# Vanishing multiplicities

## Two unsuccessful contributions to GCT

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# Orbit closure problem and obstructions

- We work over $\mathbb{C}$. Let $G$ (*e.g.* a linear group) be a complex connected reductive group acting on a vector space $V$. Given two orbits $\mathcal{O}'$ and $\mathcal{O}$ we want to get methods to prove that

$$\mathcal{O}' \not\subset \overline{\mathcal{O}}. \tag{1}$$

- Basic idea: if, by contradiction, $\mathcal{O}' \subset \overline{\mathcal{O}}$ then for any irrep. $V_G(\lambda)$

$$\mathrm{mult}(V_G(\lambda), \mathbb{C}[\overline{\mathcal{O}'}]) \le \mathrm{mult}(V_G(\lambda), \mathbb{C}[\overline{\mathcal{O}}]) \le \mathrm{mult}(V_G(\lambda), \mathbb{C}[\mathcal{O}]).$$

$$\boxed{\text{is } \mathrm{mult}(V_G(\lambda), \mathbb{C}[\mathcal{O}]) = 0 \text{ ?}}$$

Let $H$ be the isotropy of a point of $\mathcal{O}$. Then

$$\mathrm{mult}(V_G(\lambda), \mathbb{C}[\mathcal{O}]) = \dim((V_G(\lambda)^*)^H).$$

# Main example

- Let $E = \mathbb{C}^n$, $W = \mathrm{End}(E) = E^* \otimes E$, $V = S^n W^*$, $G = \mathrm{GL}(W) = \mathrm{GL}_{n^2}(\mathbb{C})$ and

$$\mathcal{O} = G.\det \subset V.$$

  Here $\dim G = n^4$ and $\dim V = \binom{n^2+n-1}{n} = 10, 165, 3876, 118755\dots$

- The isotropy if given by

$$\det(AMB) = \det(M) \quad \text{if} \quad \det(A).\det(B) = 1$$
$$\det(M^t) = \det(M).$$

  Hence $H^0 = S(\mathrm{GL}(E) \times \mathrm{GL}(E))$ and $H/H^0 = \mathbb{Z}/2\mathbb{Z}$.

- But

$$k_{\delta^n\,\delta^n\,\lambda} = \dim\left((S_\lambda V^*)^{H^0}\right) \geq \dim\left((S_\lambda V^*)^{H}\right) =: sk_{\delta^n\,\delta^n\,\lambda},$$

  where $|\lambda| = \delta n$.

# A Murnaghan's result

Let $\alpha$, $\beta$, and $\gamma$ be three partitions of the same integer $n$. The Kronecker coefficient $k_{\alpha\,\beta\,\gamma}$ is defined by

$$[\alpha] \otimes [\beta] = \sum_{\gamma} k_{\alpha\,\beta\,\gamma}[\gamma], \tag{2}$$

or

$$S_\gamma(E \otimes F) = \sum_{\alpha\,\beta} k_{\alpha\,\beta\,\gamma} S_\alpha E \otimes S_\beta F. \tag{3}$$

Similarly the Littlewood-Richardson coefficients are defined by

$$S_\alpha V \otimes S_\beta V = \sum_{\gamma} c_{\alpha\,\beta}^{\gamma} S_\gamma V. \tag{4}$$

# A Murnaghan's result

**Proposition**

1. *If* $k_{\alpha\,\beta\,\gamma} \neq 0$ *then*
$$(n - \alpha_1) + (n - \beta_1) \geq n - \gamma_1. \tag{5}$$
2. *If* $(n - \alpha_1) + (n - \beta_1) = n - \gamma_1$ *then*
$$k_{\alpha\,\beta\,\gamma} = c^{\bar{\gamma}}_{\bar{\alpha}\,\bar{\beta}}. \tag{6}$$

In particular, Kronecker coefficients extend Littlewood-Richardson's one.

# Weyl's inequalities

If $c_{\bar{\alpha}\,\bar{\beta}}^{\bar{\gamma}} \neq 0$ then

$$\bar{\gamma}_{e+j-1} \leq \bar{\beta}_{j-1}, \tag{7}$$

whenever $l(\bar{\alpha}) \leq e$ and $j \geq 2$.

