Dichotomy Theorems in Counting Complexity and Holographic Algorithms

Jin-Yi Cai
University of Wisconsin, Madison

September 17th, 2014

Supported by NSF CCF-1217549.

The P vs. NP Question

It is generally conjectured that many combinatorial problems in the class NP are not computable in P.

Conjecture: P $\neq$ NP.

P $=^?$ NP is: Is there a universal and efficient method to discover a proof when one exists?

#P

Counting problems:

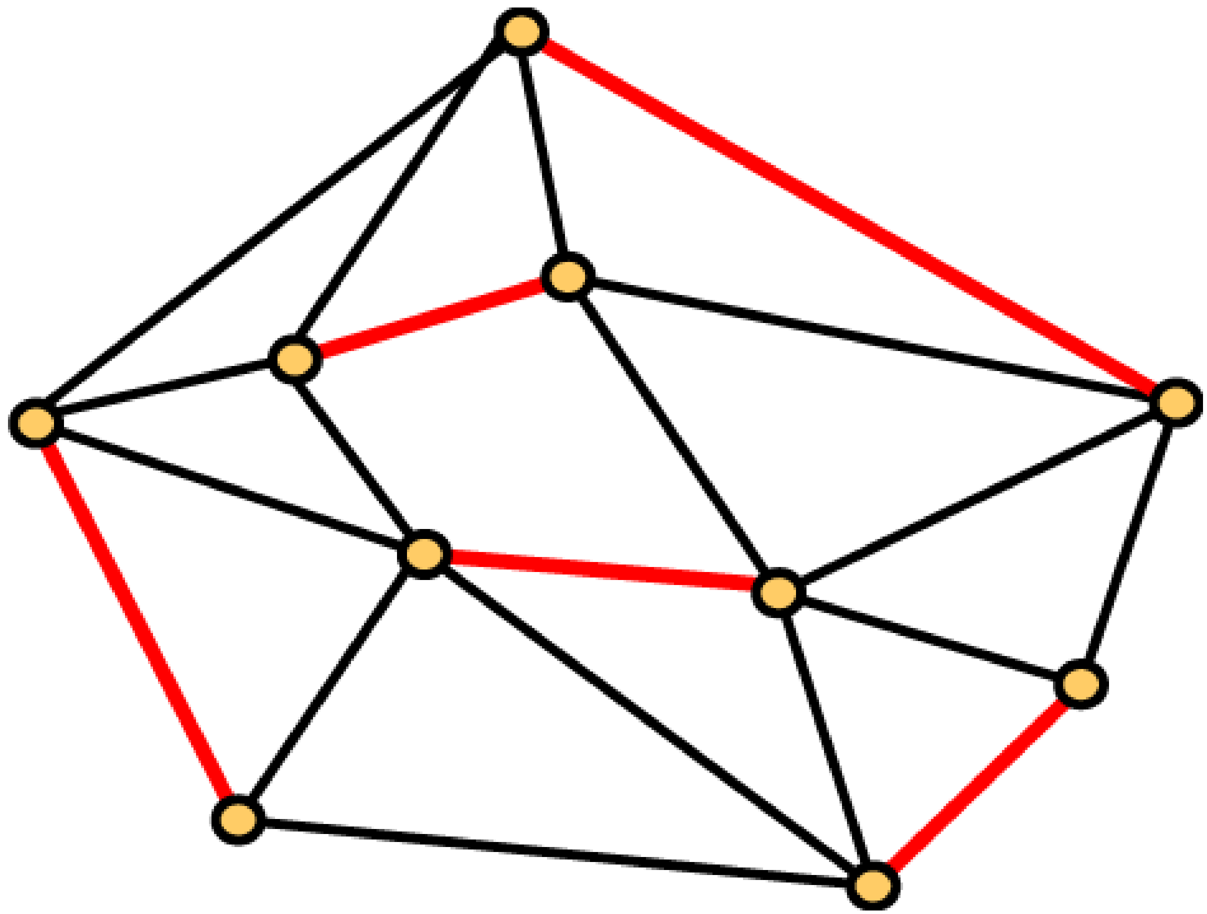
#SAT: How many satisfying assignments are there in a Boolean formula?

#PerfMatch: How many perfect matchings are there in a graph?

#P is at least as powerful as NP, and in fact subsumes the entire polynomial time hierarchy $\cup_i \Sigma_i^p$ [Toda].

#P-completeness: #SAT, #PerfMatch, Permanent, etc.

Perfect Matchings



Matchgates Based Holographic Algorithms

Valiant introduced these new algorithms.

- **Superposition of states, similar to quantum computing.**
- **Computable on classical computers, without using quantum computers.**

Two main ingredients:

(1) Use perfect matchings to encode fragments of computations.

(2) Use linear algebra to achieve exponential cancellations.

They (seem to) achieve exponential speed-ups for some problems.

Two Great Algorithms

Most #P-complete problems are counting versions of NP-complete decision problems.

But the following problems are solvable in P:

- **Whether there exists a Perfect Matching in a general graph [Edmonds].**
- **Count the number of Perfect Matchings in a planar graph [Kasteleyn].**

Note that the problem of counting the number of (not necessarily perfect) matchings in a planar graph is still #P-complete [Jerrum].

Sample Problems Solved by Holographic Algorithms

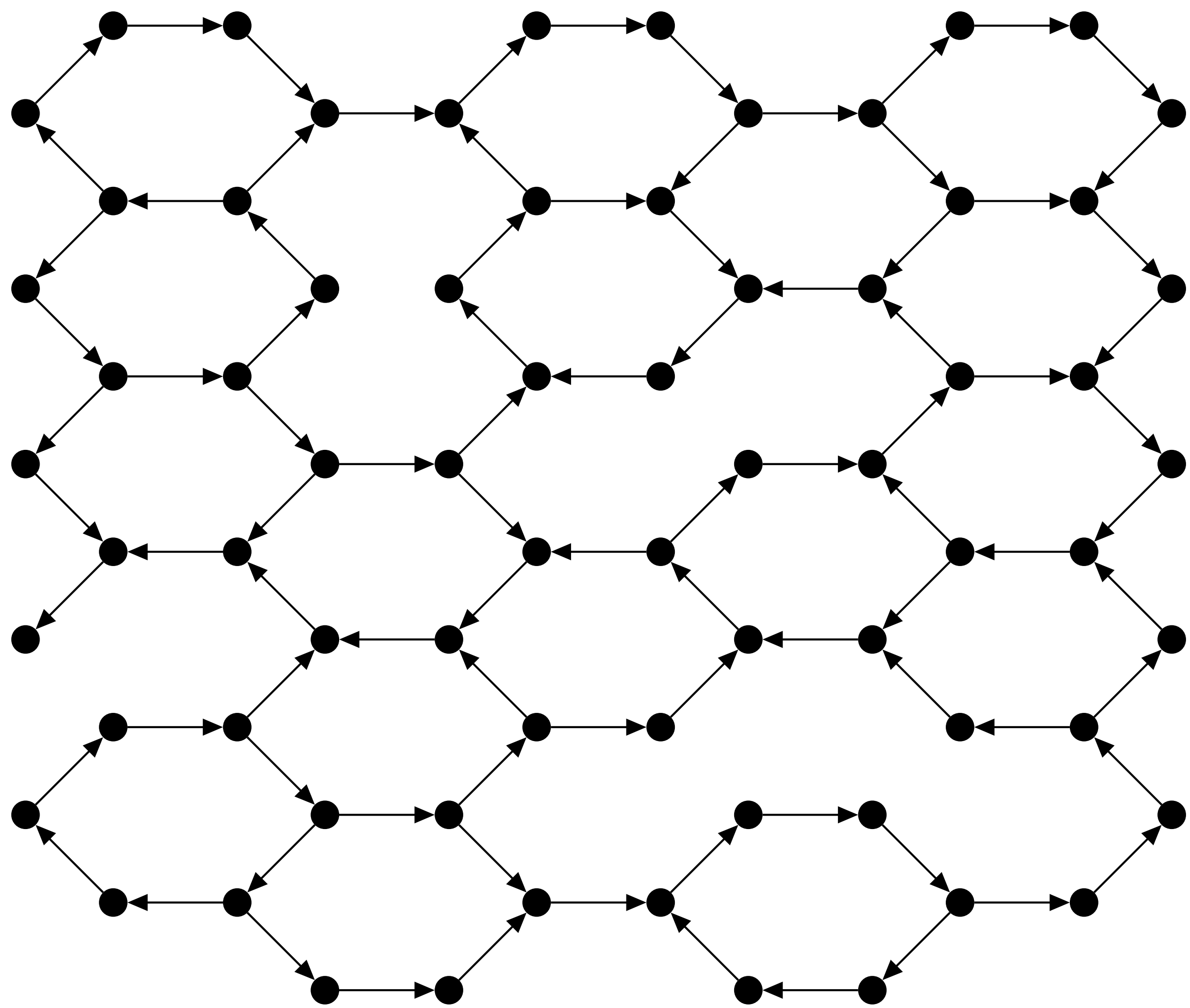
#PL-3-NAE-ICE

Input: **A planar graph $G = (V, E)$ of maximum degree 3.**

Output: **The number of orientations such that no node has all edges directed towards it or all edges directed away from it.**

