The GCT chasm I

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The main reference

GCT5 [M.]: Geometric Complexity Theory V: Equivalence between black-box derandomization of polynomial identity testing and derandomization of Noether's Normalization Lemma

Abstract: FOCS 2012.

Full version: Arxiv and the home page.

The beginning of GCT

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Why is improving on this lower bound so difficult? This talk.

The permanent vs. determinant problem

Conjecture [Valiant 1979]: The permanent of an $n \times n$ variable matrix X cannot be expressed as a symbolic determinant of size m, i.e., as the determinant of an $m \times m$ matrix whose entries are linear functions of the entries of X, if $m = \text{poly}(n)$.

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Stronger conjecture:[GCT1: M., Sohoni; 2001] The permanent of an $n \times n$ variable matrix X cannot be approximated infinitesimally closely by symbolic determinants of $O(\text{poly}(n))$ or even $O(2^{n^\epsilon})$ size, for some small enough constant $\epsilon > 0$.

The main result

Theorem [GCT5]: The stronger GCT1-conjecture implies that the problem (NNL) of derandomizing Noether's Normalization Lemma for the orbit closure of the determinant can be brought down from EXPSPACE, where it currently is, to DET $\subseteq P$, up to quasi-prefix.

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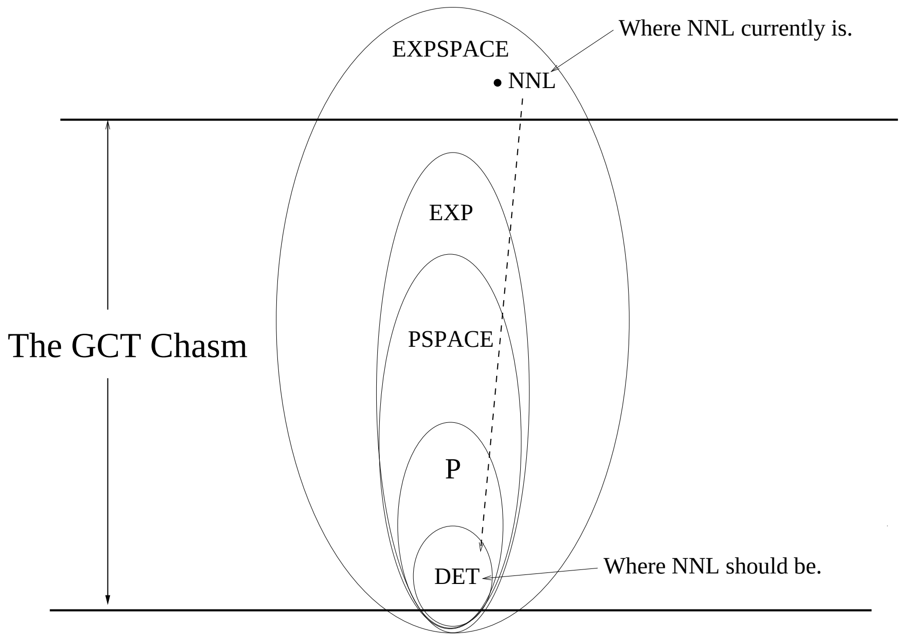

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[Under stronger assumptions, $P \neq BPP$.]

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(5) Why stengthened perm vs. det brings it to (quasi)-DET.

(6) Evidence for it may not be possible to cross the chasm.

The orbit closures

Let $V = \mathbb{C}_m[Y]$ be the space of homogeneous polynomials of degree m in the entries of a variable $m \times m$ matrix Y with the action of $G = GL_{m^2}(\mathbb{C})$ that maps $f(Y) \in V$ to $f(\sigma^{-1}Y)$ for any $\sigma \in G$ (thinking of Y as an m^2-vector).

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Let X be the lower-right $n \times n$ sub-matrix of Y, and z any element of Y outside X. Let $f(Y) = z^{m-n}\text{perm}(X) \in P(V)$, and define the orbit closure of the permanent as $\Delta[\text{perm}, n, m] = \overline{Gf} \subseteq P(V)$.

The reformulation in terms of orbit closures

The stronger permanent vs. determinant conjecture is now equivalent to:

Conjecture [GCT1]: $\Delta[\text{perm}, n, m] \not\subseteq \Delta[\det, m]$ if $m = \text{poly}(n)$, or more generally, $O(2^{n^\epsilon})$, for a small enough $\epsilon > 0$.

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The main difference between the original Valiant's conjecture and this conjecture: The original conjecture is a statement about the constructible set $M_{m^2}(\mathbb{C}) \cdot \det(Y) \subseteq V$, whereas this conjecture is a statement about the variety which is its closure.

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The basic principle of algebraic geometry: The difficulty of a constructible set is controlled by what lies on its border.

