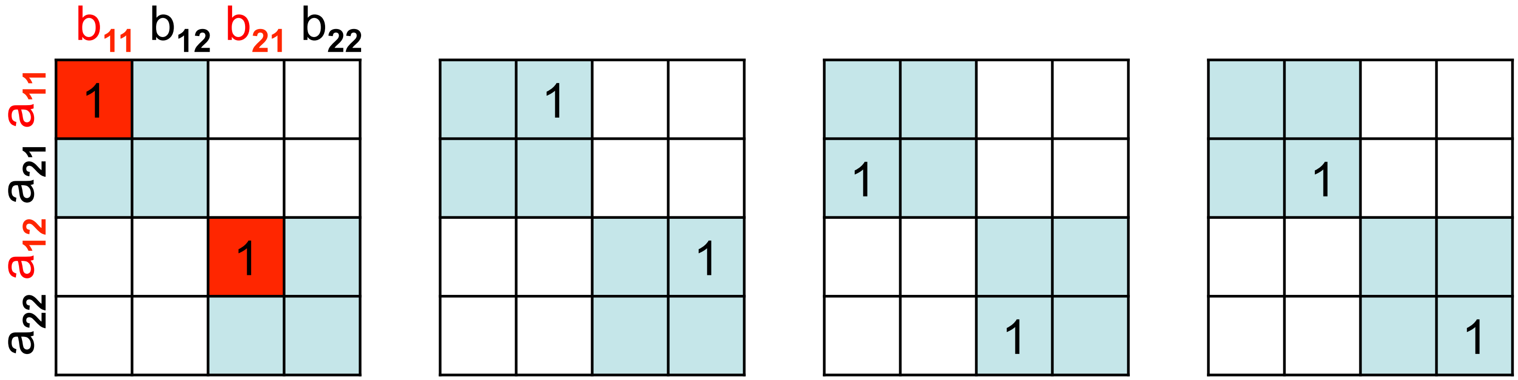

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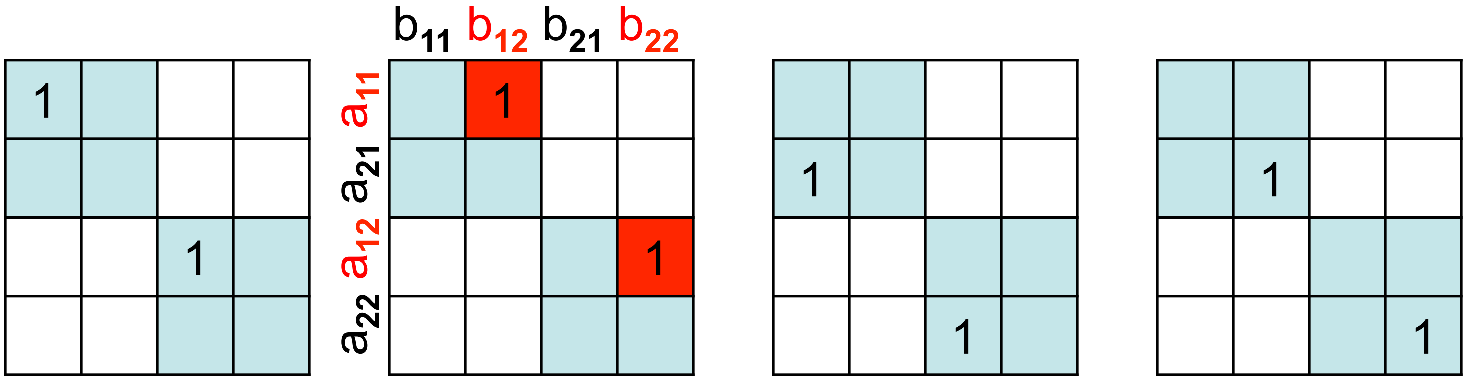

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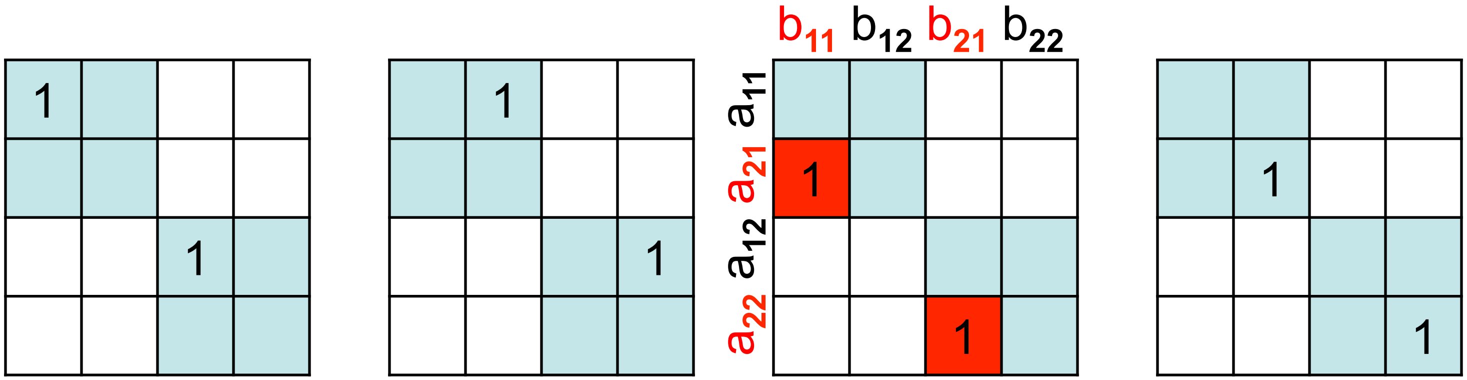