Ising problems are motivated by statistical physics.

Important contributions by Ising, Onsager, Fisher, Temperley, Kasteleyn, C.N.Yang, T.D.Lee, Baxter, Lieb, Wilson etc.



A Satisfiability Problem

#PL-3-NAE-SAT

Input: A planar formula Φ consisting of a conjunction of NOT-ALL-EQUAL clauses each of size 3.

Output: The number of satisfying assignments of Φ.

Constraint Satisfaction Problems.

e.g. PL-3-EXACTLY-ONE-SAT is NP-complete.

and

#PL-3-EXACTLY-ONE-SAT is #P-complete.

Pl-Node-Bipartition

PL-NODE-BIPARTITION
Input: A planar graph $G = (V, E)$ of maximum degree 3.
Output: The cardinality of a smallest subset $V' \subset V$ such that the deletion of V' and its incident edges results in a bipartite graph.

NP-complete for maximum degree 6.

If instead of **NODE** deletion we consider **EDGE** deletion, this is the well known **MAX-CUT** problem.

MAX-CUT is NP-hard (even NP-hard to approximate by the **PCP** Theory.)

A Particular Counting Problem

$\#_7$Pl-Rtw-Mon-3CNF

Input: A planar graph G_Φ representing a Read-twice Monotone 3CNF Boolean formula Φ.

Output: The number of satisfying assignments of Φ, modulo **7**.

Here the vertices of G_Φ represent variables x_i and clauses c_j. An edge exists between x_i and c_j iff x_i appears in c_j.

Nodes x_i have degree 2 and nodes c_j have degree 3.

#P-Hardness

Fact: **#Pl-Rtw-Mon-3CNF is #P-Complete.**

Fact: **$\#_2$Pl-Rtw-Mon-3CNF is NP-hard.**

Some Similar Counting Problems

$\#_3$Pl-Rtw-Mon-4CNF

Input: **A planar graph G_Φ representing a Read-twice Monotone 4CNF Boolean formula Φ.**

Output: **The number of satisfying assignments of Φ, modulo 3.**

$\#_5$Pl-Rtw-Mon-4CNF

Input: **A planar graph G_Φ representing a Read-twice Monotone 4CNF Boolean formula Φ.**

Output: **The number of satisfying assignments of Φ, modulo 5.**

Unexpected Algorithms

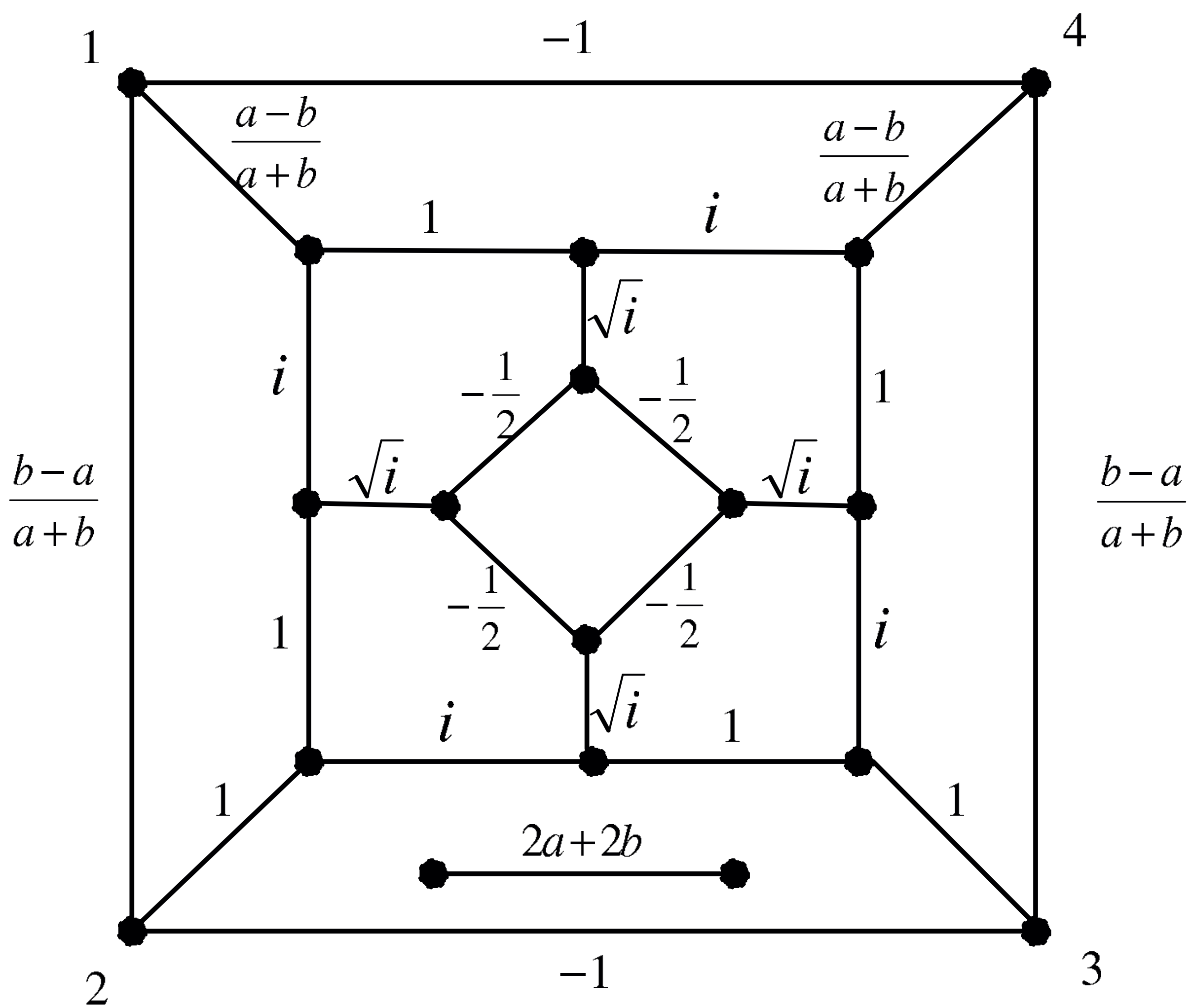
There are polynomial time algorithms for

- $\#_7$**Pl-Rtw-Mon-3CNF**
- $\#_3$**Pl-Rtw-Mon-4CNF**
- $\#_5$**Pl-Rtw-Mon-4CNF**
- $\cdots$

Using Matchgates . . .

and Holographic Algorithms.

A Matchgate


1
−1
4
$\frac{a-b}{a+b}$
$\frac{a-b}{a+b}$
1
i
$\sqrt{i}$
i
$-\frac{1}{2}$
$-\frac{1}{2}$
1
$\frac{b-a}{a+b}$
$\sqrt{i}$
$\sqrt{i}$
$\frac{b-a}{a+b}$
$-\frac{1}{2}$
$-\frac{1}{2}$
1
i
$\sqrt{i}$
i
1
1
1
$2a+2b$
2
−1
3

Matchgate

A planar matchgate $\Gamma = (G, X)$ is a weighted graph $G = (V, E, W)$ with a planar embedding, having external nodes, placed on the outer face.