## Theorem

*Let $e$ and $f$ be two postive integers. Let $\alpha$, $\beta$, and $\gamma$ be three partitions of the same integer $n$ such that*

$$l(\alpha) \leq e+1, \quad l(\beta) \leq f+1, \quad \textit{and} \quad l(\gamma) \leq e+f+1. \tag{8}$$

*Let $j \in \{2, \ldots, f+1\}$.*
*If $k_{\alpha\,\beta\,\gamma} \neq 0$ then*

$$n + \gamma_1 + \gamma_{e+j} \geq \alpha_1 + \beta_1 + \beta_j \tag{9}$$

# Horn's inequalities

To $I \in \mathcal{S}(r,n)$, associate the partition

$$\tau^I = (d - r + 1 - i_1 \geq d - r + 2 - i_2 \geq \cdots \geq d - i_r).$$

Set $|\alpha_I| := \sum_{i \in I} \alpha_i$.

**Theorem**

*Let $\alpha$, $\beta$, and $\gamma$ be three partitions of the same integer $n$ satisfying conditions (8).*
*If $k_{\alpha\,\beta\,\gamma} \neq 0$ then*

$$n + |\bar{\alpha}_I| - \alpha_1 + |\bar{\beta}_J| - \beta_1 \geq |\bar{\gamma}_K| - \gamma_1, \tag{10}$$

*for any $0 < r < e$, $0 < s < f$, $I \in \mathcal{S}(r,e)$, $J \in \mathcal{S}(s,f)$ and $K \in \mathcal{S}(r+s, e+f)$ such that*

$$c_{\tau^I\,\tau^J}^{\tau^K} = 1. \tag{11}$$

# The statements

## Theorem

*The* $\mathrm{GL}(W)$*-module* $S_\lambda W$ *is not a submodule of* $\mathbb{C}[\mathcal{O}]$ *for*

1. $\lambda = ab^{n^2-1}$ *where* $a \geq b$ *and*

$$\begin{cases} n \equiv 2 \quad [4]; \\ n \text{ devides } a-b; \\ b \text{ is odd.} \end{cases}$$

2. $\lambda = a^2 b^7$ *where* $a \geq b$, $n = 3$, *and*

$$\begin{cases} 3 \text{ devides } a-b; \\ a \text{ is odd.} \end{cases}$$

3. $\lambda = a^3 b^6$ *where* $a \geq b$ , $n = 3$, *and*

$$\{ \; a \text{ is odd.}$$

# Examples

Let $\delta \in \mathbb{Z}_{\geq 0}$ be such that $|\lambda| = n\delta$. Such an example is interesting if

1. $S^{\delta}(S^n W)$ contains $S_{\lambda} W$;
2. $k_{\delta^n\,\delta^n\,\lambda} \neq 0$.

Such an example is the case $\lambda = 7^3 3^6$. Then $\delta = 13$ and

$$mult(S_{\lambda} W, \mathbb{C}[\mathcal{O}]) = 0, \quad (12)$$

$$mult(S_{\lambda} W, S^{\delta}(S^3 W)) = 1, \quad (13)$$

$$mult(S_{\lambda} W, \mathbb{C}[G/H^{\circ}]) = k_{13^3\,13^3\,7^3 3^6} = k_{4^3\,4^3\,4^3} = 2. \quad (14)$$

**There exists a degree 13 equation for $\overline{\mathrm{GL}_9 \,.\, \det_3}$.**

# A question

- Consider

$$z^3 + xt^2 + x^2y = \begin{vmatrix} 1 & 0 & y & 0 & 0 \\ x & t & 0 & z & 0 \\ 0 & 1 & t & 0 & 0 \\ 0 & 0 & z & 0 & -x \\ 0 & 0 & 0 & 1 & z \end{vmatrix}.$$

  It is the only cubic surface containing a unique line. It is also the unique nondeterminantal cubic surface.

- Hence $\mathrm{dc}(z^3 + xt^2 + x^2y) \leq 5$.
  One can prove that $\mathrm{dc}(z^3 + xt^2 + x^2y) \geq 4$.
  The open question:

$$\mathrm{dc}(z^3 + xt^2 + x^2y) = 4 \text{ or } 5 \text{ ?}$$