ALGORITHMIC INVARIANT THEORY

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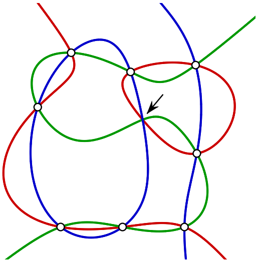

Tutorial at the Simons Institute Workshop
on Geometric Complexity Theory
September 15, 2014

Brief Personal History

My Masters Thesis ("Diplom", Darmstadt 1984)
used classical invariants ("brackets") as a tool
for geometric computations with convex polytopes.

At that time, I was inspired by *Felix Klein's Erlanger Programm*
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In Fall 1987, during my first postdoc at the IMA in Minneapolis, I was the notetaker for Gian-Carlo Rota's lectures *Introduction to Invariant Theory in Superalgebras*. This became our joint paper.

In Spring 1989, during my second postdoc at RISC-Linz, Austria, I taught a course on *Algorithms in Invariant Theory*. This was published as a book in the RISC series of Springer, Vienna.

During the year 1989-90, DIMACS at Rutgers ran a program on *Computational Geometry*. There I met Ketan Mulmuley....

Changing Coordinates

Fix a field K of characteristic zero. Consider a matrix group G inside the group $\mathrm{GL}(n, K)$ of all invertible $n \times n$-matrices.

Every matrix $g = (g_{ij})$ in G gives a linear change of coordinates on K^n. This transforms polynomials in $K[x_1, x_2, \ldots, x_n]$ via

$$x_i \mapsto g_{i1}x_1 + g_{i2}x_2 + \cdots + g_{in}x_n \qquad \text{for} \quad i = 1, 2, \ldots, n.$$

An *invariant of G* is a polynomial that is left unchanged by these transformations for all $g \in G$. These form the *invariant ring*

$$K[x_1, x_2, \ldots, x_n]^G \quad \subset \quad K[x_1, x_2, \ldots, x_n].$$

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Example

For the group S_n of $n \times n$ permutation matrices, this is the *ring of symmetric polynomials*. For instance,

$$\begin{aligned} K[x_1, x_2, x_3]^{S_3} &= K\left[x_1 + x_2 + x_3\,,\ x_1x_2 + x_1x_3 + x_2x_3\,,\ x_1x_2x_3\right] \\ &= K\left[x_1 + x_2 + x_3\,,\ x_1^2 + x_2^2 + x_3^2\,,\ x_1^3 + x_2^3 + x_3^3\right]. \end{aligned}$$

Rotating by 90 Degrees

The cyclic group

$$G \;=\; \mathbb{Z}/4\mathbb{Z} \;=\; \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

has the invariant ring

$$\begin{aligned} K[x,y]^G &= \left\{ f \in K[x,y] \,:\, f(-y,x) = f(x,y) \right\} \\ &= K\left[x^2+y^2,\, x^2y^2,\, x^3y - xy^3\right] \end{aligned}$$

This is the coordinate ring of the quotient space $K^2/\!/G$.

The three generators embed this surface into K^3 via

$$K[x,y]^G \;\simeq\; K[a,b,c]/\langle c^2 - a^2b + 4b^2 \rangle$$

Q: How can we be sure that there are no other invariants?

Scaling the Coordinates

The multiplicative group $G = K^*$ is known as the *algebraic torus*. Consider its action on $S = K[x, y, z]$ via

$$x \mapsto t^2 x\,,\;\; y \mapsto t^3 y\,,\;\; z \mapsto t^{-7} z.$$

The invariant ring equals

$$\begin{aligned} S^G &= K\{x^i y^j z^k \,:\, 2i + 3j = 7k\} \\ &= K[x^7 z^2, x^2 yz, xy^4 z^2, y^7 z^3] \end{aligned}$$

Big Question: *Is S^G always finitely generated as a K-algebra?*

True if G is an algebraic torus.

Reason: Every semigroup of the form $\mathbb{N}^n \cap L$, where $L \subset \mathbb{Q}^n$ is a linear subspace, has a finite *Hilbert basis*.

Also *true if G is a finite group.*

Averaging Polynomials

For a finite matrix group G, the *Reynolds operator* is the map

$$S \rightarrow S^G, \ \ p \mapsto p^* = \frac{1}{|G|} \sum_{g \in G} g(p)$$

Key Properties

(a) The Reynolds operator $*$ is a K-linear map.