Matchgates with only output nodes are called generators.

Matchgates with only input nodes are called recognizers.

Standard Signatures

Define $\mathrm{PerfMatch}(G) = \sum_M \prod_{(i,j)\in M} w_{ij}$**, where the sum is over all perfect matchings** M**.**

A matchgate Γ **is assigned a Standard Signature**

$$G = (G^S) \text{ and } R = (R_S),$$

for generators and recognizers respectively.

$$G^S = \mathrm{PerfMatch}(G - S).$$

$$R_S = \mathrm{PerfMatch}(G' - S).$$

Each entry is indexed by a subset S **of external nodes.**

A Mathematics Talk Must Have One Proof and One Joke

A Mathematics Talk Must Have One Proof and One Joke

But they should not be the same.

Collapsing #P to P

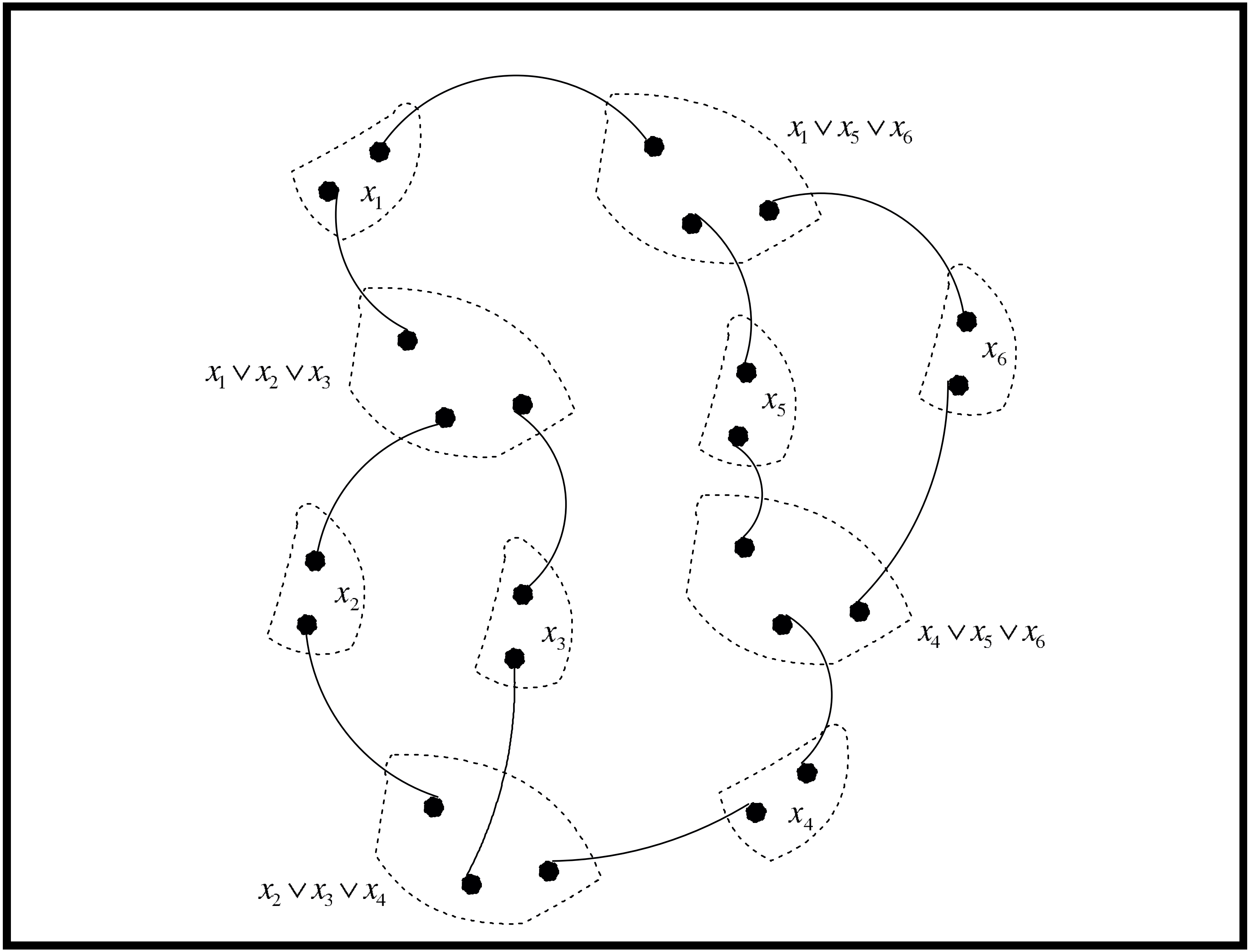
Let's try to solve the #P-hard problem in P:

#Pl-Rtw-Mon-3CNF

Input: **A planar graph G_Φ representing a Read-twice Monotone 3CNF Boolean formula Φ.**

Output: **The number of satisfying assignments of Φ.**

An Instance For #Pl-Rtw-Mon-3CNF



$x_1 \vee x_5 \vee x_6$
x_1
$x_1 \vee x_2 \vee x_3$
x_6
x_5
x_2
x_3
$x_4 \vee x_5 \vee x_6$
x_4
$x_2 \vee x_3 \vee x_4$

Recognizer Signature

Given Φ as a planar graph G_Φ.

Variables and clauses are nodes.

Edge (x, C): x appears in C.

For each clause C in Φ with 3 variables, we define

$$R_C = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

where the 8 entries are indexed by $b_1 b_2 b_3 \in \{0, 1\}^3$.

Here $b_1 b_2 b_3$ corresponds to a truth assignment to the 3 variables.

R_C **corresponds to an** $\textsc{Or}_3$ **gate.**

Generator Signature

For each variable x we want a generator G with signature $G_x^{00} = 1, G_x^{01} = 0, G_x^{10} = 0, G_x^{11} = 1$, or

$$G_x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

$\ldots$ to indicate that the fan-out value from x to C and C' must be consistent.

Exponential Sum

Now we can form the tensor product $\mathbf{R} = \bigotimes_C R_C$ and $\mathbf{G} = \bigotimes_x G_x$.

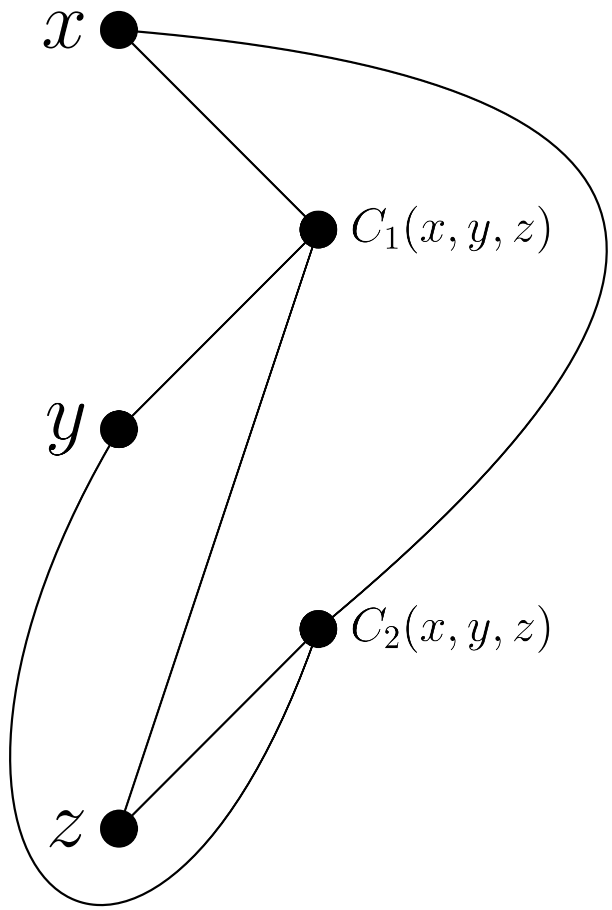
The sum

$$\langle \mathbf{R}, \mathbf{G} \rangle = \sum_{i_1, i_2, \ldots, i_e \in \{0,1\}} R_{i_1 i_2 \ldots i_e} G^{i_1 i_2 \ldots i_e}$$

