Graph Sparsification I : Effective Resistance Sampling

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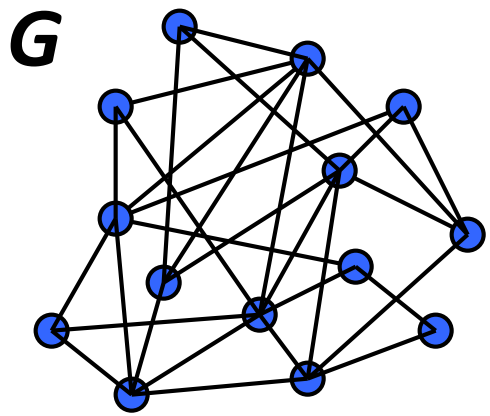
Microsoft Research India

Simons Institute, August 26 2014

Graphs



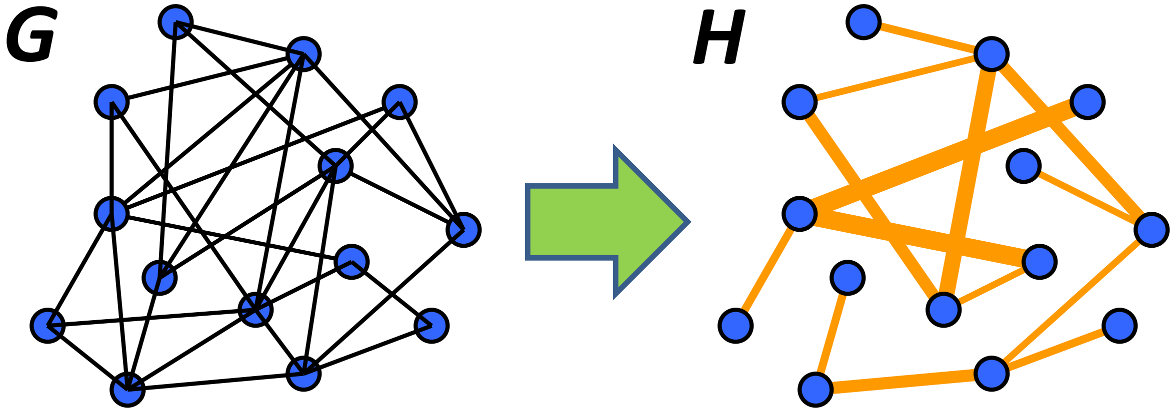
$\boldsymbol{G}$=(V,E,w) *undirected*

$|V| = n$

$w: E \rightarrow \boldsymbol{R}_{+}$

Sparsification

Approximate any graph ***G*** by a sparse graph ***H***.



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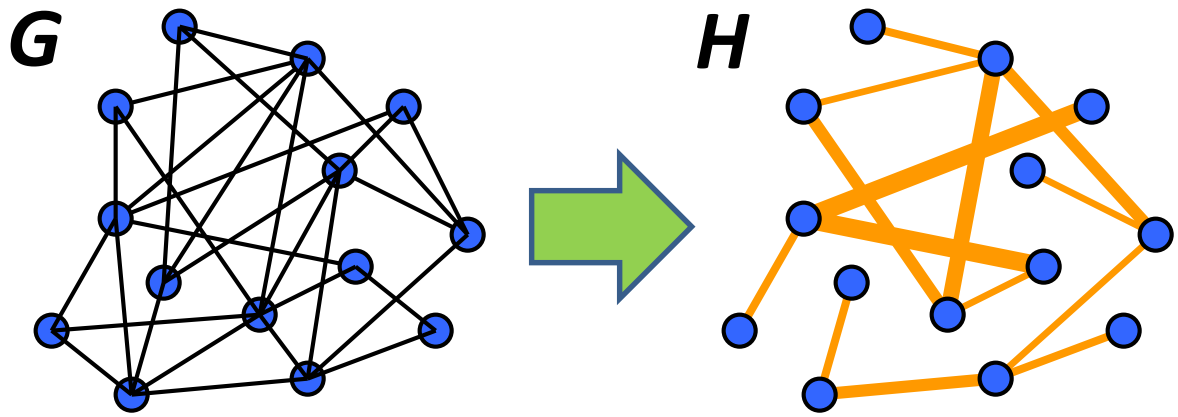
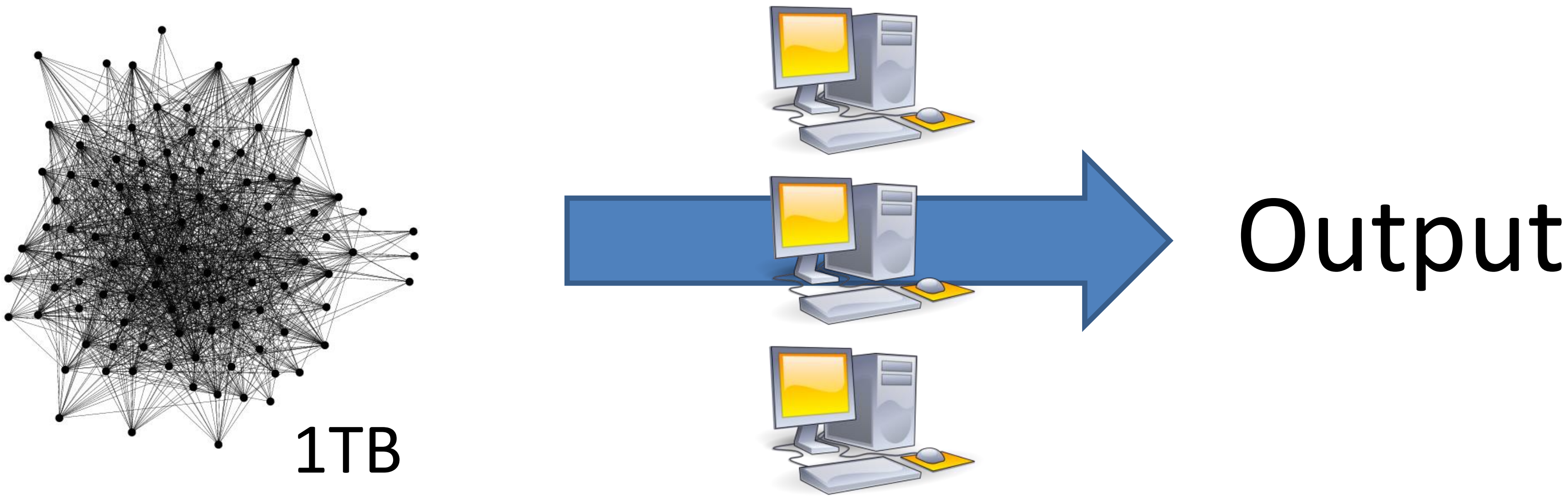