$$\begin{array}{|c|c|c|c|} \hline 1 & & & \\ \hline & & & \\ \hline & & 1 & \\ \hline & & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline & 1 & & \\ \hline & & & \\ \hline & & & 1 \\ \hline & & & \\ \hline \end{array} \quad \begin{array}{c|c|c|c|c|} & b_{11} & b_{12} & b_{21} & b_{22} \\ \hline a_{11} & & & & \\ \hline a_{21} & 1 & & & \\ \hline a_{12} & & & & \\ \hline a_{22} & & & 1 & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & 1 & & \\ \hline & & & \\ \hline & & & 1 \\ \hline \end{array}$$

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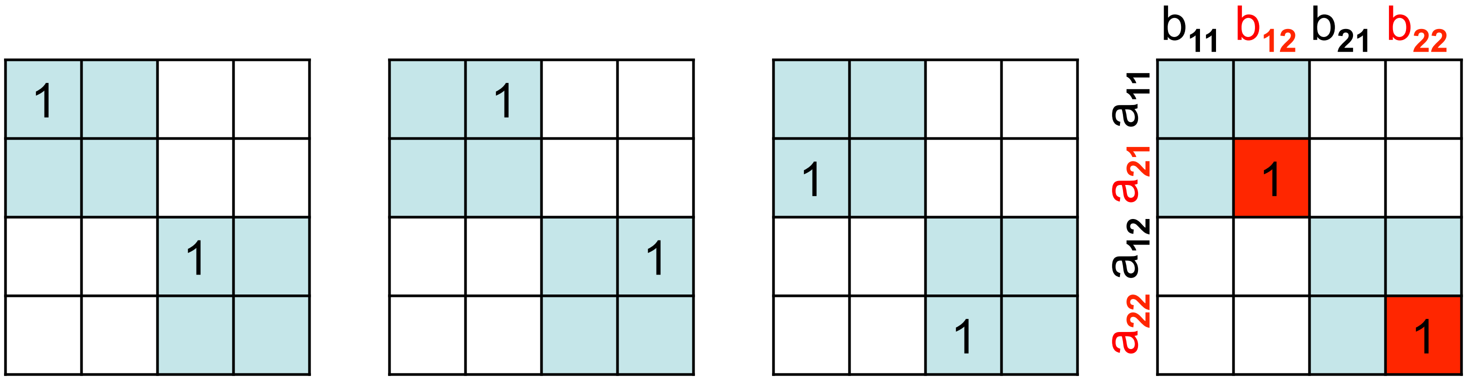

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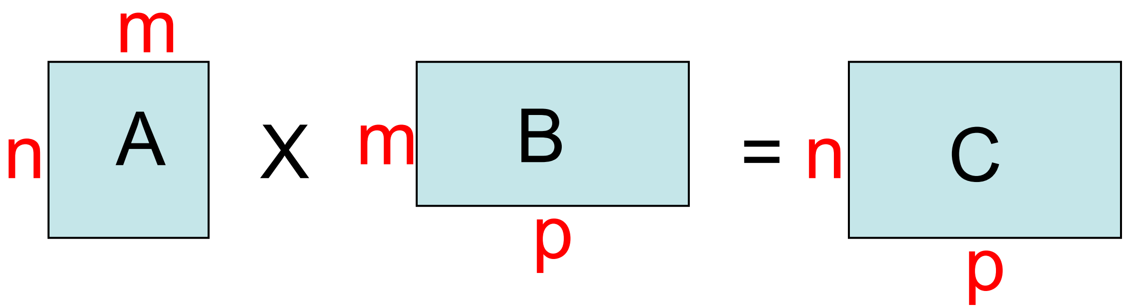
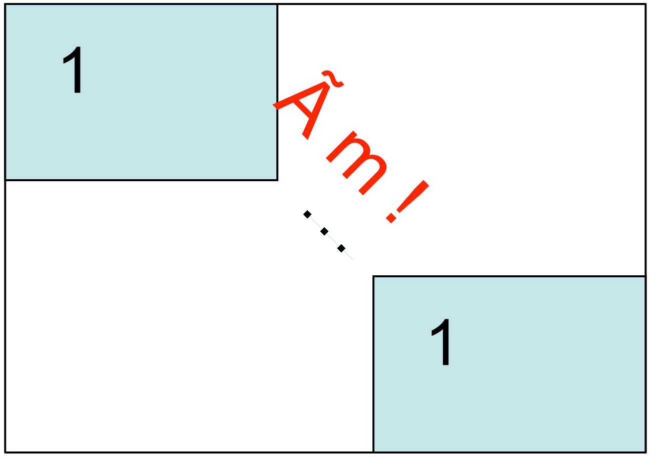
The matrix multiplication tensor