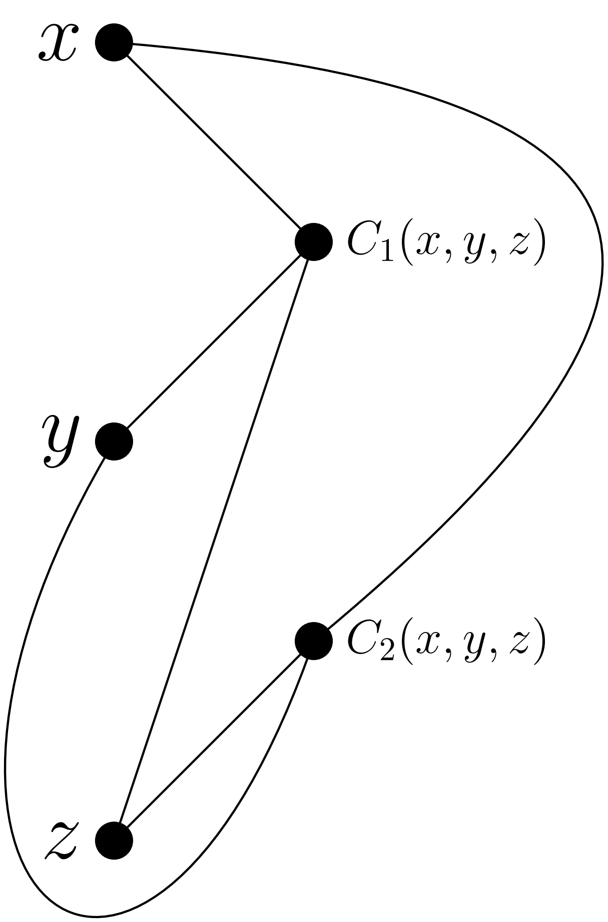
counts exactly the number of satisfying assignments to Φ.

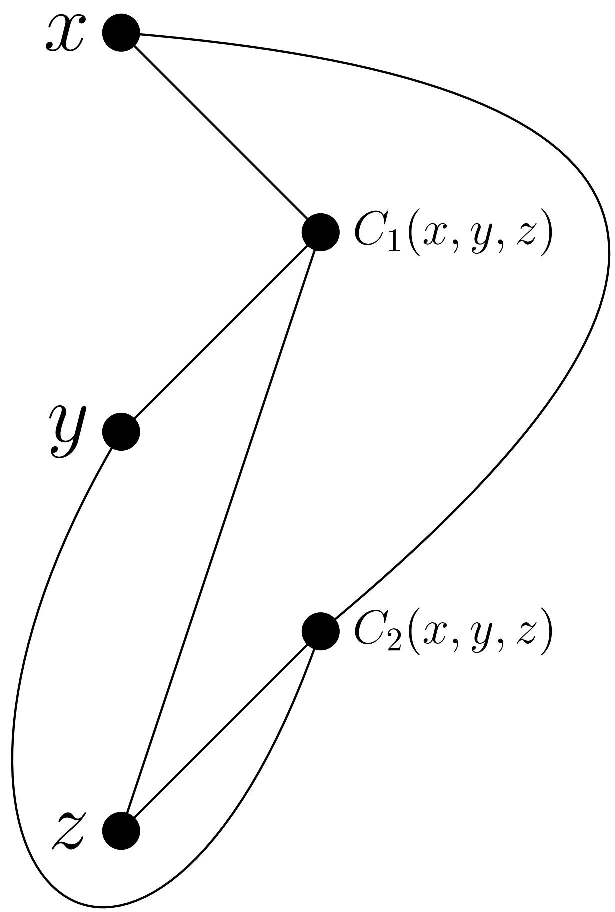
The indices of $\mathbf{R} = (R_{i_1 i_2 \ldots i_e})$ and $\mathbf{G} = (G^{i_1 i_2 \ldots i_e})$ match up one-to-one according to which x appears in which C.

A Schematic Instance


x
$C_1(x, y, z)$
y
$C_2(x, y, z)$
z

$$(1\ 0\ 0\ 1)_x \otimes (1\ 0\ 0\ 1)_y \otimes (1\ 0\ 0\ 1)_z$$





$$\begin{pmatrix}0\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}_{C_1} \otimes \begin{pmatrix}0\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}_{C_2}$$

Tensor Contraction

The Dot product counts exactly the number of satisfying assignments to Φ.

$$[(1\ 0\ 0\ 1)_x \otimes (1\ 0\ 0\ 1)_y \otimes (1\ 0\ 0\ 1)_z] \cdot \left[\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}_{C_1} \otimes \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}_{C_2} \right]$$

Realizability

If these signatures are indeed realizable as signatures of planar matchgates, then by Kasteleyn's Algorithm on planar perfect matchings, we would have shown

$$\#\mathrm{P} = \mathrm{NP} = \mathrm{P} \quad !!!$$

The above G is indeed realizable.

But R is not (realizable as standard signature).

Basis Transformations

The 1st ingredient of the theory:

Matchgates

The 2nd ingredient of the theory:

A choice of linear basis

by which the computation is manipulated/interpreted.

Transformation Matrix

So let $\boldsymbol{b}$ denote the standard basis,

$$\boldsymbol{b} = [e_0, e_1] = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right].$$

Consider another basis

$$\boldsymbol{\beta} = [t_0, t_1] = \left[\begin{pmatrix} t_0^0 \\ t_0^1 \end{pmatrix}, \begin{pmatrix} t_1^0 \\ t_1^1 \end{pmatrix} \right].$$

Let $\boldsymbol{\beta} = \boldsymbol{b}T$. Denote $T = (t_j^i)$ and $T^{-1} = (\tilde{t}_j^i)$.

(Upper index is for row and lower index is for column.)

Contravariant and Covariant Tensors

We assign to each generator Γ a contravariant tensor $\boldsymbol{G} = (G^{\alpha})$.

Under a basis transformation,

$$(G')^{i_1' i_2' \ldots i_n'} = \sum G^{i_1 i_2 \ldots i_n} \tilde{t}_{i_1}^{i_1'} \tilde{t}_{i_2}^{i_2'} \cdots \tilde{t}_{i_n}^{i_n'}$$

Correspondingly, each recognizer Γ gets a covariant tensor $\boldsymbol{R} = (R_{\alpha})$.

$$(R')_{i_1' i_2' \ldots i_n'} = \sum R_{i_1 i_2 \ldots i_n} t_{i_1'}^{i_1} t_{i_2'}^{i_2} \cdots t_{i_n'}^{i_n}$$

After this transformation, the signature

$$\textsc{Or}_3 = (0, 1, 1, 1, 1, 1, 1, 1)$$

IS realizable.

