Combinatorics and complexity of Kronecker coefficients

Greta Panova

University of Pennsylvania

September 2014

The Kronecker coefficients

Irreducible representations of the symmetric group S_n:

(group homomorphisms $S_n \to GL_N(\mathbb{C})$)

— the **Specht modules** $\mathbb{S}_\lambda$, indexed by partitions $\lambda \vdash n$

Tensor product decomposition:

$$\mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_{\nu \vdash n} \mathbb{S}_\nu^{\oplus ??}$$

The Kronecker coefficients

Irreducible representations of the symmetric group S_n:

(group homomorphisms $S_n \to GL_N(\mathbb{C})$)

— the **Specht modules** $\mathbb{S}_\lambda$, indexed by partitions $\lambda \vdash n$

Tensor product decomposition:

$$\mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_{\nu \vdash n} \mathbb{S}_\nu^{\oplus g(\lambda,\mu,\nu)}$$

Kronecker coefficients: $g(\lambda, \mu, \nu)$ – multiplicity of $\mathbb{S}_\nu$ in $\mathbb{S}_\lambda \otimes \mathbb{S}_\mu$

The Kronecker coefficients

Irreducible representations of the symmetric group S_n:

(group homomorphisms $S_n \to GL_N(\mathbb{C})$)

— the **Specht modules** $\mathbb{S}_\lambda$, indexed by partitions $\lambda \vdash n$

Tensor product decomposition:

$$\mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_{\nu \vdash n} \mathbb{S}_\nu^{\oplus g(\lambda,\mu,\nu)}$$

Kronecker coefficients: $g(\lambda,\mu,\nu)$ – multiplicity of $\mathbb{S}_\nu$ in $\mathbb{S}_\lambda \otimes \mathbb{S}_\mu$

$$g(\lambda,\mu,\nu) = \dim \mathrm{Hom}_{S_n}(\mathbb{S}_\nu, \mathbb{S}_\lambda \otimes \mathbb{S}_\mu)$$

In terms of $GL(\mathbb{C}^m)$ modules V_λ, V_μ and $GL(\mathbb{C}^{m^2})$ module V_ν:

$$g(\lambda,\mu,\nu) = \dim \mathrm{Hom}_{GL(\mathbb{C}^m)\times GL(\mathbb{C}^m)}(V_\lambda \otimes V_\mu, V_\nu)$$

The Kronecker coefficients

Irreducible representations of the symmetric group S_n:

(group homomorphisms $S_n \to GL_N(\mathbb{C})$)

— the **Specht modules** $\mathbb{S}_\lambda$, indexed by partitions $\lambda \vdash n$
Tensor product decomposition:

$$\mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \oplus_{\nu \vdash n} \mathbb{S}_\nu^{\oplus g(\lambda,\mu,\nu)}$$

Kronecker coefficients: $g(\lambda, \mu, \nu)$ – multiplicity of $\mathbb{S}_\nu$ in $\mathbb{S}_\lambda \otimes \mathbb{S}_\mu$

Littlewood-Richardson coefficients $c^\lambda_{\mu\nu}$: Tensor products of GL_N representations:

$$V_\mu \otimes V_\nu = \oplus_{\lambda \vdash |\mu|+|\nu|} V_\lambda^{\oplus c^\lambda_{\mu\nu}}$$

The combinatorial problem

Problem (Murnaghan, 1938)

Find a positive combinatorial interpretation for $g(\lambda,\mu,\nu)$, i.e. a family of combinatorial objects $\mathcal{O}_{\lambda,\mu,\nu}$, s.t. $g(\lambda,\mu,\nu) = \#\mathcal{O}_{\lambda,\mu,\nu}$.

Motivation: Littlewood–Richardson
$c^{\lambda}_{\mu,\nu}$, $\mathcal{O}_{\lambda,\mu,\nu} = \{$ LR tableaux of shape λ/μ, type $\nu\}$
$\lambda = (6,4,3), \mu = (3,1), \nu = (4,3,2)$:

			1	1	1
	1	2	2		
2	3	3			

			1	1	1
	2	2	2		
1	3	3			

$\Rightarrow c^{\lambda}_{\mu\nu} = 2.$

Theorem (Murnaghan)

If $|\nu| + |\mu| = |\lambda|$ and $n > |\nu|$, then

$$g((n+|\mu|,\nu),(n+|\nu|,\mu),(n,\lambda)) = c^{\lambda}_{\mu\nu}.$$

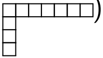
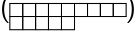
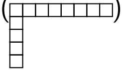
The combinatorial problem

Problem (Murnaghan, 1938)

Find a positive combinatorial interpretation for $g(\lambda, \mu, \nu)$, i.e. a family of combinatorial objects $\mathcal{O}_{\lambda,\mu,\nu}$, s.t. $g(\lambda, \mu, \nu) = \#\mathcal{O}_{\lambda,\mu,\nu}$.

Results since then:

Combinatorial formulas for $g(\lambda, \mu, \nu)$, when:

- μ and ν are hooks (), [*Remmel, 1989*]
- $\nu = (n-k, k)$ () and $\lambda_1 \geq 2k-1$, [*Ballantine–Orellana, 2006*]
- $\nu = (n-k, k)$, $\lambda = (n-r, r)$ [*Blasiak–Mulmuley–Sohoni*,2013]
- $\nu = (n-k, 1^k)$ (), [*Blasiak, 2012*]

Kronecker coefficients and GCT

Input: Integers N, ℓ, partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $\mu = (\mu_1, \ldots, \mu_\ell)$, $\nu = (\nu_1, \ldots, \nu_\ell)$, where $0 \leq \lambda_i, \mu_i, \nu_i \leq N$, and $|\lambda| = |\mu| = |\nu|$.
Size(Input)=$O(\ell \log N)$.