- ***H*** is faster to compute with than ***G***
- Nontrivial statement about ***G***

Sample Application



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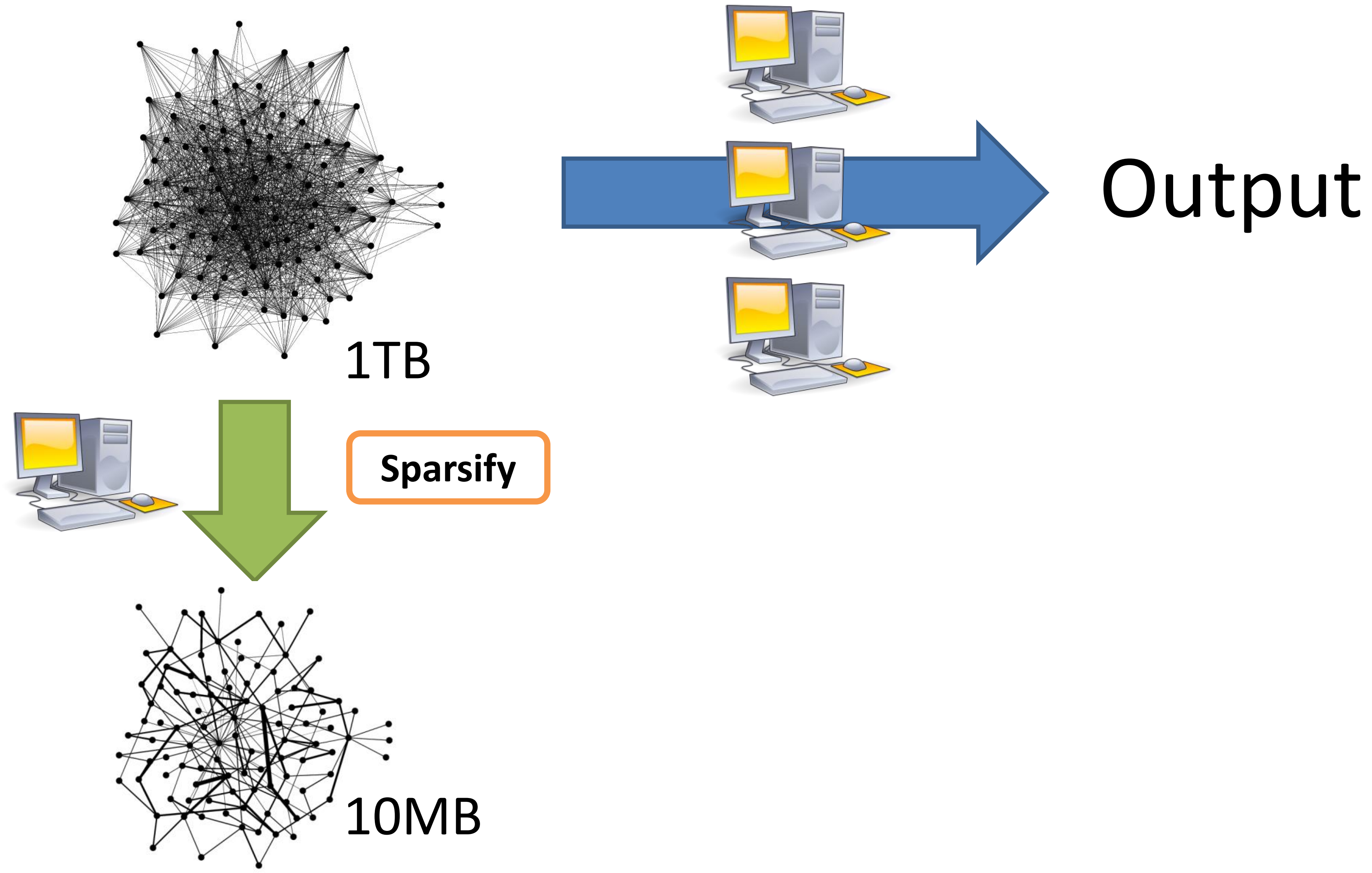