Tensor Contraction

Recall that the Dot product counts exactly the number of satisfying assignments to Φ.

$$\left[(1\ 0\ 0\ 1)_x \otimes (1\ 0\ 0\ 1)_y \otimes (1\ 0\ 0\ 1)_z\right] \cdot \left[\begin{pmatrix}0\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}_{C_1} \otimes \begin{pmatrix}0\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}_{C_2}\right]$$

$$\left[(1\ \ 0\ \ 0\ \ 1)_x \otimes (1\ \ 0\ \ 0\ \ 1)_y \otimes (1\ \ 0\ \ 0\ \ 1)_z\right] T^{\otimes 6} \cdot (T^{-1})^{\otimes 6} \left[\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}_{C_1} \otimes \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}_{C_2} \right]$$

$$= \quad (1\ \ 0\ \ 0\ \ 1)_x T^{\otimes 2} \otimes (1\ \ 0\ \ 0\ \ 1)_y T^{\otimes 2} \otimes (1\ \ 0\ \ 0\ \ 1)_z T^{\otimes 2} \cdot (T^{-1})^{\otimes 3} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}_{C_1} \otimes (T^{-1})^{\otimes 3} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}_{C_2}$$

Realization for the OR_3 gate

So we want the following

$$(0, 1, 1, 1, 1, 1, 1, 1)$$

as a non-standard signature under some basis.

i.e., for some matchgate standard signature
$R = (R_{000}, R_{001}, R_{010}, R_{011}, R_{100}, R_{101}, R_{110}, R_{111})$**, such that**

$$(0, 1, 1, 1, 1, 1, 1, 1) = R\boldsymbol{\beta}^{\otimes 3}$$

or

$$(0, 1, 1, 1, 1, 1, 1, 1)(\boldsymbol{\beta}^{-1})^{\otimes 3} = R$$

Let

$$\boldsymbol{\beta} = \left[\begin{pmatrix} 1+\omega \\ 1-\omega \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right],$$

where $\omega = e^{2\pi i/3}$ **is a primitive third root of unity.**

The Transformation Matrix from R' to R

$$\left(\begin{pmatrix} 1+\omega & 1 \\ 1-\omega & 1 \end{pmatrix}^{-1}\right)^{\otimes 3}$$ **is $\frac{1}{8}$ times**

$$\begin{pmatrix}
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
-1+\omega & 1+\omega & 1-\omega & -1-\omega & 1-\omega & -1-\omega & -1+\omega & 1+\omega \\
-1+\omega & 1-\omega & 1+\omega & -1-\omega & 1-\omega & -1+\omega & -1-\omega & 1+\omega \\
-3\omega & -2-\omega & -2-\omega & \omega & 3\omega & 2+\omega & 2+\omega & -\omega \\
-1+\omega & 1-\omega & 1-\omega & -1+\omega & 1+\omega & -1-\omega & -1-\omega & 1+\omega \\
-3\omega & -2-\omega & 3\omega & 2+\omega & -2-\omega & \omega & 2+\omega & -\omega \\
-3\omega & 3\omega & -2-\omega & 2+\omega & -2-\omega & 2+\omega & \omega & -\omega \\
3+6\omega & 3 & 3 & -1-2\omega & 3 & -1-2\omega & -1-2\omega & -1
\end{pmatrix}$$

The Transformation Matrix from R' to R

$$\left(\begin{pmatrix} 1+\omega & 1 \\ 1-\omega & 1 \end{pmatrix}^{-1}\right)^{\otimes 3}$$ **is $\frac{1}{8}$ times**

$$\begin{pmatrix}
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
-1+\omega & 1+\omega & 1-\omega & -1-\omega & 1-\omega & -1-\omega & -1+\omega & 1+\omega \\
-1+\omega & 1-\omega & 1+\omega & -1-\omega & 1-\omega & -1+\omega & -1-\omega & 1+\omega \\
-3\omega & -2-\omega & -2-\omega & \omega & 3\omega & 2+\omega & 2+\omega & -\omega \\
-1+\omega & 1-\omega & 1-\omega & -1+\omega & 1+\omega & -1-\omega & -1-\omega & 1+\omega \\
-3\omega & -2-\omega & 3\omega & 2+\omega & -2-\omega & \omega & 2+\omega & -\omega \\
-3\omega & 3\omega & -2-\omega & 2+\omega & -2-\omega & 2+\omega & \omega & -\omega \\
3+6\omega & 3 & 3 & -1-2\omega & 3 & -1-2\omega & -1-2\omega & -1
\end{pmatrix}$$

The Transformation Matrix from R' to R

$$\left(\begin{pmatrix} 1+\omega & 1 \\ 1-\omega & 1 \end{pmatrix}^{-1}\right)^{\otimes 3}$$ **is** $\frac{1}{8}$ **times**

$$\begin{pmatrix}
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
-1+\omega & 1+\omega & 1-\omega & -1-\omega & 1-\omega & -1-\omega & -1+\omega & 1+\omega \\
-1+\omega & 1-\omega & 1+\omega & -1-\omega & 1-\omega & -1+\omega & -1-\omega & 1+\omega \\
-3\omega & -2-\omega & -2-\omega & \omega & 3\omega & 2+\omega & 2+\omega & -\omega \\
-1+\omega & 1-\omega & 1-\omega & -1+\omega & 1+\omega & -1-\omega & -1-\omega & 1+\omega \\
-3\omega & -2-\omega & 3\omega & 2+\omega & -2-\omega & \omega & 2+\omega & -\omega \\
-3\omega & 3\omega & -2-\omega & 2+\omega & -2-\omega & 2+\omega & \omega & -\omega \\
3+6\omega & 3 & 3 & -1-2\omega & 3 & -1-2\omega & -1-2\omega & -1
\end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 & 2 & 0 & 2 & 0 & 0 & 2 \end{pmatrix}$$

Back to Standard Signature

By covariant transformation, (adding the last 7 rows),

$$(R_{i_1 i_2 i_3}) = \frac{1}{4}(0,1,1,0,1,0,0,1).$$

There indeed exists a matchgate with three external nodes with the standard signature $= \frac{1}{4}(0,1,1,0,1,0,0,1)$**.**

Thus,

$$R'_C = (0,1,1,1,1,1,1,1) = \frac{1}{4}(0,1,1,0,1,0,0,1)\left(\begin{pmatrix} 1+\omega & 1 \\ 1-\omega & 1 \end{pmatrix}\right)^{\otimes 3}.$$

Over Finite Fields

Over the field $\mathbb{Z}_7$ (but not $\mathbb{Q}$) both the generators and recognizers are simultaneously realizable. They are realizable as non-standard signatures.

This gives $\#_7$Pl-Rtw-Mon-3CNF $\in$ P.

Mersenne numbers $2^k - 1$

For each k, there is a holographic transformation and suitable matchgates such that $\#_{2^k-1}$Pl-Rtw-Mon-kCNF is computable in polynomial time.

This includes

- **$\#_7$Pl-Rtw-Mon-3CNF**
- **$\#_3$Pl-Rtw-Mon-4CNF**
- **$\#_5$Pl-Rtw-Mon-4CNF**
- $\cdots$

Exactness of Some Proofs

$$\begin{aligned}
A &= x^4y^4t + t + 4x^3y^2 + 4x + 4x^2y + \frac{2cx^2}{t} \\
B &= 2y^2t + 12y + \frac{2c}{t} \\
C &= 2xy^2t + 4x^2y^2 + 4 + 4xy + \frac{2cx}{t} \\
D &= x^2y^3t + yt + 3x^2y^2 + 3 + 6xy + \frac{2cx}{t}.
\end{aligned}$$

For any $c \neq 1$, there are x, y and $t \neq 0$, such that $A = B = C \neq 0$, and $D = 0$.

And for $c = 1$, it corresponds to a matchgate signature.

Complexity Dichotomy Theorems

Three Frameworks for Counting Problems

1. **Graph Homomorphisms**
2. **Constraint Satisfaction Problems (CSP)**
3. **Holant Problems**

In each framework, there has been remarkable progress in the classification program of the complexity of counting problems.

Graph Homomorphism

L. Lovász:

Operations with structures, Acta Math. Hung. 18 (1967), 321-328.

`http://www.cs.elte.hu/~lovasz/hom-paper.html`

Let $\mathbf{A} = (A_{i,j}) \in \mathbb{C}^{\kappa \times \kappa}$ **be a symmetric complex matrix.**

The Graph Homomorphism problem is:

INPUT: **An undirected graph** $G = (V, E)$**.**

OUTPUT:

$$Z_{\mathbf{A}}(G) = \sum_{\xi : V \to [\kappa]} \prod_{(u,v) \in E} A_{\xi(u),\xi(v)}.$$

Examples of Graph Homomorphism

Let

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

then $Z_{\mathbf{A}}(G)$ **counts the number of** Vertex Covers **in** G**.**

Let

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix}$$

then $Z_{\mathbf{A}}(G)$ **counts the number of vertex** κ-Colorings **in** G**.**

Dichotomy Theorem for Graph Homomorphism

Theorem[C., Xi Chen and Pinyan Lu] There is a complexity dichotomy for $Z_{\mathbf{A}}(\cdot)$:

For any symmetric complex valued matrix $\mathbf{A} \in \mathbb{C}^{\kappa \times \kappa}$, the problem of computing $Z_{\mathbf{A}}(G)$, for any input G, is either in P or #P-hard.