<n,m,p> is a nm £ mp £ pn tensor described by trilinear form $\sum_{i,j,k} X_{i,j} Y_{j,k} Z_{k,i}$



Each of np slices of <n,m,p>:



Strategies
for upper bounding the rank
# of the
matrix multiplication tensor

Upper bounds on rank

- Observation: $<n,n,n>^{-i} = <n^i, n^i, n^i>$

) $R(<n^i, n^i, n^i>) \cdot R(<n,n,n>)^i$

- Strategy I: bound rank for small n by hand
  - $R(<2,2,2>) = 7$! < 2.81
  - $R(<3,3,3>)$ 2 [19..23] (worse bound)

  - even computer search infeasible…

Upper bounds on rank

- Border rank = rank of sequence of tensors approaching target tensor entrywise

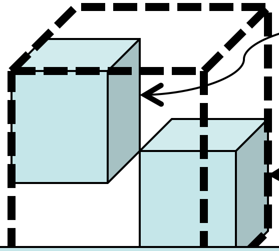
$\begin{array}{|c|c|}\hline 1 & \\ \hline & \\ \hline \end{array}$ $\begin{array}{|c|c|}\hline & 1 \\ \hline 1 & 2 \\ \hline \end{array}$ rank = 3
border rank = 2: $\begin{array}{|c|c|}\hline 2\text{-}1 & 1 \\ \hline 1 & 2 \\ \hline \end{array}$ $\begin{array}{|c|c|}\hline 1 & \\ \hline & \\ \hline \end{array}$

- Strategy II: bound *border rank* for small n

- Lemma: $\underline{R}(<n,n,n>) < r$) ! $< \log_n r$
  - – $\underline{R}(<2,2,3>)$ · 10 ! < 2.79

Upper bounds on rank

- Direct sum of tensors $\langle n,n,n\rangle$ © $\langle m,m,m\rangle$



(multiple matrix multi

“Asymptotic Sum Inequality” and example (Schönhage 1981)

- Strategy III: bound (border) rank of *direct sums* of small matrix multiplication tensors

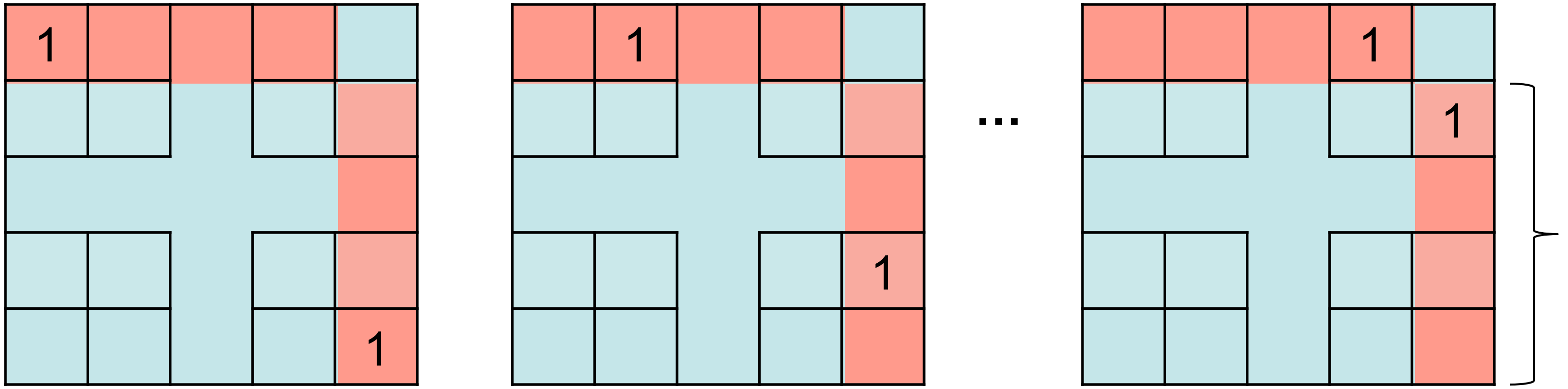
$$\underline{R}(\langle n_1,n_1,n_1\rangle \text{ © } \ldots \text{ © } \langle n_k,n_k,n_k\rangle) < r\ )\ \sum_i n_i^{!} < r$$

– $\underline{R}(\langle 4,1,3\rangle \text{ © } \langle 1,6,1\rangle) \cdot 13$! < 2.55

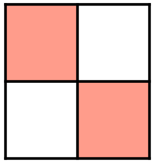
Upper bounds on rank

- Strategy IV: Strassen “laser method”
  - tensor with “coarse structure” of MM and “fine structure” components isomorphic to MM