POSITIVITY OF KRONECKER COEFFICIENTS (KP):

Decide: whether $g(\lambda, \mu, \nu) > 0$

KRONECKER COEFFICIENTS (KRON):

Compute: $g(\lambda, \mu, \nu)$.

Kronecker coefficients and GCT

Input: Integers N, ℓ, partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $\mu = (\mu_1, \ldots, \mu_\ell)$, $\nu = (\nu_1, \ldots, \nu_\ell)$, where $0 \leq \lambda_i, \mu_i, \nu_i \leq N$, and $|\lambda| = |\mu| = |\nu|$.
Size(Input)$=O(\ell \log N)$.

POSITIVITY OF KRONECKER COEFFICIENTS (KP):

Decide: whether $g(\lambda, \mu, \nu) > 0$

Conjecture (Mulmuley)

KP *is in* P .

KRONECKER COEFFICIENTS (KRON):

Compute: $g(\lambda, \mu, \nu)$.

Conjecture (Mulmuley)

When the input is binary, KRON *is in* #P .

Kronecker coefficients and GCT

Input: Integers N, ℓ, partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $\mu = (\mu_1, \ldots, \mu_\ell)$, $\nu = (\nu_1, \ldots, \nu_\ell)$, where $0 \leq \lambda_i, \mu_i, \nu_i \leq N$, and $|\lambda| = |\mu| = |\nu|$.
Size(Input)$=O(\ell \log N)$.

POSITIVITY OF KRONECKER COEFFICIENTS (KP):

Decide: whether $g(\lambda, \mu, \nu) > 0$

Conjecture (Mulmuley)

KP *is in* P . *(No positive combinatorial interpretation, not known: "KP is in NP "??)*

KRONECKER COEFFICIENTS (KRON):

Compute: $g(\lambda, \mu, \nu)$.

Conjecture (Mulmuley)

When the input is binary, KRON *is in* #P .

Theorem:[Littlewood-Richardson rule/Narayanan + Knutson-Tao]
Conjectures hold for the Littlewood-Richardson coefficients $c^{\bar\lambda}_{\bar\mu\bar\nu} = g(\lambda, \mu, \nu)$, where $\lambda = (n - |\bar\lambda|, \bar\lambda)$, etc, and $|\bar\lambda| = |\bar\mu| + |\bar\nu|$.

Kronecker coefficients and GCT

Input: Integers N, ℓ, partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $\mu = (\mu_1, \ldots, \mu_\ell)$, $\nu = (\nu_1, \ldots, \nu_\ell)$, where $0 \leq \lambda_i, \mu_i, \nu_i \leq N$, and $|\lambda| = |\mu| = |\nu|$.
Size(Input)$=O(\ell \log N)$.

POSITIVITY OF KRONECKER COEFFICIENTS (KP):

Decide: whether $g(\lambda, \mu, \nu) > 0$

Conjecture (Mulmuley)

KP *is in* P . *(No positive combinatorial interpretation, not known: "*KP *is in* NP *"??)*

KRONECKER COEFFICIENTS (KRON):

Compute: $g(\lambda, \mu, \nu)$.

Conjecture (Mulmuley)

When the input is binary, KRON *is in* #P *.*

Theorem[Bürgisser-Ikenmeyer]. KRON is in GapP .
(GapP $= \{F \ : \ F = F_1 - F_2 \ , F_1, F_2 \in$ #P$\}$)
Theorem[Narayanan]. KRON is #P –hard.

Positivity problems

Conjecture (Mulmuley)

KP *is in* P *.*
(Given input (λ, μ, ν)*, deciding whether* $g(\lambda, \mu, \nu) > 0$ *is in* P *.)*

Positivity problems

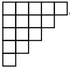
Conjecture (Mulmuley)

KP *is in* P .
(Given input (λ, μ, ν)*, deciding whether* $g(\lambda, \mu, \nu) > 0$ *is in* P *.)*

Theorem (Heide, Saxl, Tiep, Zalesski)

Let G *be a finite simple group of Lie type, other than* $PSU_n(q)$ *with* $n \geq 3$ *coprime to* $2(q+1)$*. Then every irreducible character of* G *is a constituent of the tensor square* $St \otimes St$ *of the Steinberg character* St *of* G*.*

Conjecture (Saxl)

For every $n > 9$ *there is an irreducible character* χ^μ*, s.t.* $\chi^\mu \otimes \chi^\mu$ *contains every irreducible character of* S_n *as a constituent. When* $n = \binom{k+1}{2}$*, such character is* χ^{ρ_k} *for* $\rho_k = (k, k-1, \ldots, 2, 1) \vdash n$*:* $\rho_k =$ *.*

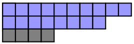
In other words: $g(\rho_k, \rho_k, \nu) > 0$ *for all* $\nu \vdash n$*.*

Positivity results I: Saxl conjecture

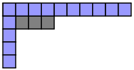
Theorem (Pak-P-Vallejo)

There is a universal constant L, such that for every $k \geq L$, the tensor square $\chi^{\rho_k} \otimes \chi^{\rho_k}$ contains the characters χ^{λ} as constituents, for all

$\lambda = (n-\ell, \ell)$, $0 \leq \ell \leq n/2$, *or* $\lambda = (n-r, 1^r)$, $0 \leq r \leq n-1$,



$\lambda = (n-\ell-m, \ell, m)$, $m \in \{1, 3, 5, 7, 8, 9\}$, $L \leq \ell+m \leq n/2$,



$\lambda = (n-r-m, m, 1^r)$, $1 \leq m \leq 10$, $L \leq r < n/2 - 5$.

Remark. Holds for other partitions besides ρ_k, e.g. a chopped square $(k^{k-1}, k-1)$.