Given A, whether $Z_{\mathbf{A}}(\cdot)$ is in P or #P-hard can be decided in polynomial time in the size of A.

SIAM J. Comput. 42(3): 924-1029 (2013) (106 pages)

Many partial results: Dyer, Greenhill, Bulatov, Grohe, Goldberg, Jerrum, Thurley, $\ldots$

Dichotomy Theorem for #CSP

Theorem[C., Xi Chen] Every finite set $\mathcal{F}$ of complex valued constraint functions on any finite domain set $[\kappa]$ defines a counting CSP problem #CSP$(\mathcal{F})$ that is either computable in P or #P-hard.

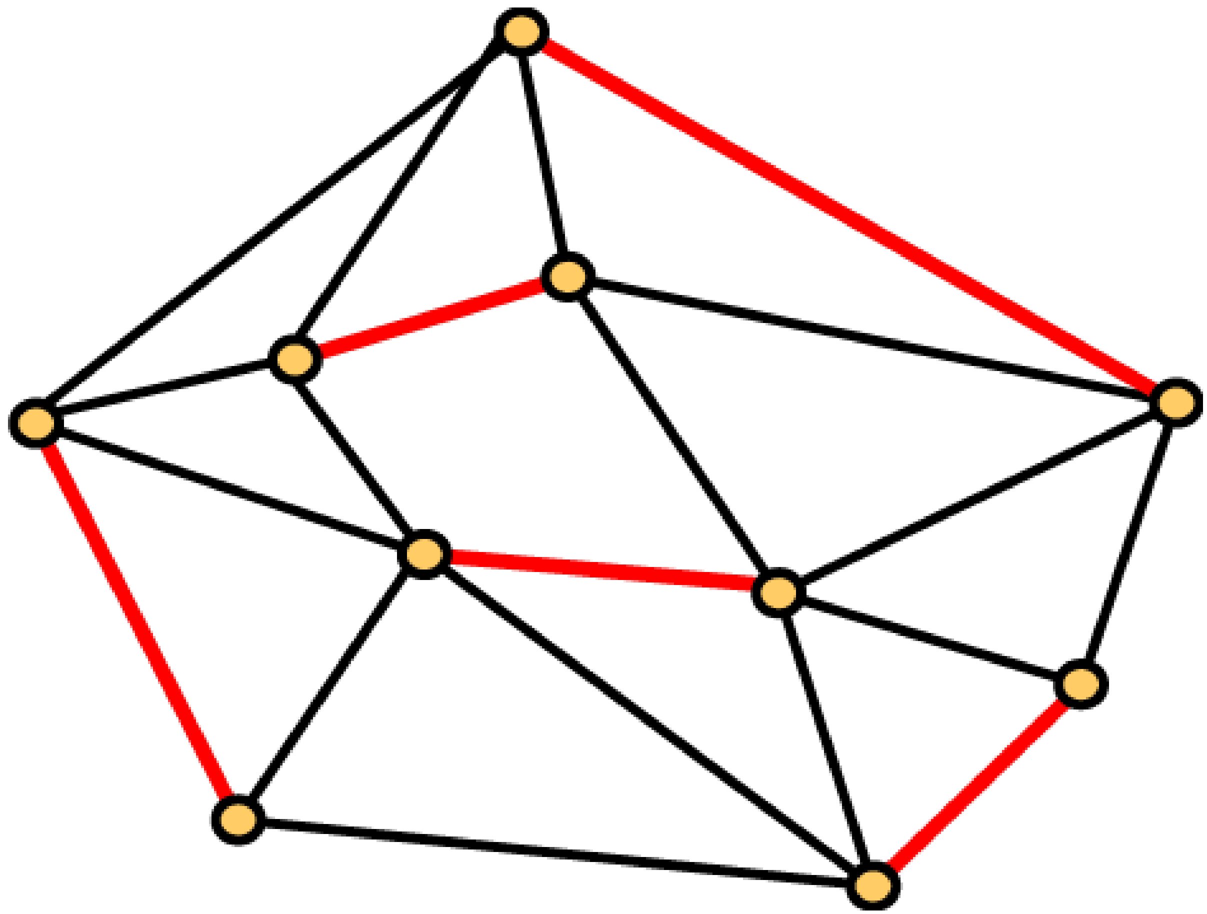
The decision version of this is open.

The decidability of this #CSP Dichotomy is open.

Creignou, Hermann, ..., Bulatov, Dalmau, Dyer, Richerby, ...

Creignou, Khanna, Sudan: Complexity Classifications of Boolean Constraint Satisfaction Problems, SIAM.

Perfect Matchings



Matching as Holant

Think of edges as variables, and assign vertices with a local constraint function.

$$\text{Holant}_\Omega = \sum_{\sigma:E\to\{0,1\}} \prod_{v\in V} f_v\left(\sigma\mid_{E(v)}\right).$$

The problem of counting Perfect Matching**s on** G **corresponds to attaching the Exact-One function at every vertex of** G**.**

The problem of counting all Matching**s on** G **is to attach the At-Most-One function at every vertex of** G**.**

#κ-EdgeColoring as a Holant Problem

Consider a 3-regular graph G.

Let AD_3 denote the following local constraint function

$$\mathrm{AD}_3(x, y, z) = \begin{cases} 1 & \textbf{if } x, y, z \in [\kappa] \textbf{ are all distinct} \\ 0 & \textbf{otherwise} \end{cases}$$

Now place AD_3 at each vertex v, with incident edges x, y, z.

Then we evaluate the sum of product

$$\mathrm{Holant}(G; \mathrm{AD}_3) = \sum_{\sigma: E(G) \to [\kappa]} \prod_{v \in V(G)} \mathrm{AD}_3 \left(\sigma \mid_{E(v)} \right).$$

Theorem[C., Guo, Williams] #κ-EdgeColoring on r-regular (planar) graphs is #P-hard for all $\kappa \geq r \geq 3$.

Dichotomy for Boolean #CSP

$\mathcal{A}$ **denotes functions of an Affine type:**

$$f(x_1, x_2, \ldots, x_n) = \lambda \cdot \chi_S \cdot \mathbf{i}^{Q(x_1, x_2, \ldots, x_n)}.$$

$\mathcal{A}$ **denotes functions of a Product type.**

Theorem (C., Pinyan Lu, Mingji Xia)
Suppose $\mathcal{F}$ **is a set of functions mapping Boolean inputs to complex numbers. If** $\mathcal{F} \subseteq \mathcal{A}$ **or** $\mathcal{F} \subseteq \mathcal{P}$**, then #CSP(**$\mathcal{F}$**) is computable in P. Otherwise, #CSP(**$\mathcal{F}$**) is** $\#\mathrm{P}$**-hard.**

Many partial results: Bulatov, Dyer, Goldberg, Jalsenius, Jerrum, Richerby, ...

Dichotomy Theorem for Holant

Theorem[C., Heng Guo, Tyson Williams] Let $\mathcal{F}$ be any set of symmetric, complex-valued signatures in Boolean variables. Then $\text{Holant}(\mathcal{F})$ **is #P-hard unless $\mathcal{F}$ satisfies one of the following conditions, in which case the problem is in P:**

1. **All non-degenerate signatures in $\mathcal{F}$ have arity ≤ 2;**
2. **$\mathcal{F}$ is $\mathcal{A}$-transformable;**
3. **$\mathcal{F}$ is $\mathcal{P}$-transformable;**
4. **$\mathcal{F} \subseteq \mathcal{V}^\sigma \cup \{f \in \mathcal{R}_2^\sigma \mid \text{arity}(f) = 2\}$ for $\sigma \in \{+, -\}$;**
5. **All non-degenerate signatures in $\mathcal{F}$ are in $\mathcal{R}_2^\sigma$ for $\sigma \in \{+, -\}$.**

A Complexity Trichotomy Theorem

Theorem[C., Pinyan Lu, Mingji Xia] Let $\mathcal{F}$ be any finite set of symmetric constraint functions mapping Boolean variables to $\mathbb{R}$. Then there are precisely three classes of #CSP($\mathcal{F}$) problems, depending on $\mathcal{F}$.