What lies on the border of $G \cdot \det(Y)$?

Defn: A family of points $\{p_m\}$, $p_m \in \Delta[\det, m]$, is called a family of bad exterior points if $\{p_m\} \notin VP$ (i.e., p_m cannot be computed by a small circuit over $\mathbb{C}$).

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Next: The complexity-theoretic evidence and natural (constructive) candidates from representation theory.

Newton degeneration

Given any symbolic matrix Z of size $m = \text{poly}(n)$ over the variables $z_1, \ldots, z_n$, let $Newton(Z)$ be the Newton polytope of $\det(Z) = \sum_\alpha c_\alpha z^\alpha$. Given any face $F \subseteq Newton(Z)$, let $\det_F(Z) = \sum_{\alpha \in F} c_\alpha z^\alpha$. Call it the Newton degeneration of $\det(Z)$ associated with the face F.

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(2) Qiao: The membership problem for $Newton(Z)$ is P-hard.

Newton degeneration of VP

Given any family $\{\det(Z_n)\} \in VP_s$ and any sequence $\{F_n \subseteq Newton(Z_n)\}$, the family $\{\det_{F_n}(Z_n)\}$ is called a Newton degeneration of $\{\det(Z_n)\}$. Let $Newton(VP_s) \subseteq \overline{VP_s} \cap VNP$ be the set of all Newton degenerations of the elements in VP_s.

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To each quiver Q without oriented cycles, one can associate using the representation theory of quivers a subclass $VP_s[Q] \subseteq VP_s$. Conjecturally $Newton(VP_s[Q]) \not\subseteq VP$, Q wild.

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(3) Q is $\overrightarrow{\Longrightarrow}$ (wild): $VP_s[Q]$ consists of the families $\{\det(Z_n)\}$, where Z_n is a $d \times d$ block matrix, with $d = p(n)$ (a fixed polynomial), and its (i,j)-th block is the symbolic sum $x_{ij}^1 Z_1 + x_{ij}^2 Z_2 + x_{ij}^3 Z_3$, where Z_1, Z_2 and Z_3 are $n \times n$ variable matrices, and x_{ij}^k's are variables. $Newton(VP_s[Q]) \subseteq \overline{VP_s}$, and conjecturally it is not in VP.

Noether's Normalization Lemma [Hilbert]

Lemma [NNL]: Given any projective variety $X \subseteq P(\mathbb{C}^k)$ of dimension n, there exists a homogeneous linear map $\psi : \mathbb{C}^k \to \mathbb{C}^m$, $m =$ poly(n), that induces a regular (well defined) map on X, called a normalizing map.

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If we use the standard representations of ψ and X, then the specification of ψ itself requires exponential space in n. So we only consider the case when X is an explicit variety, such as $\Delta[\det, m]$, that has a specification of bit-length polynomial in its dimension (a circuit for the determinant).

The problem NNL for $\Delta[\det, m]$

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Given an $m \times m$ matrix B, let $\psi_B : V \to \mathbb{C}$ denote the linear evaluation map that maps $f(Y) \in V$ to $f(B)$. Given a set $\mathcal{B} = \{B_1, \ldots, B_l\}$ of $m \times m$ matrices, let $\psi_{\mathcal{B}} : V \to \mathbb{C}^l$ denote the map $(\psi_{B_1}, \ldots, \psi_{B_l})$.

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Lemma: There exists a small set $\mathcal{B}$ of integer matrices of poly(m) total bit-size such that $\psi_{\mathcal{B}} : V \to \mathbb{C}^l$ induces a regular (normalizing) map on $\Delta[\det, m] \subseteq P(V)$.

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The problem NNL: Given m (specified in unary), construct a small set $\mathcal{B}$ such that $\psi_{\mathcal{B}}$ is a normalizing map on $\Delta[\det, m]$.

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The main obstacle to the existing techniques: the existence of bad exterior (including wild) points in $\Delta[\det, m]$.

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Equivalence Theorem: There exists an exponential time computable multilinear polynomial in n variables which cannot be approximated infinitesimally closely by symbolic determinants of size $m = O(2^{n^\epsilon})$ iff (ignoring a quasi-prefix) NNL for $\Delta[\mathrm{det}, m]$ is in P.

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Theorem [Shallow circuits]: If there exists an exponential time computable multilinear polynomial in n variables that cannot be approximated infinitesimally closely by depth three (or depth four homogeneous) circuits of size $O(2^{n^{1/2+\epsilon}})$, for some $\epsilon > 0$, then NNL for $\Delta[\mathrm{det}, m]$ is in quasi-DET.

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(a) the Hardness vs. randomness principle: Nisan-Wigderson and Kabanets-Impagliazzo, and

(b) efficient factorization of Multi-variate polynomials: Kaltofen. This lies at the heart of the proof.