(many independent MMs in high tensor powers)



coarse structure <1,2,1>



fine = scalar **x** row vector
col vector **x** scalar

Upper bounds on rank

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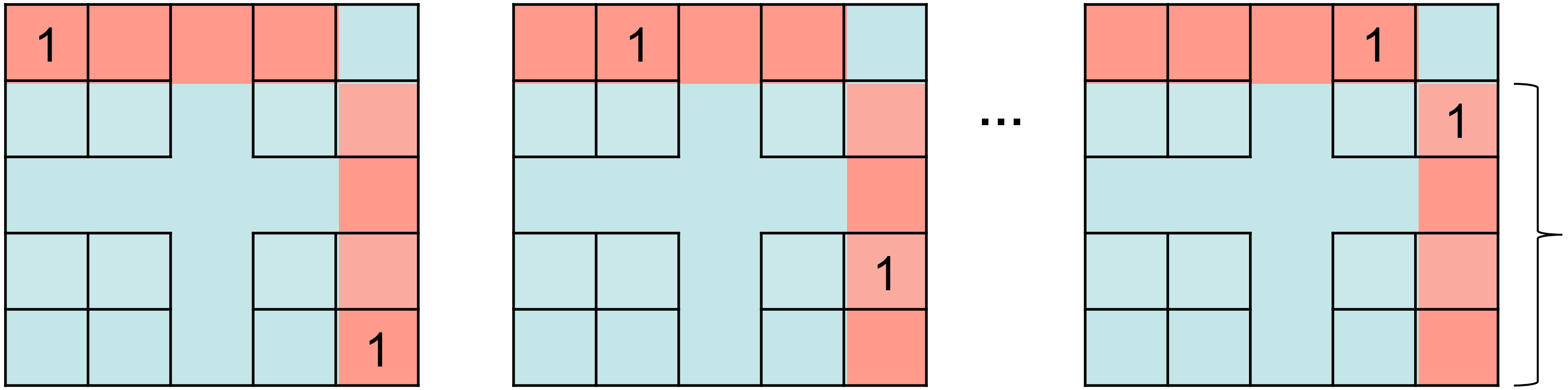

border rank = q + 1; q = 5 yields ! < 2.48

Upper bounds on rank

- Coppersmith-Winograd and beyond: border rank of this tensor is q+2:

$$\boxed{\begin{aligned}&\textstyle\sum_{i=1\ldots q} X_0Y_iZ_i + X_iY_0Z_i + X_iY_iZ_0 + \\ &\qquad X_0Y_0Z_{q+1} + X_0Y_{q+1}Z_0 + X_{q+1}Y_0Z_0\end{aligned}}$$

  - 6 “pieces”: target proportions in high tensor power affect # and size of independent MMs
  - q = 6 yields ! < 2.388

Upper bounds on rank

- Coppersmith-Winograd and beyond: analyze tensor powers of this tensor

$$T_q = \sum_{i=1\ldots q} X_0Y_iZ_i + X_iY_0Z_i + X_iY_iZ_0 + X_0Y_0Z_{q+1} + X_0Y_{q+1}Z_0 + X_{q+1}Y_0Z_0$$

Tensor power	# "pieces"	bound	reference
2	36	2.375	C-W
4	1296	2.3737	Stothers
8	1679616	2.3729	Williams
16	2.82 x 10^12	2.3728640	Le Gall
32	7.95 x 10^24	2.3728639	Le Gall

Upper bounds on rank

- Coppersmith-Winograd and beyond

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- Ambainis-Filmus 2014: N-th tensor power cannot beat bound of 2.3078

A different approach

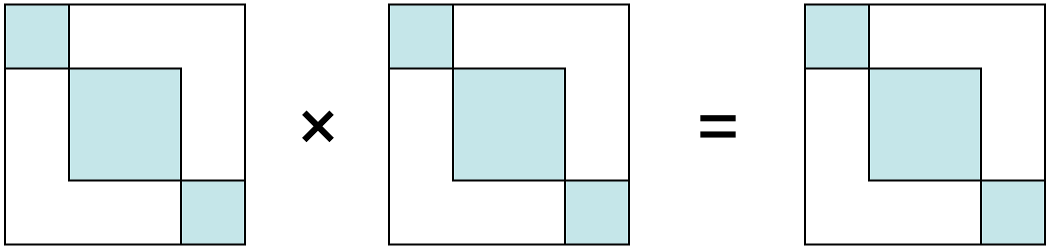
- So far...
  - – bound border rank of small tensor (by hand)
  - – asymptotic bound from high tensor powers

- Disadvantages
  - – limited universe of “starting” tensors
  - – high tensor powers hard to analyze

matrix multiplication via groups and coherent configurations / association schemes

The general approach

- Cohn-Umans 2003, 2012:
  - *embed* n x n matrix multiplication into semi-simple algebra multiplication
  - semi-simple: isomorphic to block-diagonal MM



commutative, diagonal

  - key hope: “nice basis” w/ combinatorial structure
  - reduce n x n MM to smaller MMs; recurse