(b) The Reynolds operator $*$ restricts to the identity on S^G.

(c) The Reynolds operator $*$ is an S^G-module homomorphism, i.e.

$$(p \cdot q)^* = p \cdot q^* \quad \text{for all invariants} \quad p \in S^G.$$

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Definition

More generally, a matrix group G is called reductive if it admits an operator $* : S \to S^G$ with these three properties.

Remark

Finite matrix groups in characteristic zero are reductive.

Finite Generation

Theorem (David Hilbert, 1890)

The invariant ring S^G of a reductive group G is finitely generated.

Proof.

By (a), the invariant ring S^G is the K-vector space spanned by all symmetrized monomials $(x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n})^*$. Let I_G be the **ideal** in S generated by these invariants, for $(e_1, \ldots, e_n) \neq (0, \ldots, 0)$.

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By Hilbert's Basis Theorem, the ideal I_G is generated by a finite subset of these invariants, say, $I_G = \langle p_1, p_2, \ldots, p_m \rangle$. We claim that $S^G = K[p_1, p_2, \ldots, p_m]$.

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Suppose not, and pick $q \in S^G \backslash K[p_1, \ldots, p_m]$ of minimum degree. Since $q \in I_G$, we can write $q = f_1 p_1 + f_2 p_2 + \cdots + f_m p_m$, where $f_i \in S$ are homogeneous of strictly smaller degree. By (b) and (c),

$$q \; = \; q^* \; = \; f_1^* \cdot p_1 + f_2^* \cdot p_2 + \cdots + f_m^* \cdot p_m.$$

By minimality, each f_i^* lies in $K[p_1, \ldots, p_m]$. Hence so does q. □

Finite Groups

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The Hilbert series of the invariant ring S^G is the average of the inverted characteristic polynomials of all group elements, i.e.

$$\sum_{d=0}^{\infty} \dim_K(S_d^G) \cdot z^d \quad = \quad \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(\mathrm{Id} - z \cdot g)}.$$

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Example (Rotations by 90 Degrees)

$$\begin{vmatrix} 1-z & 0 \\ 0 & 1-z \end{vmatrix}^{-1} + \begin{vmatrix} 1+z & 0 \\ 0 & 1+z \end{vmatrix}^{-1} + \begin{vmatrix} 1 & z \\ -z & 1 \end{vmatrix}^{-1} + \begin{vmatrix} 1 & -z \\ z & 1 \end{vmatrix}^{-1}$$

$$= \frac{1 - z^8}{(1 - z^2)^2 \cdot (1 - z^4)} = 1 + z^2 + 3z^4 + 3z^6 + \cdots$$

Two Algorithms

Crude Algorithm

1. Compute the Molien series.
2. Produce invariants of low degree using the Reynolds operator.
3. Compute the Hilbert series of the current subalgebra of S.
4. If that Hilbert series equals the Molien series, we are done.
5. If not, increase the degree and go back to 2.

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Derksen's Algorithm (1999)

1. Introduce three sets of variables: $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ for K^n, and $\mathbf{g} = (g_1, \ldots, g_r)$ for $G \subset \mathrm{GL}(n, K)$.
2. Consider the ideal $J = \langle \mathbf{y} - \mathbf{g} \cdot \mathbf{x} \rangle + \langle \mathbf{g} \in G \rangle$ in $K[\mathbf{x}, \mathbf{y}, \mathbf{g}]$.
3. Compute generators $p_1, \ldots, p_m$ for $I_G = (J \cap K[\mathbf{x}, \mathbf{y}])\big|_{\mathbf{y}=0}$.
4. Output: The invariants $p_1^*, \ldots, p_m^*$ generate $K[\mathbf{x}]^G$.

Torus Action Example:

$$\langle u - t^2 x, v - t^3 y, w - s^7 z, st - 1 \rangle \cap K[x, y, z, u, v, w]\big|_{u=v=w=0}$$

Classical Invariant Theory

We fix a polynomial representation of the special linear group:

$$\mathrm{SL}(d, K) \xrightarrow{\rho} G \subset \mathrm{GL}(V) \qquad \text{where} \quad V \simeq K^n.$$

Fact: The matrix group G is reductive. **What is the Reynolds operator?**

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Example

Let $d = 2, n = 4$ and consider the *adjoint representation* where $g \in \mathrm{SL}(2, K)$ acts on matrix space $V = K^{2\times 2}$ via $g \mapsto g \cdot \mathbf{x} \cdot g^{-1}$.