Character lemma

Theorem (Pak-P,2014+ (extends earlier [Pak-P-Vallejo]))

Let $\mu = \mu'$ *be a self-conjugate partition and let* $\widehat{\mu} = (2\mu_1 - 1, 2\mu_2 - 3, \ldots) \vdash n$ *be the partition of its principal hooks. Then:*

$$g(\nu, \mu, \mu) \geq \left|\chi^{\nu}[\widehat{\mu}]\right|, \quad \textit{for every} \quad \nu \vdash n.$$

Character lemma

Theorem (Pak-P,2014+ (extends earlier [Pak-P-Vallejo]))

Let $\mu = \mu'$ be a self-conjugate partition and let $\widehat{\mu} = (2\mu_1 - 1, 2\mu_2 - 3, \ldots) \vdash n$ be the partition of its principal hooks. Then:

$$g(\nu, \mu, \mu) \geq \left|\chi^\nu[\widehat{\mu}]\right|, \quad \textit{for every} \quad \nu \vdash n.$$

Proof idea: reducing to A_n representations.

Character lemma

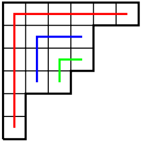
Theorem (Pak-P,2014+ (extends earlier [Pak-P-Vallejo]))

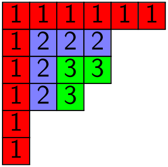
Let $\mu = \mu'$ be a self-conjugate partition and let $\widehat{\mu} = (2\mu_1 - 1, 2\mu_2 - 3, \ldots) \vdash n$ be the partition of its principal hooks. Then:

$$g(\nu, \mu, \mu) \geq \left|\chi^{\nu}[\widehat{\mu}]\right|, \quad \textit{for every} \quad \nu \vdash n.$$

Proof idea: reducing to A_n representations.

Example:

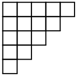
$\mu =$ 

$\lambda = \nu =$ 

$|\chi^{\nu}[(\widehat{\nu}]| = 1$

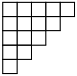
Corollary:[Bessenrodt–Behns] For every μ, s.t. $\mu = \mu'$ we have $\langle \chi^{\mu} \otimes \chi^{\mu}, \chi^{\mu} \rangle \neq 0$.

Computing $\chi^\nu[(2k-1, 2k-5, \ldots)]$

$\rho_k = (k, k-1, \ldots, 1) =$  , $A = \{2k-1, 2k-5, \ldots\}$

$$\sum_{m=0}^{N} p_A(m) t^m = \prod_{s \in A} (1 + t^s)$$

Computing $\chi^{\nu}[(2k-1, 2k-5, \ldots)]$

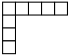
$\rho_k = (k, k-1, \ldots, 1) =$  , $A = \{2k-1, 2k-5, \ldots\}$

$$\sum_{m=0}^{N} p_A(m) t^m = \prod_{s \in A} (1 + t^s)$$

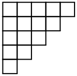
Two-rows:

 $\chi^{(n-\ell,\ell)}[(2k-1, 2k-5, \ldots)] = p_A(\ell) - p_A(\ell-1)$

Hooks:

 $$\chi^{(n-\ell,1^{\ell})}[(2k-1, \ldots)] = \begin{cases} p_A(\ell) \quad , & k\text{–odd} \\ p_A(\ell) - p_A(\ell-1) + p_A(\ell-2), & k\text{–even} \end{cases}$$

Computing $\chi^\nu[(2k-1, 2k-5, \ldots)]$

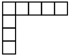
$\rho_k = (k, k-1, \ldots, 1) =$  , $A = \{2k-1, 2k-5, \ldots\}$

$$\sum_{m=0}^{N} p_A(m) t^m = \prod_{s \in A} (1 + t^s)$$

Two-rows:

 $\chi^{(n-\ell,\ell)}[(2k-1, 2k-5, \ldots)] \;=\; p_A(\ell) \;-\; p_A(\ell-1)$

Hooks:

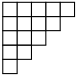
 $$\chi^{(n-\ell,1^\ell)}[(2k-1,\ldots)] = \begin{cases} p_A(\ell) > 0, & k\text{–odd} \\ \underbrace{p_A(\ell) - p_A(\ell-1) + p_A(\ell-2)}_{>0}, & k\text{–even} \end{cases}$$

Theorem (Odlyzko-Richmond)

For every $A = \{a, a+b, a+2b, \ldots, a+rb\}$, $\gcd(a,b) = 1$, *there exists* $L = L(a,b)$ *s.t.*

$$p_A(m+1) \; > \; p_A(m) \; > \; 0, \quad \text{for all } L \le m < \lfloor N/2 \rfloor .$$

Computing $\chi^\nu[(2k-1, 2k-5, \ldots)]$

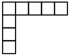
$\rho_k = (k, k-1, \ldots, 1) =$  $, A = \{2k-1, 2k-5, \ldots\}$

$$\sum_{m=0}^{N} p_A(m) t^m = \prod_{s \in A} (1 + t^s)$$

Two-rows:

 $\chi^{(n-\ell,\ell)}[(2k-1, 2k-5, \ldots)] = p_A(\ell) - p_A(\ell-1) > 0$

Hooks:

 $$\chi^{(n-\ell,1^\ell)}[(2k-1, \ldots)] = \begin{cases} p_A(\ell) > 0, & k\text{–odd} \\ \underbrace{p_A(\ell) - p_A(\ell-1) + p_A(\ell-2)}_{>0}, & k\text{–even} \end{cases}$$

$\Rightarrow$ **Theorem** (Saxl conj for special ν):
$g(\nu, \rho_k, \rho_k) \geq \chi^\nu[(2k-1, 2k-5, \ldots)] > 0$ for ν – 2-row, hook, etc.