The Group Algebra

- given finite group G, group algebra C[G] has elements $\Sigma_g a_g g$

with multiplication

$$(\Sigma_g a_g g)(\Sigma_h b_h h) = \Sigma_f (\Sigma_{gh = f} a_g b_h) f$$

- structure: C[G] ' $(C^{d_1 \times d_1}) \times \ldots \times (C^{d_k \times d_k})$
- group elements are “nice basis”

“Nice basis” embedding:

Subgroups X, Y, Z of G satisfy the **triple product property** if for all $x \in X$, $y \in Y$, $z \in Z$:

$$xyz = 1 \quad \text{iff} \quad x = y = z = 1.$$

The embedding:

$Q(S) = \{s^{-1}t : s, t \in S\}$

Subsets X, Y, Z of G satisfy the

triple product property

if for all $x \in Q(X)$, $y \in Q(Y)$, $z \in Q(Z)$:

$xyz = 1$ iff $x = y = z = 1$.

$\underline{\mathbf{A}} = \Sigma a_{x\ ,y}\ (x\ y^{\ -1})$ $\underline{\mathbf{B}} = \Sigma b_{y\ ,z}\ (y\ z^{\ -1})$

<u>Claim:</u> $(AB)_{x\ ,z}$ = coeff. on $(x\ z^{\ -1})$ in $\underline{\mathbf{A}}{*}\underline{\mathbf{B}}$.

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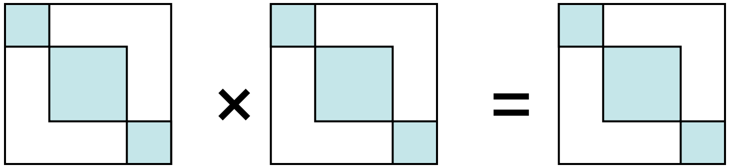
$\underline{\mathbf{A}} = \Sigma a_{x_1,y_1}(x_1y_1^{-1})$ $\underline{\mathbf{B}} = \Sigma b_{y_2,z_2}(y_2z_2^{-1})$

<u>Claim:</u> $(AB)_{x_3,z_3}$ = coeff. on $(x_3z_3^{-1})$ in $\underline{\mathbf{A}}*\underline{\mathbf{B}}$.

$(x_1y_1^{-1})(y_2z_2^{-1}) = x_3z_3^{-1}$) $x_3^{-1}x_1y_1^{-1}y_2z_2^{-1}z_3=1$

How many multiplications?

Embedding + structure of C[G] yields bound on rank (´ # multiplications):



- we use $m \leq \Sigma d_i^3$ mults
- really $m = \Sigma d_i^{!}$ mults
- *at least* $m \geq \Sigma d_i^2 = |G|$ mults

First Challenge: embed $k \times k$ matrix multiplication in group of size ¼ k^2

The embedding

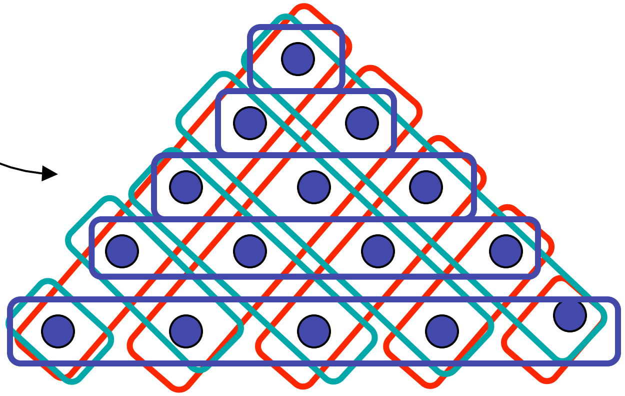
First Challenge: embed $k \times k$ matrix multiplication in group of size ¼ k^2

- simple pigeonhole argument:
  - embedding in an abelian group requires group to have size k^3

The triangle construction

Theorem: can embed k × k matrix multiplication in symmetric group of size $k^{2 + o(1)}$

n objects

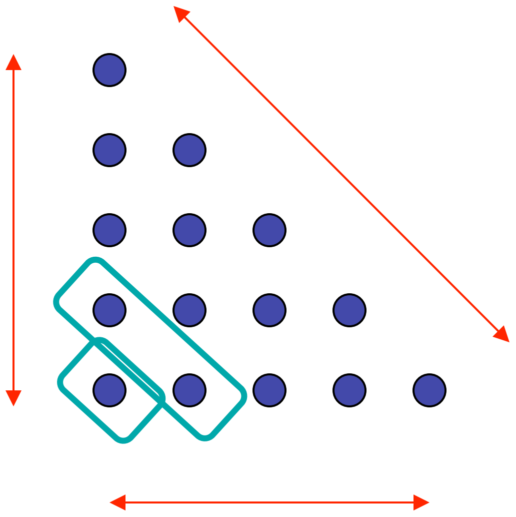


- subgroup X
- subgroup Y
- subgroup Z

need X, Y, Z in S_n all with size ≈ $|S_n|^{1/2}$

The triangle construction

- X moves points within rows
- Y moves points within columns
- Z moves points within diagonals
- want: $xyz = 1 \quad \Rightarrow \quad x = y = z = 1$