Explicitly, this is the quadratic representation given by

$$\rho(g) = \begin{pmatrix} g_{11}g_{22} & -g_{11}g_{21} & g_{12}g_{22} & -g_{12}g_{21} \\ -g_{11}g_{12} & g_{11}^2 & -g_{12}^2 & g_{11}g_{12} \\ g_{21}g_{22} & -g_{21}^2 & g_{22}^2 & -g_{21}g_{22} \\ -g_{12}g_{21} & g_{11}g_{21} & -g_{12}g_{22} & g_{11}g_{22} \end{pmatrix}$$

The vectorization of the 2×2-matrix $g \cdot \mathbf{x} \cdot g^{-1}$ equals the 4×4-matrix $\rho(g)$ times the vectorization of $\mathbf{x} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$.

Orbits

The invariant ring for the adjoint action on 2×2-matrices $\mathbf{x}$ is

$$\mathbb{C}[\mathbf{x}]^{\mathrm{SL}(2,\mathbb{C})} \;=\; \mathbb{C}[\mathrm{trace}(\mathbf{x}), \det(\mathbf{x})].$$

The invariants are constant along orbits and their closures.

Example

The orbit of $\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$ is closed. It is the variety defined by the ideal

$$\langle\, \mathrm{trace}(\mathbf{x}) - 9,\, \det(\mathbf{x}) + 1 \,\rangle \;=\; \langle\, \mathrm{trace}(\mathbf{x}) - 9,\, \mathrm{trace}(\mathbf{x}^2) - 83 \,\rangle.$$

Question: Are all orbits closed? Do the invariants separate orbits?

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Question: Are all orbits closed? Do the invariants separate orbits?

Answer: Not quite. The nullcone $V(\langle \mathrm{trace}(\mathbf{x}), \det(\mathbf{x}) \rangle)$ contains many orbits (of nilpotent matrices) that cannot be separated.

Recall the *Jordan canonical form*, and consider the orbits of

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \ldots$$

Brackets ... back to Felix Klein

Let $n = dm$ and $V = K^{d \times m}$ = the space of $d \times m$-matrices.
Our group $G = \mathrm{SL}(d, K)$ acts on V by left multiplication.

First Fundamental Theorem

$K[V]^G$ is generated by the $\binom{m}{d}$ maximal minors of $\mathbf{x} = (x_{ij})$.

Second Fundamental Theorem

The relations among these generators, which are denoted by $[i_1 i_2 \cdots i_d]$, are generated by the quadratic Plücker relations.

Example

For $d = 2, m = 4$, the generators are $[\,i\,j\,] = x_{1i} \cdot x_{2j} - x_{1j} \cdot x_{2i}$ and the ideal of relations is $\big\langle [12] \cdot [34] - [13] \cdot [24] + [14] \cdot [23] \big\rangle$.

Example

For $d = 3, m = 6$, our matrix $\mathbf{x}$ represents six points in $\mathbb{P}^2$. These lie on a conic if and only if $[123][145][246][356] = [124][135][236][456]$.

Algebraic Geometry

Let $n = \binom{d+m-1}{m-1}$ and consider the action of $G = \mathrm{SL}(d, K)$ on $V = S^d K^m = \{$homog. polynomials of degree d in m variables$\}$.

The invariant ring $K[V]^G$ is finitely generated. Its generators express geometric properties of hypersurfaces of degree d in $\mathbb{P}^{m-1}$.

This is the point of departure for **G**eometric **I**nvariant **T**heory.

Example

Let $d = m = 2, n = 3$, so V is the 3-dim'l space of binary quadrics

$$f(t_0, t_1) \quad = \quad x_1 \cdot t_0^2 + x_2 \cdot t_0 t_1 + x_3 \cdot t_1^2$$

Pop Quiz: Can **you** write down the 3×3-matrix $\rho(g)$? *Do now.*

Check: The invariant ring is generated by the discriminant

$$K[x_1, x_2, x_3]^G \;\; = \;\; K\left[x_2^2 - 4x_1x_3\right].$$

Plane Cubics

The case $d = m = 3$ corresponds to cubic curves in the plane $\mathbb{P}^2$.
A ternary cubic has $n = 10$ coefficients:

$$x_1 t_0^3 + x_2 t_1^3 + x_3 t_2^3 + x_4 t_0^2 t_1 + x_5 t_0^2 t_2 + x_6 t_0 t_1^2 + x_7 t_0 t_2^2 + x_8 t_1^2 t_2 + x_9 t_1 t_2^2 + x_{10} t_0 t_1 t_2$$