Symmetric function techniques

$\Lambda(x) = \{f \in \mathbb{Q}[[x_1, x_2, \ldots]] \ , \ f(x_1, x_2, \ldots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots)$ for all permutations $\sigma\}$

Schur functions s_λ – (orthonormal basis) for Λ,

$s_\lambda(x_1, \ldots, x_n) = \text{trace}\rho_\lambda[diag(x_1, \ldots, x_n)]$, where $\rho_\lambda : GL_n \to GL(V_\lambda)$.

$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$

Weyl character formula:

$$s_\lambda(x_1, \ldots, x_n) = \frac{\det \left[x_i^{\lambda_j + n - j} \right]_{i,j=1}^{n}}{\det \Delta(x_1, \ldots, x_n)}$$

Semi-Standard Young tableaux of shape λ :

$$s_{(2,2)}(x_1, x_2, x_3) = s_{\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline\end{array}}(x_1, x_2, x_3) = \underset{\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline\end{array}}{x_1^2 x_2^2} + \underset{\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & 3 \\ \hline\end{array}}{x_1^2 x_3^2} + \underset{\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 3 \\ \hline\end{array}}{x_2^2 x_3^2}$$

$$+ \underset{\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline\end{array}}{x_1^2 x_2 x_3} + \underset{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline\end{array}}{x_1 x_2^2 x_3} + \underset{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 3 \\ \hline\end{array}}{x_1 x_2 x_3^2}.$$

Symmetric function techniques II

Kronecker product $*$ on Λ:

$$\lambda, \mu \vdash n: \qquad s_\lambda * s_\mu = \sum_{\nu \vdash n} g(\lambda, \mu, \nu) s_\nu$$

Symmetric function techniques II

Kronecker product $*$ on Λ:

$$\lambda, \mu \vdash n : \qquad s_\lambda * s_\mu = \sum_{\nu \vdash n} g(\lambda, \mu, \nu) s_\nu$$

generalized Cauchy:

$$\sum_{\lambda,\mu,\nu} g(\lambda, \mu, \nu) s_\lambda(x) s_\mu(y) s_\nu(z) = \prod_{i,j,k} \frac{1}{1 - x_i y_j z_k}$$

"plethystic" version:

$g(\lambda, \mu, \nu) = [s_\lambda(x) s_\mu(y)] s_\nu(xy)$ where $xy = (x_1 y_1, x_1 y_2, \ldots, x_2 y_1, x_2 y_2, \ldots)$

$(\langle f(xy), g(x)h(y) \rangle = \langle f, g * h \rangle)$

Symmetric function techniques II

Kronecker product $*$ on Λ:

$$\lambda, \mu \vdash n : \qquad s_\lambda * s_\mu = \sum_{\nu \vdash n} g(\lambda, \mu, \nu) s_\nu$$

generalized Cauchy:

$$\sum_{\lambda,\mu,\nu} g(\lambda, \mu, \nu) s_\lambda(x) s_\mu(y) s_\nu(z) = \prod_{i,j,k} \frac{1}{1 - x_i y_j z_k}$$

"plethystic" version:

$g(\lambda, \mu, \nu) = [s_\lambda(x) s_\mu(y)] s_\nu(xy)$ where $xy = (x_1 y_1, x_1 y_2, \ldots, x_2 y_1, x_2 y_2, \ldots)$

$(\langle f(xy), g(x) h(y) \rangle = \langle f, g * h \rangle)$

Schur expansion:

$$\langle s_\lambda, f \rangle = [x_1^{\lambda_1 + n - 1} \cdots x_n^{\lambda_n}] \Delta(x_1, \ldots, x_n) f(x_1, \ldots, x_n)$$

Easy example

MathOverflow question [Jiarui Fei]: Is there an easy way to see why

$$g(\lambda, \mu, \nu) = 1$$

for $\lambda = m^{n\ell},\ \mu = (\ell m)^n,\ \nu = (nm)^\ell$?

Easy example

MathOverflow question [Jiarui Fei]: Is there an easy way to see why

$$g(\lambda, \mu, \nu) = 1$$

for $\lambda = m^{n\ell},\ \mu = (\ell m)^n,\ \nu = (nm)^\ell$?

Answer: [P]

$$(\diamond) \qquad s_\lambda(xy) = \sum_{\alpha,\beta} g(\lambda, \alpha, \beta) s_\alpha(x) s_\beta(y) = g(\lambda, \mu, \nu) s_\mu(x) s_\nu(y) + (\geq 0).$$

Easy example

MathOverflow question [Jiarui Fei]: Is there an easy way to see why

$$g(\lambda,\mu,\nu)=1$$

for $\lambda = m^{n\ell},\ \mu=(\ell m)^n,\ \nu=(nm)^\ell$?

Answer: [P]

$$(\diamond) \qquad s_\lambda(xy)=\sum_{\alpha,\beta} g(\lambda,\alpha,\beta)s_\alpha(x)s_\beta(y) = g(\lambda,\mu,\nu)s_\mu(x)s_\nu(y)+(\geq 0).$$

Set $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_\ell)$, $xy=(x_1y_1,\ldots,x_ny_\ell)$, so

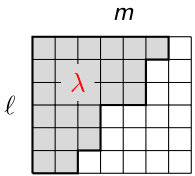
$$s_{m^{n\ell}}(xy)=\prod_{i,j}(x_iy_j)^m = \left(\prod x_i\right)^{\ell m}\left(\prod y_j\right)^{nm} = s_{(\ell m)}(x)s_{(nm)^\ell}(y)$$

...compare with $(\diamond)$

Applications: partitions inside a rectangle

$$p_n(\ell, m) = \#\{\lambda \vdash n, \ell(\lambda) \leq \ell, \lambda_1 \leq m\}$$

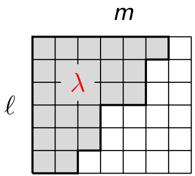
$$\sum_{n \geq 0} p_n(\ell, m) q^n = \prod_{i=1}^{\ell} \frac{1 - q^{m+i}}{1 - q^i} = \binom{m+\ell}{m}_q$$