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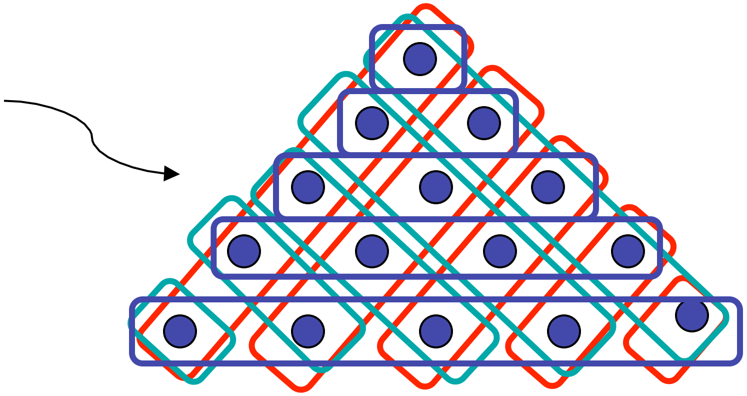


- subgroup X
- subgroup Y
- subgroup Z

unfortunately, $d_{max} > |X| \; (= |Y| = |Z|)$

What should we be aiming for?

Theorem: in group G supporting k x k matrix multiplication with character degrees d_1, d_2, d_3,…, we obtain:

$$k^{\omega} \cdot \sum_i d_i^{\omega}$$

- If X, Y, Z µ G satisfy T.P.P. and
  - $(|X|\text{¢}|Y|\text{¢}|Z|)^{1/3} = k \text{ ¸ } |G|^{1/2 - o(1)}$
  - $d_{max} \cdot |G|^{1/2 - \text{²}}$

  then ! = 2

$\sum_i d_i^{!} \cdot d_{max}^{! - 2}|G|$

Constructions in linear groups

- Good candidate family:

  SL(n, q) for fixed dimension n

- In SL(n, R) these three subgroups satisfy the triple product property:
  - upper-triangular with ones on the diagonal
  - lower-triangular with ones on the diagonal
  - the special orthogonal group SO(n, R)

  and dim. of each is ½ dim. of G as n ! 1

Group algebra approach

- [CKSU 2005] wreath product groups yield :
  - ! < 2.48, ! < 2.41
  - key part of construction is combinatorial
  - two conjectures implying ! = 2

- Main disadvantage:
  - non-trivial results *require* non-abelian groups
  - most ideas foiled by too-large char. degrees

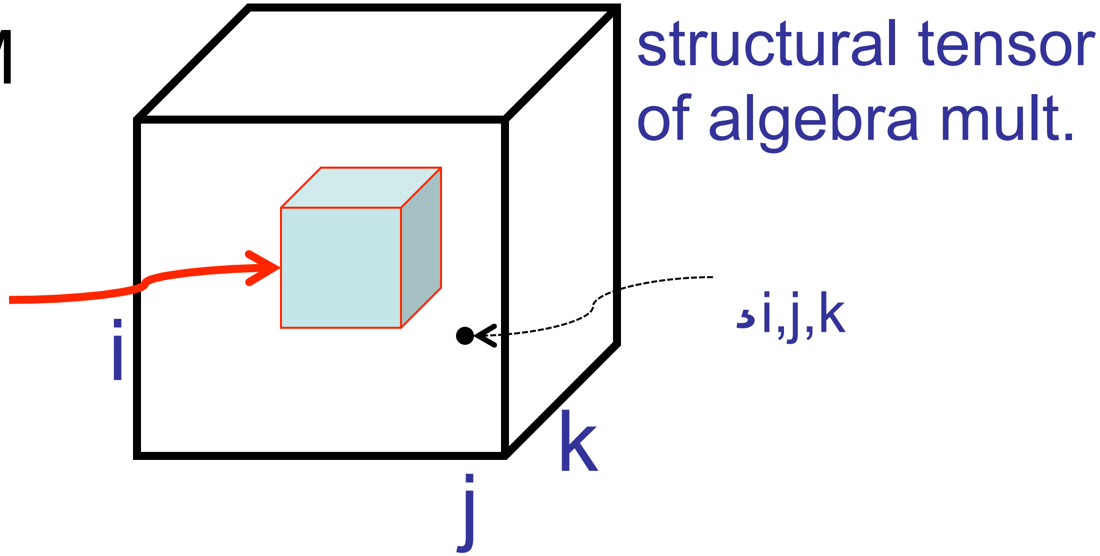
General semi-simple algebras

- (finite dimensional, complex) algebra specified by
  - “nice basis” $e_1, e_2, \ldots, e_r$
  - structure constants د$_{i,j,k}$ satisfying

$$e_i \, e_j = \sum_k \text{د}_{i,j,k} \, e_k$$

“realizes” MM
if contains*:

MM tensor
<n,n,n>



structural tensor
of algebra mult.