The invariant ring $K[V]^G$ is a subring of $K[V] = K[x_1, x_2, \ldots, x_{10}]$.
It is generated by two classical invariants:

- a quartic S with 26 terms; $\leftarrow$ the Aronhold invariant
- a sextic T with 103 terms.

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- a sextic T with 103 terms.

Another important invariant is the discriminant $\Delta = T^2 - 64S^3$ which has 2040 terms of degree 12. It vanishes if and only if the cubic curve is singular. If $\Delta \neq 0$ then the cubic is an elliptic curve.

Number theorists love the j-invariant:

$$j = \frac{S^3}{\Delta}$$

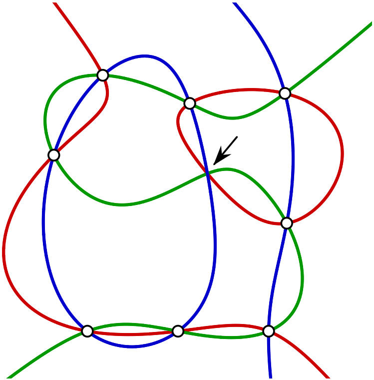
This serves as the coordinate on the *moduli space*

$$V/\!/G \; = \; \mathrm{Proj}(K[V]^G) \; = \; \mathrm{Proj}(K[S, T])$$

Old and New

Theorem (Cayley-Bacharach)

Let $P_1, \ldots, P_8$ be eight distinct points in the plane, no three on a line, and no six on a conic. There exists a unique ninth point P_9 such that every cubic curve through $P_1, \ldots, P_8$ also contains P_9.



My paper with Qingchun Ren and Jürgen Richter-Gebert (May 2014) gives an explicit formula (in brackets) for P_9 in terms of $P_1, P_2, \ldots, P_8$.

Hilbert's 14th Problem

Given any matrix group G,
is the invariant ring $K[V]^G$ always finitely generated?

Does Hilbert's 1890 Theorem extend to non-reductive groups?

Note: Subalgebras of a polynomial ring
need not be finitely generated, e.g.

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A negative answer was given by Masayoshi Nagata in 1959.

We shall describe Nagata's counterexample, following the exposition in
[*SAGBI bases of Cox-Nagata Rings* (with Z. Xu, JEMS 2010)]

Additive Groups

Fix $n = 2m$. The group $(K^m, +)$ is not reductive.
It acts on $K[\mathbf{x}, \mathbf{y}] = K[x_1, \ldots, x_m, y_1, \ldots, y_m]$ via

$$\begin{array}{rclr} x_i & \mapsto & x_i & \text{and} \\ y_i & \mapsto & y_i + u_i x_i & \text{for } \; u \in K^m. \end{array}$$

Let $d \leq m$ and fix a generic $d \times m$-matrix U.

Let $G = \mathrm{rowspace}(U) \subset K^m$. The additive group $(G, +) \simeq (K^d, +)$ acts on $K[\mathbf{x}, \mathbf{y}]$ by the rule above.

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Theorem

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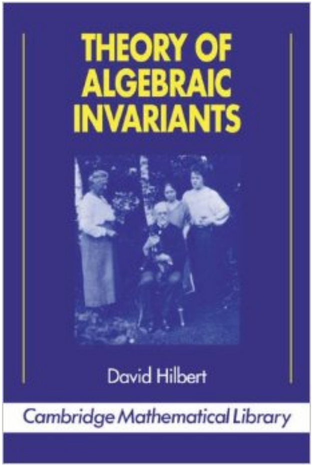
Proof.

Blow up $m = 5, 6, 7, 8, 9, \ldots$ general points in the plane $\mathbb{P}^2$ and you will discover the Weyl groups $D_5, E_6, E_7, E_8, E_9, \ldots$. □

Conclusion

Invariant theory is timeless, relevant and fun.



Reinhard Laubenbacher and I had lots of **fun** when translating and editing the notes from Hilbert's course (Summer Semester 1897 at Göttingen)