Applications: partitions inside a rectangle

$$p_n(\ell, m) = \#\{\lambda \vdash n, \ell(\lambda) \le \ell, \lambda_1 \le m\}$$

$$\sum_{n\ge 0} p_n(\ell,m) q^n = \prod_{i=1}^{\ell} \frac{1-q^{m+i}}{1-q^i} = \binom{m+\ell}{m}_q$$



Theorem (Sylvester 1878, Cayley's conjecture 1856)

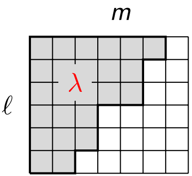
The sequence $p_0(\ell, m), \ldots, p_{\ell m}(\ell, m)$ *is unimodal, i.e.*

$$p_0(\ell,m) \le p_1(\ell,m) \le \ldots \le p_{\lfloor \ell m/2 \rfloor}(\ell,m) \ge \cdots \ge p_{\ell m}(\ell,m)$$

Applications: partitions inside a rectangle

$$p_n(\ell, m) = \#\{\lambda \vdash n, \ell(\lambda) \leq \ell, \lambda_1 \leq m\}$$

$$\sum_{n \geq 0} p_n(\ell, m) q^n = \prod_{i=1}^{\ell} \frac{1-q^{m+i}}{1-q^i} = \binom{m+\ell}{m}_q$$



Theorem (Sylvester 1878, Cayley's conjecture 1856)

The sequence $p_0(\ell, m), \ldots, p_{\ell m}(\ell, m)$ *is unimodal, i.e.*

$$p_0(\ell, m) \leq p_1(\ell, m) \leq \ldots \leq p_{\lfloor \ell m/2 \rfloor}(\ell, m) \geq \cdots \geq p_{\ell m}(\ell, m)$$

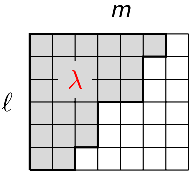
"*I am about to demonstrate a theorem which has been waiting proof for the last quarter of a century and upwards. [...] I accomplished with scarcely an effort a task which I had believed lay outside the range of human power.*"

J.J. Sylvester, 1878.

Applications: partitions inside a rectangle

$$p_n(\ell, m) = \#\{\lambda \vdash n, \ell(\lambda) \le \ell, \lambda_1 \le m\}$$

$$\sum_{n \ge 0} p_n(\ell, m) q^n = \prod_{i=1}^{\ell} \frac{1 - q^{m+i}}{1 - q^i} = \binom{m+\ell}{m}_q$$



Theorem (Sylvester 1878, Cayley's conjecture 1856)

The sequence $p_0(\ell, m), \ldots, p_{\ell m}(\ell, m)$ *is unimodal, i.e.*

$$p_0(\ell, m) \le p_1(\ell, m) \le \ldots \le p_{\lfloor \ell m/2 \rfloor}(\ell, m) \ge \cdots \ge p_{\ell m}(\ell, m)$$

Proofs:

Sylvester, 1878: *"by aid of a construction drawn from the resources of Imaginative Reason"* (Lie algebras, $\mathfrak{sl}_2$ representations)

Stanley, 1978: hard Lefschetz theorem (alg. geom.), gives Sperner property; 1982: Linear Algebra Paradigm.

Proctor, 1982: explicit linear operators.

O'Hara, 1990: constructive combinatorial proof.

Kronecker and partitions inside a rectangle

Let $\tau^k = (N-k, k)$, $a_k(\lambda, \mu) = \sum_{\alpha \vdash k, \beta \vdash N-k} c^{\lambda}_{\alpha\beta} c^{\mu}_{\alpha\beta}$.

Lemma (Pak-P, Vallejo)

- ♣ *We have that* $g(\lambda, \mu, \tau^k) = a_k(\lambda, \mu) - a_{k-1}(\lambda, \mu)$.
- ♠ *The sequence* $a_0(\lambda, \mu), a_1(\lambda, \mu), \ldots, a_N(\lambda, \mu)$ *is unimodal for all* $\lambda, \mu \vdash N$.

Kronecker and partitions inside a rectangle

Let $\tau^k = (N-k, k)$, $a_k(\lambda,\mu) = \sum_{\alpha\vdash k, \beta\vdash N-k} c^\lambda_{\alpha\beta} c^\mu_{\alpha\beta}$.

Lemma (Pak-P, Vallejo)

- ♣ *We have that* $g(\lambda,\mu,\tau^k) = a_k(\lambda,\mu) - a_{k-1}(\lambda,\mu)$.
- ♠ *The sequence* $a_0(\lambda,\mu), a_1(\lambda,\mu), \ldots, a_N(\lambda,\mu)$ *is unimodal for all* $\lambda,\mu\vdash N$.

Proof: $g(\lambda,\mu,\tau^k) = [s_\lambda(x)s_\mu(y)]s_{\tau^k}(xy)$

$$s_{\tau^k}(xy) = s_{N-k}(xy)s_k(xy) - s_{N-k+1}(xy)s_{k-1}(xy)$$

$$s_{N-k}(xy)s_k(xy) = \left(\sum_{\alpha\vdash N-k} s_\alpha(x)s_\alpha(y)\right)\left(\sum_{\beta\vdash k} s_\beta(x)s_\beta(y)\right)$$

$$[s_\lambda(x)s_\mu(y)] \sum_{\alpha\vdash N-k, \beta\vdash k} (s_\alpha(x)s_\beta(x))(s_\alpha(y)s_\beta(y))$$

$$= \sum_{\alpha\vdash k, \beta\vdash N-k} c^\lambda_{\alpha\beta}c^\mu_{\alpha\beta} = a_k(\lambda,\mu) \Rightarrow ♣$$

Kronecker and partitions inside a rectangle

Let $\tau^k = (N-k, k)$, $a_k(\lambda, \mu) = \sum_{\alpha \vdash k, \beta \vdash N-k} c^{\lambda}_{\alpha\beta} c^{\mu}_{\alpha\beta}$.