Weighted vs. unweighted MM

- Technical problem:
  - MM tensor <n,n,n> given by $\sum_{i,j,k}X_{i,j}Y_{j,k}Z_{k,i}$
  - embedding into algebra bounds rank of tensor given by

$$\sum_{i,j,k\text{ﺪ}i,j,k}X_{i,j}Y_{j,k}Z_{k,l}$$

(with $_{\text{ﺪ}i,j,k} \neq 0$)

  - group algebra: $_{\text{ﺪ}i,j,k}$ always 0 or 1

Weighted vs. unweighted MM

s-rank of tensor T: minimum rank of tensor with same ***s**upport* as T

Does upper bound on s-rank of MM tensor imply upper bound on ordinary rank?

Example:

$$\begin{array}{|c|c|} \hline a_{11} & a_{12} \\ \hline a_{21} & a_{22} \\ \hline \end{array} \times \begin{array}{|c|c|} \hline b_{11} & b_{12} \\ \hline b_{21} & b_{22} \\ \hline \end{array} = \begin{array}{|c|c|} \hline a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ \hline a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \\ \hline \end{array}$$

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does it help if can compute this in 6 multiplications?

Weighted vs. unweighted MM

- s-rank can be much smaller than rank:

0	1	1	1
1	0	1	1
1	1	0	1
1	1	1	0

rank n

same support:

® = n-th root of unity

$\underset{0}{\text{®}}$	$\underset{1}{\text{®}}$	$\underset{2}{\text{®}}$	$\underset{3}{\text{®}}$
$\underset{3}{\text{®}}$	$\underset{0}{\text{®}}$	$\underset{1}{\text{®}}$	$\underset{2}{\text{®}}$
$\underset{2}{\text{®}}$	$\underset{3}{\text{®}}$	$\underset{0}{\text{®}}$	$\underset{1}{\text{®}}$
$\underset{1}{\text{®}}$	$\underset{2}{\text{®}}$	$\underset{3}{\text{®}}$	$\underset{0}{\text{®}}$

rank 1

-

1	1	1	1
1	1	1	1
1	1	1	1
1	1	1	1

rank 1

maybe it's easy to show s-rank of n £ n matrix multiplication is n^2 (!!)

Weighted vs. unweighted MM

! = inf {¿ : rank(<n,n,n>) · O(n$^{¿}$)}

!$_s$ = inf{¿ : s-rank(<n,n,n>) · O(n$^{¿}$)}

Theorem: ! · (3!$_s$ – 2)/2

in particular, !$_s$ · 2 + ^{2}) ! · 2 + (3/2)^{2}

- Proof idea:
  - find ¼ n^2 copies of <n,n,n> in 3rd tensor power
  - when broken up this way, can rescale

A promising family of semisimple algebras

Coherent configurations

“group theory without groups”

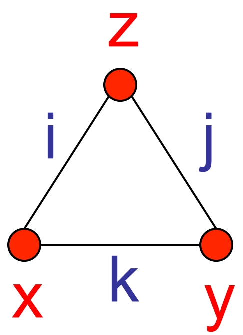
- points X, partition $R_1, R_2, \ldots, R_r$ of X^2
  - diagonal $\{(x,x) : x\ 2\ X\}$ is union of some classes
  - for each i, there is i^*

    $$R_{i^*} = \{(y,x) : (x,y)\ 2\ R_i\}$$
  - exist integers $p_{i,j}^k$ such that for all $(x,y)\ 2\ R_k$:

$$\#\{z: (x,z)\ 2\ R_i \text{ and } (z,y)\ 2\ R_j\} = p_{i,j}^k$$

if *one* class: “association scheme”

$p_{i,j}^k = p_{j,i}^k$: commutative



Coherent configs: examples

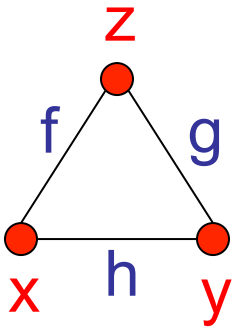
- Hamming scheme:
  - points 0/1 vectors
  - classes determined by hamming distance

- distance-regular graph:
  - points = vertices
  - classes determined by distance in graph metric

Coherent configs: examples

- scheme based on finite group G
  - set X = finite group G
  - classes R_g = {(x,xg) : x 2 X}

$p_{f,g}^{h}$ = 1 if fg=h, 0 otherwise



- "Schurian":
  - group G acts on set X
  - classes = orbits of (diagonal) G-action on X^2

Coherent configs: examples

- “Schurian”:
  - group G acts on set X
  - classes = orbits of (diagonal) G-action on X^2

- one Schurian scheme: “group scheme”
  - group G x G acts on G via (g,h)¢$x = gxh^{-1}$
  - orbits all of the form $\{(x,y): xy^{-1}$ 2 $C_i\}$ for conjugacy class C_i
  - always commutative!