(1) #CSP($\mathcal{F}$) is in P.
(2) #CSP($\mathcal{F}$) is #P-hard, but solvable in P for planar inputs.
(3) #CSP($\mathcal{F}$) is #P-hard even for planar inputs.

Furthermore $\mathcal{F}$ is in class (2) iff there is a holographic algorithm based on matchgates and the planar problems are solved by the FKT algorithm.

A Complexity Trichotomy Theorem

Theorem[Heng Guo, Tyson Williams] Let $\mathcal{F}$ be any finite set of symmetric constraint functions mapping Boolean variables to $\mathbb{C}$. Then there are precisely three classes of #CSP($\mathcal{F}$) problems, depending on $\mathcal{F}$.

(1) #CSP($\mathcal{F}$) is in P.
(2) #CSP($\mathcal{F}$) is #P-hard, but solvable in P for planar inputs.
(3) #CSP($\mathcal{F}$) is #P-hard even for planar inputs.

Furthermore $\mathcal{F}$ is in class (2) iff there is a holographic algorithm based on matchgates and the planar problems are solved by the FKT algorithm.

A mathematics talk without a proof is like a movie without a love scene.

Hendrik Lenstra

Eulerian Orientation

An orientation of G is Eulerian if for each vertex v of G, the number of incoming edges of v equals the number of outgoing edges of v.

#EO **is the problem of counting the number of Eulerian orientations.**

Theorem #EO **is #P-hard over planar 4-regular graphs.**

Tutte Polynomial

For an undirected graph $G = (V, E)$, its Tutte polynomial is

$$T(G; x, y) = \sum_{A \subseteq E} (x-1)^{\kappa(A)-\kappa(E)} (y-1)^{\kappa(A)+|A|-|V|},$$

where $\kappa(A)$ is the number of connected components of the graph (V, A).

Theorem[Vertigan and Jaeger,Vertigan, Welsh] Evaluating $T(G; 3, 3)$ for the Tutte polynomial $T(G; x, y)$ is #P-hard, over planar graphs.

Medial Graph

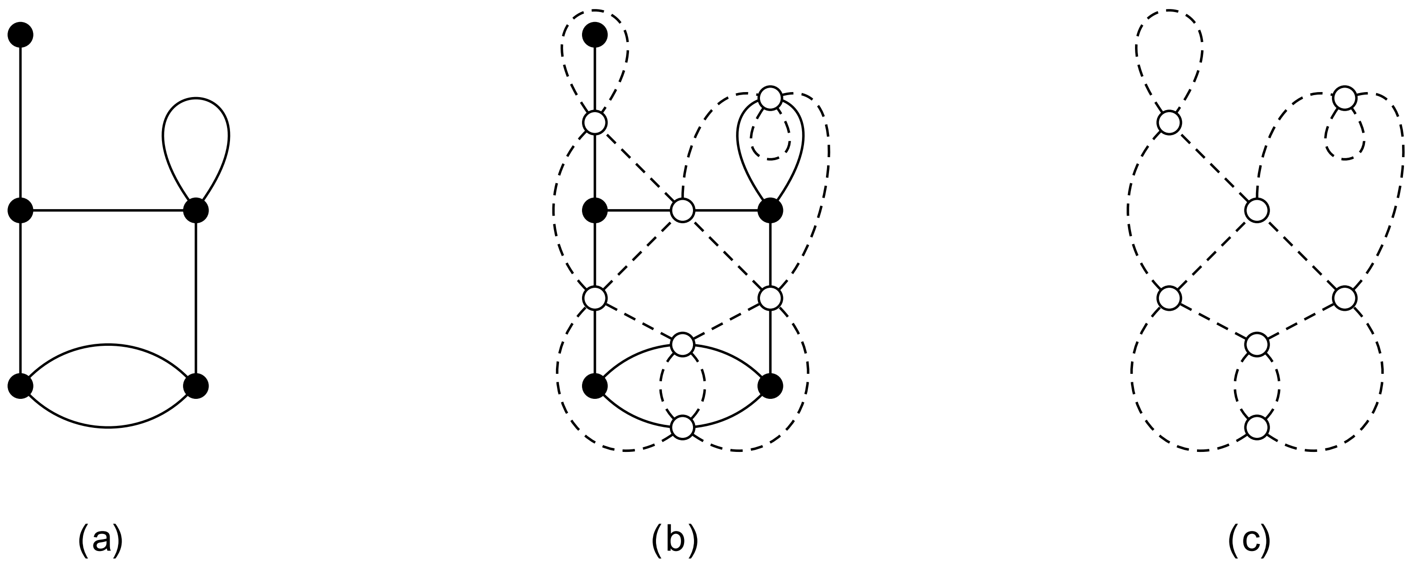


Figure 1: A plane graph (a), its medial graph (c), and the two graphs superimposed (b).

The medial graph is 4-regular.

Signature Matrix

The signature matrix of an arity 4 signature g is

$$M_g = \begin{bmatrix} g^{0000} & g^{0010} & g^{0001} & g^{0011} \\ g^{0100} & g^{0110} & g^{0101} & g^{0111} \\ g^{1000} & g^{1010} & g^{1001} & g^{1011} \\ g^{1100} & g^{1110} & g^{1101} & g^{1111} \end{bmatrix}.$$

Medial Graph and Tutte Polynomial

Theorem[Las Vergnas]

$$2 \cdot T(G; 3, 3) = \sum_{O \in \mathcal{O}(G_m)} 2^{\beta(O)},$$

Ths sum is the bipartite planar Holant problem Pl-Holant $(\neq_2 \mid f)$**, where**

$$M_f = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Z-transformation

We perform a holographic transformation by $Z = \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]$ **to get**

$$\begin{aligned} \text{Pl-Holant}\,(\neq_2 \mid f) &\equiv_T \text{Pl-Holant}\left([0,1,0](Z^{-1})^{\otimes 2} \mid Z^{\otimes 4} f\right) \\ &\equiv_T \text{Pl-Holant}\left([1,0,1] \mid \hat{f}\right) \\ &\equiv_T \text{Pl-Holant}(\hat{f}), \end{aligned}$$