Lemma (Pak-P, Vallejo)

- ♣ *We have that* $g(\lambda, \mu, \tau^k) = a_k(\lambda, \mu) - a_{k-1}(\lambda, \mu)$.
- ♠ *The sequence* $a_0(\lambda, \mu), a_1(\lambda, \mu), \ldots, a_N(\lambda, \mu)$ *is unimodal for all* $\lambda, \mu \vdash N$.

Proof: ♣ + $g(\lambda, \mu, \tau^k) \geq 0 \Rightarrow a_k(\lambda, \mu) \geq a_{k-1}(\lambda, \mu) \Leftrightarrow$ ♠

Kronecker and partitions inside a rectangle

Let $\tau^k = (N-k,k)$, $a_k(\lambda,\mu) = \sum_{\alpha\vdash k, \beta\vdash N-k} c^{\lambda}_{\alpha\beta} c^{\mu}_{\alpha\beta}$.

Lemma (Pak-P, Vallejo)

- ♣ *We have that* $g(\lambda,\mu,\tau^k) = a_k(\lambda,\mu) - a_{k-1}(\lambda,\mu)$.
- ♠ *The sequence* $a_0(\lambda,\mu), a_1(\lambda,\mu),\ldots,a_N(\lambda,\mu)$ *is unimodal for all* $\lambda,\mu\vdash N$.

Let $\lambda=\mu=(m^\ell) \Rightarrow$

$$c^{m^\ell}_{\alpha\beta} = \begin{cases} 1, & \alpha\subset(m^\ell), \beta = m^\ell - (\alpha_\ell,\ldots,\alpha_1) \\ 0, & \text{o.w.} \end{cases}$$

$\Rightarrow a_k(m^\ell,m^\ell) = p_k(m,\ell)$

Corollary

$g(m^\ell,m^\ell,\tau^k) = p_k(m,\ell) - p_{k-1}(m,\ell)$ *and*
[Sylvester's Theorem:] $p_0(m,\ell), p_1(m,\ell),\ldots,p_{m\ell}(m,\ell)$ *is unimodal.*

Further: effective bounds

Theorem (Pak-P, 2014+)

For all $m \geq \ell \geq 8$ *and* $2 \leq k \leq \ell m/2$*, we have:*

$$p_k(\ell, m) - p_{k-1}(\ell, m) > A \frac{2^{\sqrt{s}}}{s^{9/4}},$$

where $s = \min\{2k, \ell^2\}$ *and* $A = 0.00449$*.*

Further: effective bounds

Theorem (Pak-P, 2014+)

For all $m \geq \ell \geq 8$ and $2 \leq k \leq \ell m/2$, we have:

$$p_k(\ell, m) - p_{k-1}(\ell, m) > A \frac{2^{\sqrt{s}}}{s^{9/4}},$$

where $s = \min\{2k, \ell^2\}$ and $A = 0.00449$.

Proof: Case 1 – if $m = \ell$

$$\prod_{i=1}^{m} \left(1 + q^{2i-1}\right) =: \sum_{k=0}^{m^2} b_k(m)\, q^k .$$

$$p_k(m,m) - p_{k-1}(m,m) = g(m^m, m^m, \tau^k) \underbrace{\geq}_{\text{Character Lemma}}$$

$$|\chi^{\tau^k}[(2m-1, 2m-3, \ldots)]| = |b_k(m) - b_{k-1}(m)|$$

+ asymptotics of $b_k(m)$

Further: effective bounds

Theorem (Pak-P, 2014+)

For all $m \geq \ell \geq 8$ and $2 \leq k \leq \ell m/2$, we have:

$$p_k(\ell, m) - p_{k-1}(\ell, m) > A \frac{2^{\sqrt{s}}}{s^{9/4}},$$

where $s = \min\{2k, \ell^2\}$ and $A = 0.00449$.

Proof: Case 2 – $m > \ell$:
Kronecker monotonicity property [Manivel]:

$$g(\alpha^1 + \alpha^2, \beta^1 + \beta^2, \gamma^1 + \gamma^2) \geq g(\alpha^1, \beta^1, \gamma^1) \quad \text{when } g(\alpha^2, \beta^2, \gamma^2) > 0$$

$$p_k(m, m) - p_{k-1}(m, m) = g(m^\ell, m^\ell, (m\ell - k, k))$$
$$\geq g(\ell^\ell, \ell^\ell, (\ell^2 - r, r)) = p_r(\ell, \ell) - p_{r-1}(\ell, \ell) \text{...case 1}$$

NP and #P from combinatorics

Theorem (Pak-P,2014+)

Let r be fixed and $\lambda = (m^\ell, 1^r)$ and $\mu = (m+r, m^{\ell-1})$. Then $g(\lambda, \mu, (m\ell + r - k, k))$ is equal to the number of certain trees with local conditions of depth $O(\log \ell)$, width $O(\ell)$, and entries $O(m\ell)$. Thus computing $g(\lambda, \mu, (m\ell + r - k, k))$ is in #P (input size is $O(\ell \log m)$).

Proof: formulas in terms of q-binomial coefficients (partitions inside rectangle) + O'Hara's combinatorial proof of Sylvester's theorem.

Theorem (Pak-P, corollary of Blasiak's combinatorial interpretation)

When ν is a hook, $\mathrm{KP} \in \mathrm{NP}$ and $\textsc{Kron} \in \#\mathrm{P}$.