Adjacency algebra

CC: points X, partition R_1, R_2, …, R_r of X^2

- for each class R_i, matrix A_i with

  $A_i[x,y]$ = 1 iff (x,y) 2 R_i

- 3 CC axioms)

  {A_i} generate a semisimple algebra

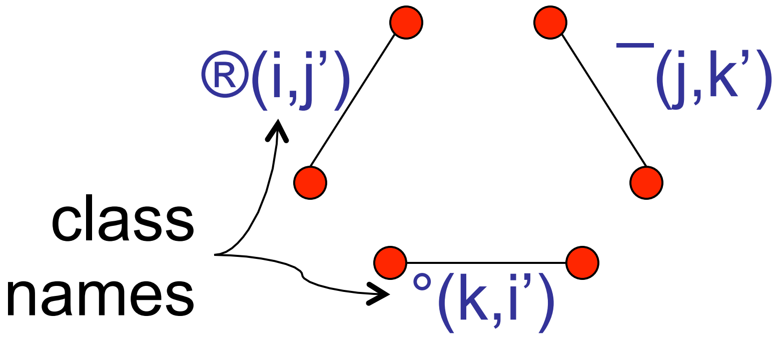
  – e.g., 3rd axiom implies $A_i A_j = \sum_k p_{ij}^k A_k$

  – if the CC based on group G, algebra is C[G]

Nice basis conditions

- group algebra C[G]: “nice basis” yields triple product property

- adjacency algebras of CCs: “nice basis” yields triangle condition:



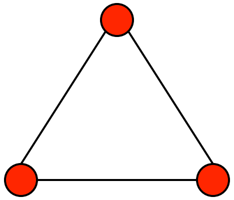
®(i,j’)

¯(j,k’)

°(k,i’)

class names

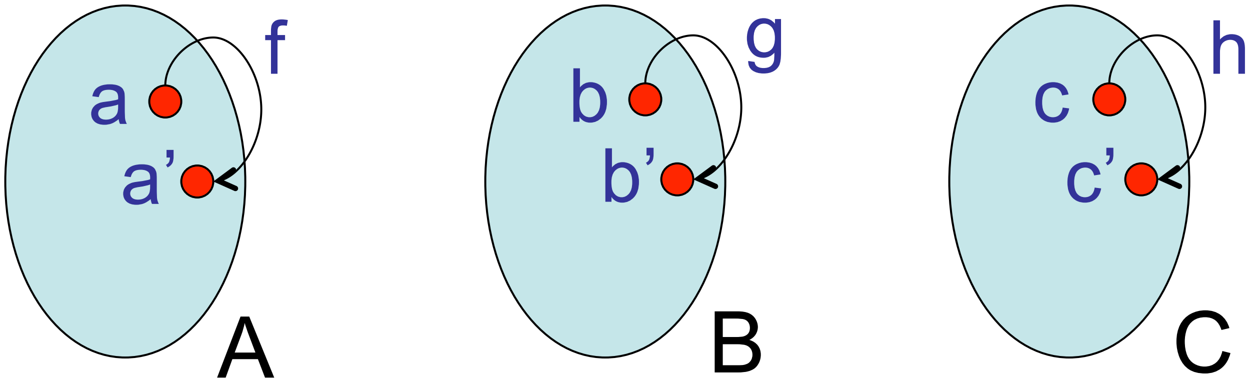
can look like



iff i = i’, j = j’, k = k’

Nice basis conditions

- Schurian CCs: “nice basis” yields
  - group G acts on set X
  - subsets A,B,C of X realize <|A|, |B|, |C|> if:



fgh = 1 implies a = a’, b = b’, c = c’

Coherent configs vs. groups

Generalization for generalization's sake?

- recall group framework:
  - – non-commutative necessary

Theorem: in group G realizing n£n matrix multiplication, with character degrees d_1, d_2, d_3,…, we obtain:

$$R(<n,n,n>) \cdot \sum_i d_i^{\omega} \cdot d_{max}^{\omega-2} \text{¢} |G|$$

goals: $|G|$ ¼ n^2 *and* small d_{max}

Coherent configs vs. groups

Generalization for generalization’s sake?

- coherent configuration framework:
  - commutative suffices!
  - combinatorial constructions from old setting yield
    $!_s < 2.48$, $!_s < 2.41$
  - conjectures from old setting (if true) would imply $!_s = 2$

in commutative Schurian CC’s

even group schemes

even symmetric

Proof idea

we prove a general transformation:

if can realize several independent matrix multiplications in CC…

- can do this in abelian groups
- conjectures: can "pack optimally"

… then high symmetric power of CC realizes *single* matrix multiplication

preserves commutativity

– reproves Schönhage's Asymptotic Sum Inequality

Commutative CCs suffice

Main point

embedding n x n matrix multiplication into a commutative coherent configuration of rank ¼ n^2 is a viable route to ! = 2

(no representation theory needed)

Open problems

- find a construction in new framework that
  - proves non-trivial bound on $!_s$
  - is not based on constructions from old setting

- is the (border) s-rank of <2,2,2> = 6?

- embed n £ n MM into commutative coherent configuration of rank ¼ n^2