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Step 1: Polynomial time Monte-Carlo algorithm: Hilbert et al. + Heintz and Schnorr.

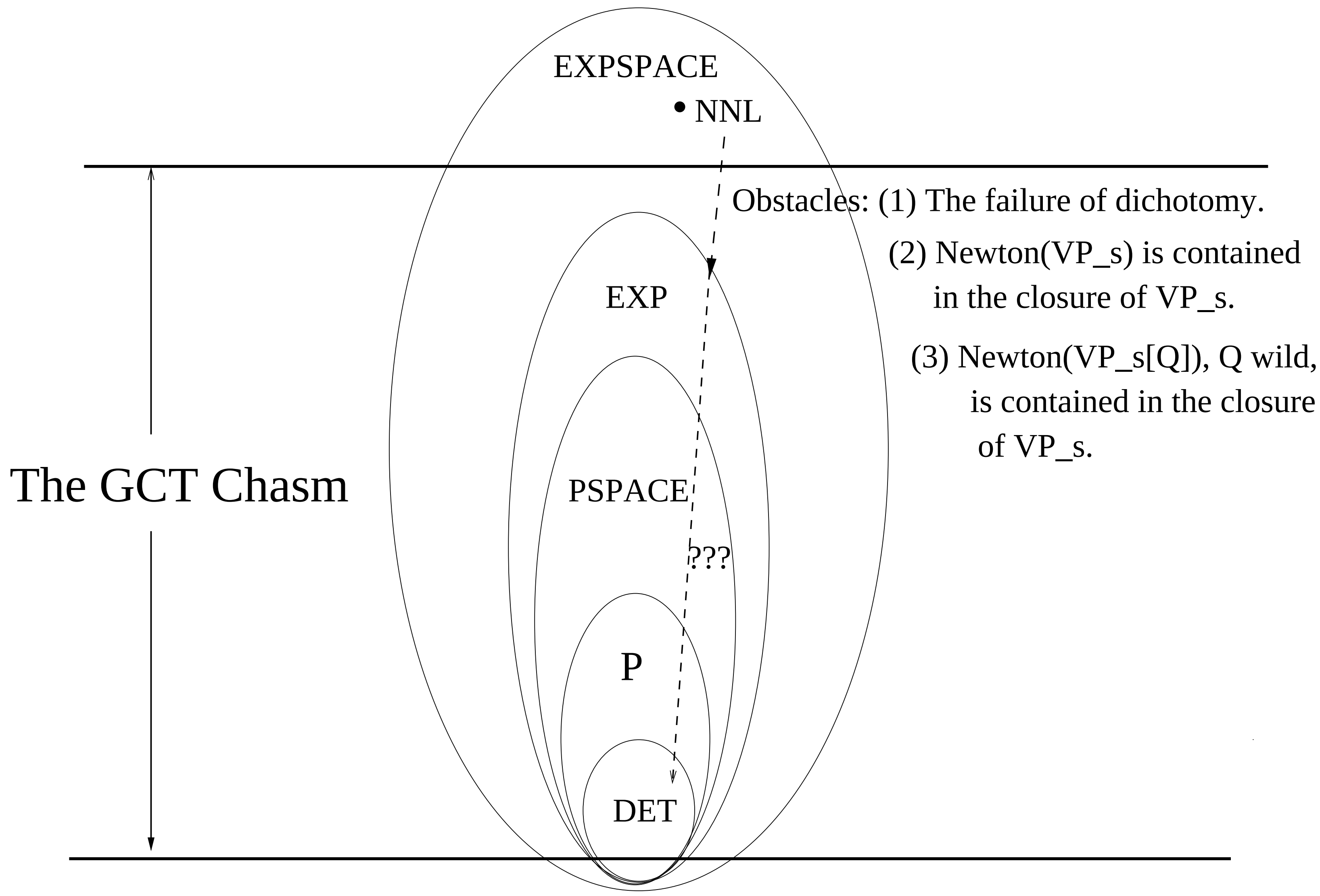
Step 2: Derandomize this algorithm using the GCT1-conjecture in conjunction with:

(a) the Hardness vs. randomness principle: Nisan-Wigderson and Kabanets-Impagliazzo, and

(b) efficient factorization of Multi-variate polynomials: Kaltofen. This lies at the heart of the proof.

All this works only in the models in which the determinant and multi-variate factorization can be computed efficiently.

The GCT chasm



Can the chasm be crossed? (contd.)

All the evidence supports that: (1) $Newton(VP_s) \not\subseteq VP$ (or even its subexponential analogue), as conjectured, and (2) the size of the circuit may not be beaten by the derandomization procedures. Hence, NNL for $\Delta[\mathrm{det}, m]$ may not be in SUBEXP.

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Yes, with implications in Klein's Erlangen program. Tomorrow.