Complexity of KRON and KP

Theorem (Pak-P)

Let $\lambda, \mu, \nu \vdash n$ be partitions with lengths $\ell(\lambda), \ell(\mu), \ell(\nu) \le \ell$, the largest parts $\lambda_1, \mu_1, \nu_1 \le N$, and $\nu_2 \le M$. Then the Kronecker coefficients $g(\lambda, \mu, \nu)$ can be computed in time

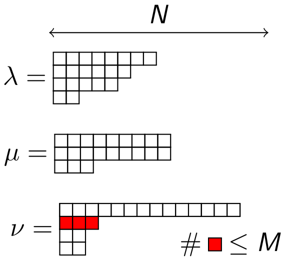
$$O(\ell \log N) + (\ell \log M)^{O(\ell^3 \log \ell)}.$$

Complexity of KRON and KP

Theorem (Pak-P)

Let $\lambda, \mu, \nu \vdash n$ be partitions with lengths $\ell(\lambda), \ell(\mu), \ell(\nu) \leq \ell$, the largest parts $\lambda_1, \mu_1, \nu_1 \leq N$, and $\nu_2 \leq M$. Then the Kronecker coefficients $g(\lambda, \mu, \nu)$ can be computed in time

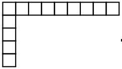
$$O(\ell \log N) + (\ell \log M)^{O(\ell^3 \log \ell)} .$$



Corollary. *Suppose*

$$\log M, \ell \;=\; O\left(\frac{(\log\log N)^{1/3}}{(\log\log\log N)^{2/3}}\right).$$

Then there is a **polynomial time** *algorithm to compute $g(\lambda, \mu, \nu)$.*

Example: ℓ small and $\nu =$  .

Complexity of KRON and KP

Theorem (Pak-P)

Let $\lambda, \mu, \nu \vdash n$ be partitions with lengths $\ell(\lambda), \ell(\mu)$, $\ell(\nu) \leq \ell$, the largest parts $\lambda_1, \mu_1, \nu_1 \leq N$, and $\nu_2 \leq M$. Then the Kronecker coefficients $g(\lambda, \mu, \nu)$ can be computed in time

$$O(\ell \log N) + (\ell \log M)^{O(\ell^3 \log \ell)}.$$

Corollary (Christandl-Doran-Walter)

When ℓ is fixed, the Kronecker coefficients can be computed in polynomial time, i.e. $\mathrm{KRON} \in FP$ (this case: Mulmuley's conjecture ✓)

Theorem (Pak-P)

When the number of parts (ℓ) is fixed, there exists a **linear time** *algorithm to decide whether $g(\lambda, \mu, \nu) > 0$ (i.e. solve KP).*

Proofs I: the Reduction Lemma

Lemma (Pak-P)

Let $\lambda, \mu, \nu \vdash n$ *and* $\ell(\lambda), \ell(\mu), \ell(\nu) \leq \ell$. *Set* $s = n - \nu_1$. *Then:*

(i) *If* $|\lambda_i - \mu_i| > s$ *for some* i, *then* $g(\lambda, \mu, \nu) = 0$,

(ii) *If* $|\lambda_i - \mu_i| \leq s$ *for all* i, $1 \leq i \leq \ell$, *there* $\exists$ *an* $r \leq 2s\ell^2$, *s.t.*

$$g(\lambda, \mu, \nu) = g\big(\phi(\lambda), \phi(\mu), \phi(\nu)\big)$$

for certain explicitly defined partitions $\phi(\lambda), \phi(\mu), \phi(\nu) \vdash r$.

Proofs I: the Reduction Lemma

Lemma (Pak-P)

Let $\lambda, \mu, \nu \vdash n$ *and* $\ell(\lambda), \ell(\mu), \ell(\nu) \leq \ell$. *Set* $s = n - \nu_1$. *Then:*

(i) *If* $|\lambda_i - \mu_i| > s$ *for some* i, *then* $g(\lambda, \mu, \nu) = 0$,

(ii) *If* $|\lambda_i - \mu_i| \leq s$ *for all* i, $1 \leq i \leq \ell$, *there* $\exists$ *an* $r \leq 2s\ell^2$, *s.t.*

$$g(\lambda, \mu, \nu) = g\big(\phi(\lambda), \phi(\mu), \phi(\nu)\big)$$

for certain explicitly defined partitions $\phi(\lambda), \phi(\mu), \phi(\nu) \vdash r$.

\+ **Semigroup property** [Manivel, Brion]:

$$g(\alpha, \beta, \gamma) > 0, g(\lambda, \mu, \nu) > 0 \Rightarrow g(\alpha + \lambda, \beta + \mu, \gamma + \nu) \geq g(\lambda, \mu, \nu)$$

Corollary ([1])

For any m *and partition* $\alpha \vdash m$, *we have that*

$$g(\lambda + n\alpha, \mu + n\alpha, \nu + (nm))$$

is bounded and increasing as a function of $n \in \mathbb{N}$, *i.e.* **stable**.

[1] [Pak-P], indep in [Vallejo], [Stembridge]

Proofs II: Explicit bounds on KRON complexity

Lemma: [2]

$$g(\alpha,\beta,\gamma) = \sum_{\sigma^1,\sigma^2,\sigma^3\in S_\ell} \operatorname{sgn}(\sigma^1\sigma^2\sigma^3)\, C(\alpha+1-\sigma^1,\beta+1-\sigma^2,\gamma+1-\sigma^3)\,,$$

where $C(u,v,w)$ is the number of $\ell\times\ell\times\ell$ contingency arrays $[A_{i,j,k}]$:

$$\sum_{j,k} A_{i,j,k} = u_i, \quad \sum_{i,k} A_{i,j,k} = v_j, \quad \sum_{i,j} A_{i,j,k} = w_k$$

[2] In [Christandl-Doran-Walter],[Pak-P]

Proofs II: Explicit bounds on KRON complexity

Lemma: [2]

$$g(\alpha,\beta,\gamma) = \sum_{\sigma^1,\sigma^2,\sigma^3\in S_\ell} \operatorname{sgn}(\sigma^1\sigma^2\sigma^3)\, C(\alpha+1-\sigma^1, \beta+1-\sigma^2, \gamma+1-\sigma^3)\,,$$

where $C(u,v,w)$ is the number of $\ell\times\ell\times\ell$ contingency arrays $[A_{i,j,k}]$:

$$\sum_{j,k} A_{i,j,k} = u_i, \quad \sum_{i,k} A_{i,j,k} = v_j, \quad \sum_{i,j} A_{i,j,k} = w_k$$

$+$ Barvinok's algorithm for counting integer points in polytopes:
$C(u,v,w)$ *can be computed in time*
$(\ell \max\{u_1,\ldots,v_1,\ldots,w_1,\ldots\})^{O(\ell^3\log\ell)}$.