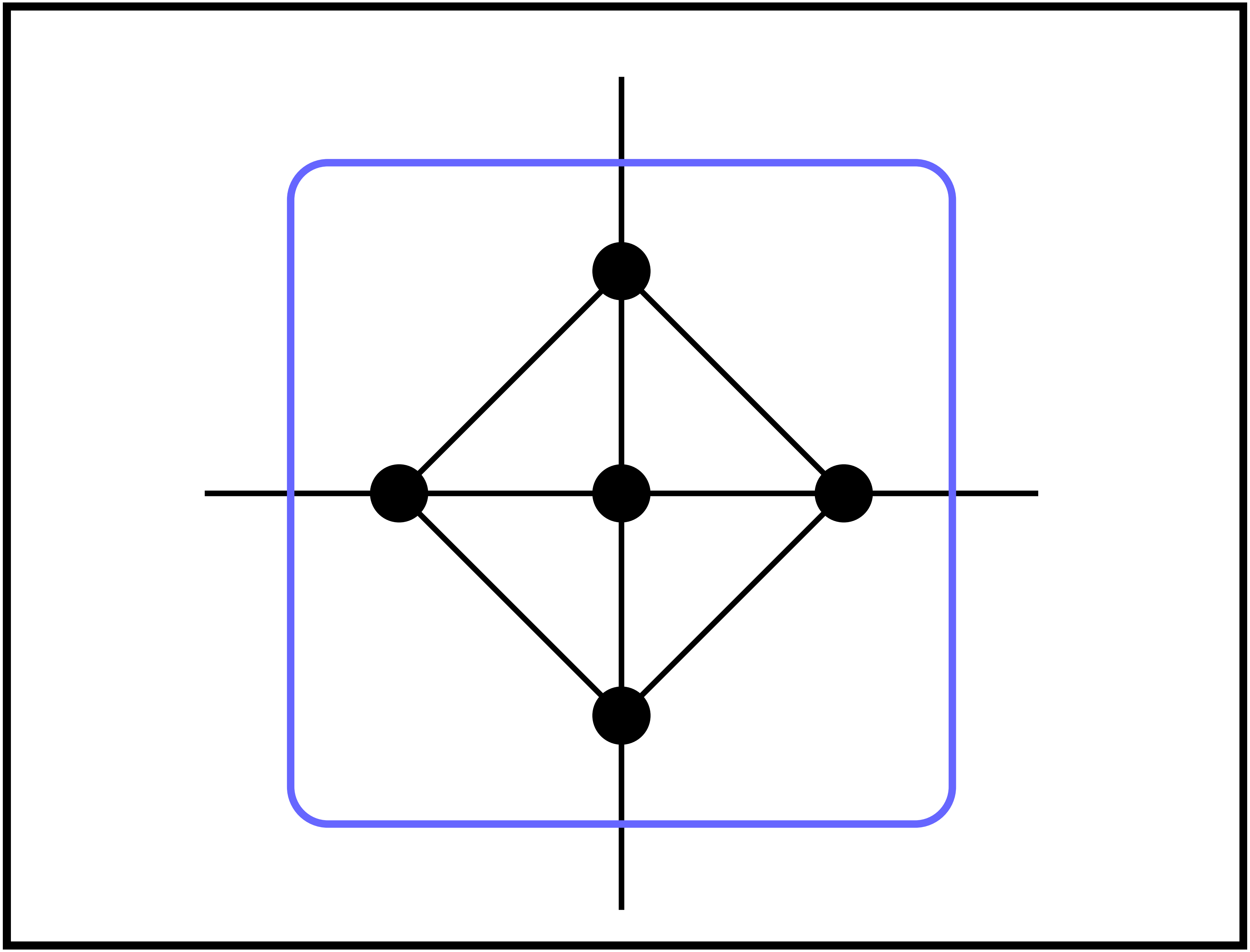
where the signature matrix of $\hat{f}$ **is**

$$M_{\hat{f}} = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix},$$

#EO Under Z-transformation

$$\begin{aligned}
& \text{Pl-Holant}\,(\neq_2 \mid [0,0,1,0,0]) \\
\equiv_T \quad & \text{Pl-Holant}\,\big([0,1,0](Z^{-1})^{\otimes 2} \mid Z^{\otimes 4}[0,0,1,0,0]\big) \\
\equiv_T \quad & \text{Pl-Holant}\,([1,0,1] \mid \tfrac{1}{2}[3,0,1,0,3]) \\
\equiv_T \quad & \text{Pl-Holant}([3,0,1,0,3]).
\end{aligned}$$

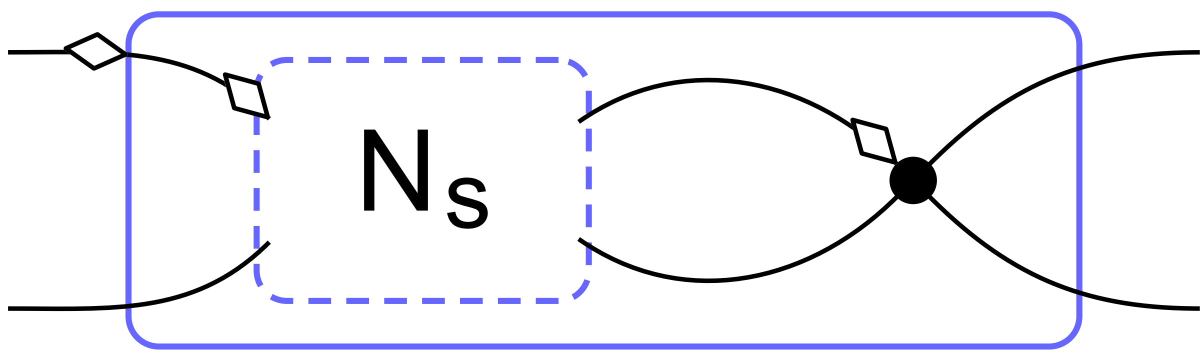
Tetrahedron Gadget



Using the planar tetrahedron gadget, we assign $[3, 0, 1, 0, 3]$ **to every vertex and obtain a signature** $32\hat{g}$**, where the signature matrix of** $\hat{g}$ **is**

$$M_{\hat{g}} = \frac{1}{2}\begin{bmatrix} 19 & 0 & 0 & 7 \\ 0 & 7 & 5 & 0 \\ 0 & 5 & 7 & 0 \\ 7 & 0 & 0 & 19 \end{bmatrix}.$$

Recursive Construction Using Tetrahedron Gadget



A Reduction

To reduce Pl-Holant$(\hat{f})$ **to** Pl-Holant$(\hat{g})$**:**

Let Ω **be an instance of** Pl-Holant$(\hat{f})$**.**

Suppose that $\hat{f}$ **appears** n **times in** Ω**.**

We construct from Ω **a sequence of instances** Ω_s **of** Holant$(\hat{g})$ **indexed by** $s \geq 1$**.**

We obtain Ω_s **from** Ω **by replacing each occurrence of** $\hat{f}$ **with the gadget** N_s **with** $\hat{g}$ **assigned to all vertices.**

To obtain Holant_{Ω_s} **from** Holant_{Ω}**, we replace** $M_{\hat{f}}$ **with** $M_{N_s} = (M_{\hat{g}})^s$**.**

Let

$$T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}.$$

Then

$$M_{\hat{f}} = T\Lambda_{\hat{f}}T^{-1} = T \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} T^{-1}$$

$$M_{\hat{g}} = T\Lambda_{\hat{g}}T^{-1} = T \begin{bmatrix} 13 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} T^{-1}.$$

We can view our construction of Ω_s as first replacing each $M_{\hat{f}}$ by $T\Lambda_{\hat{f}}T^{-1}$ to obtain a signature grid Ω', which does not change the Holant value, and then replacing each $\Lambda_{\hat{f}}$ with $\Lambda_{\hat{g}}^s$. We stratify the assignments in Ω' based on the assignment to $\Lambda_{\hat{f}}$. We only need to consider the assignments to $\Lambda_{\hat{f}}$ that assign

- $(00, 00)$ j **many times,**
- $(01, 10)$ **or** $(11, 11)$ k **many times, and**
- $(10, 01)$ ℓ **many times.**

Let $c_{jk\ell}$ be the sum over all such assignments of the products of evaluations from T and T^{-1} but excluding $\Lambda_{\hat{f}}$ on Ω'.

Then

$$\text{Pl-Holant}_{\Omega} = \sum_{j+k+\ell=n} 3^j c_{jk\ell}$$

and the value of the Holant on Ω_s, for $s \geq 1$, is

$$\text{Pl-Holant}_{\Omega_s} = \sum_{j+k+\ell=n} (13^j 6^k)^s c_{jk\ell}.$$

This coefficient matrix in the linear system is Vandermonde and has full rank since for any $j, k, j', k' \geq 0$, if $(j, k) \neq (j', k')$ then $13^j 6^k \neq 13^{j'} 6^{k'}$. Therefore, we can solve the linear system for the unknown $c_{jk\ell}$'s and obtain the value of $\text{Pl-Holant}_{\Omega}$.

Some References

Some papers can be found on my web site

`http://www.cs.wisc.edu/~jyc`

THANK YOU!

Some References

Some papers can be found on my web site

`http://www.cs.wisc.edu/~jyc`

`http://arxiv.org/abs/1303.6729` **Matchgate Theory**

`http://arxiv.org/abs/1204.6445` **Holant Dichotomy**

`http://arxiv.org/abs/1212.2284` **Planar #CSP**

`http://arxiv.org/abs/0903.4728` **Graph Homomorphism**

`http://arxiv.org/abs/1111.2384` **Full #CSP Dichotomy**