[2] In [Christandl-Doran-Walter],[Pak-P]

Proofs II: Explicit bounds on KRON complexity

Lemma: [2]

$$g(\alpha,\beta,\gamma) = \sum_{\sigma^1,\sigma^2,\sigma^3 \in S_\ell} \mathrm{sgn}(\sigma^1\sigma^2\sigma^3)\, C(\alpha+1-\sigma^1, \beta+1-\sigma^2, \gamma+1-\sigma^3),$$

where $C(u,v,w)$ is the number of $\ell \times \ell \times \ell$ contingency arrays $[A_{i,j,k}]$:

$$\sum_{j,k} A_{i,j,k} = u_i, \quad \sum_{i,k} A_{i,j,k} = v_j, \quad \sum_{i,j} A_{i,j,k} = w_k$$

$+$ Barvinok's algorithm for counting integer points in polytopes: *$C(u,v,w)$ can be computed in time $(\ell \max\{u_1,\ldots,v_1,\ldots,w_1,\ldots\})^{O(\ell^3 \log \ell)}$.*

Lemma

Let $\alpha,\beta,\gamma \vdash n$ be partitions of the same size, such that $\alpha_1,\beta_1,\gamma_1 \le m$ and $\ell(\alpha),\ell(\beta),\ell(\gamma) \le \ell$. Then $g(\alpha,\beta,\gamma)$ can be computed in time $(\ell \log m)^{O(\ell^3 \log \ell)}$.

[2] In [Christandl-Doran-Walter],[Pak-P]

The theorem

Reduction Lemma:

Let $\lambda, \mu, \nu \vdash n$ and $\ell(\lambda), \ell(\mu), \ell(\nu) \leq \ell$. Set $s = n - \nu_1$. Then:

(i) If $|\lambda_i - \mu_i| > s$ for some i, then $g(\lambda, \mu, \nu) = 0$,

(ii) If $|\lambda_i - \mu_i| \leq s$ for all i, $1 \leq i \leq \ell$, there $\exists$ an $r \leq 2s\ell^2$, s.t.

$$g(\lambda, \mu, \nu) = g\big(\phi(\lambda), \phi(\mu), \phi(\nu)\big)$$

for certain explicitly defined partitions $\phi(\lambda), \phi(\mu), \phi(\nu) \vdash r$.

Lemma: Complexity of Kron

Let $\alpha, \beta, \gamma \vdash n$ be partitions of the same size, s.t. $\alpha_1, \beta_1, \gamma_1 \leq m$ and $\ell(\alpha), \ell(\beta), \ell(\gamma) \leq \ell$. Then $g(\alpha, \beta, \gamma)$ can be computed in time $(\ell \log m)^{O(\ell^3 \log \ell)}$.

The theorem

Reduction Lemma:

Let $\lambda, \mu, \nu \vdash n$ and $\ell(\lambda), \ell(\mu), \ell(\nu) \leq \ell$. Set $s = n - \nu_1$. Then:

(i) If $|\lambda_i - \mu_i| > s$ for some i, then $g(\lambda, \mu, \nu) = 0$,

(ii) If $|\lambda_i - \mu_i| \leq s$ for all i, $1 \leq i \leq \ell$, there $\exists$ an $r \leq 2s\ell^2$, s.t.

$$g(\lambda, \mu, \nu) = g\big(\phi(\lambda), \phi(\mu), \phi(\nu)\big)$$

for certain explicitly defined partitions $\phi(\lambda), \phi(\mu), \phi(\nu) \vdash r$.

Lemma: Complexity of Kron

Let $\alpha, \beta, \gamma \vdash n$ be partitions of the same size, s.t. $\alpha_1, \beta_1, \gamma_1 \leq m$ and $\ell(\alpha), \ell(\beta), \ell(\gamma) \leq \ell$. Then $g(\alpha, \beta, \gamma)$ can be computed in time $(\ell \log m)^{O(\ell^3 \log \ell)}$.

Theorem (Pak-P)

Let $\lambda, \mu, \nu \vdash n$ be partitions with lengths $\ell(\lambda), \ell(\mu), \ell(\nu) \leq \ell$, the largest parts $\lambda_1, \mu_1, \nu_1 \leq N$, and $\nu_2 \leq M$. Then the Kronecker coefficients $g(\lambda, \mu, \nu)$ can be computed in time

$$O(\ell \log N) + (\ell \log M)^{O(\ell^3 \log \ell)}.$$

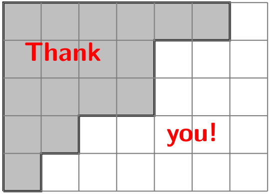
The analogous question for characters

Input: Integers N, ℓ, partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $\mu = (\mu_1, \ldots, \mu_\ell)$, where $0 \leq \lambda_i, \mu_i \leq N$, and $|\lambda| = |\mu|$.

Decide: whether $\chi^\lambda[\mu] = 0$

Proposition (Pak-P)

This problem is NP *–hard.*


Thank
you!