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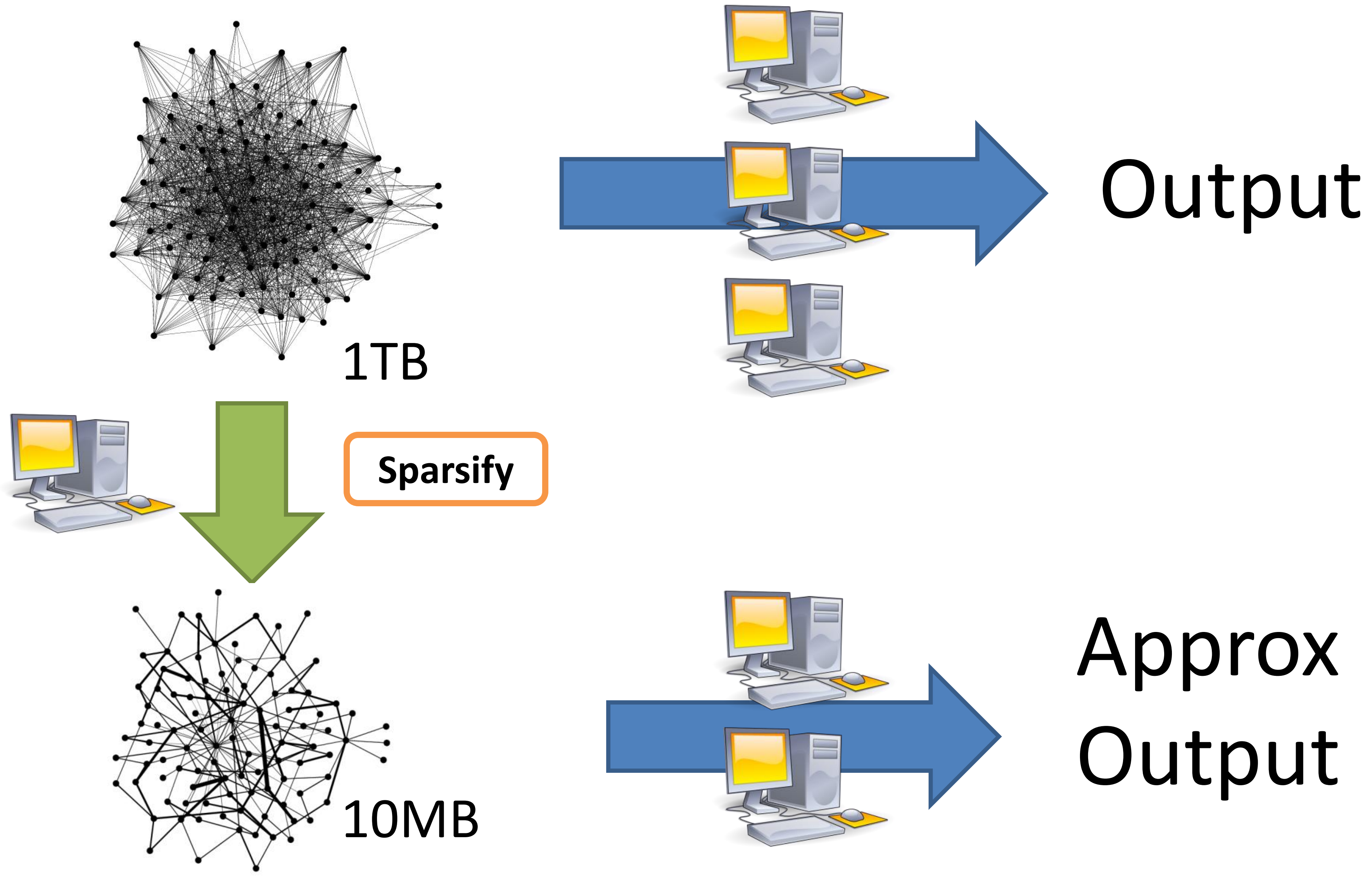

Some properties of interest

Sizes of cuts — "bottlenecks"

Clusters — "communities"

Distances

Random walks

Single / multicommodity flows

Electrical flows + other physical processes

Coloring

Hamiltonian / Eulerian cycle

Subgraph counts — e.g. triangles

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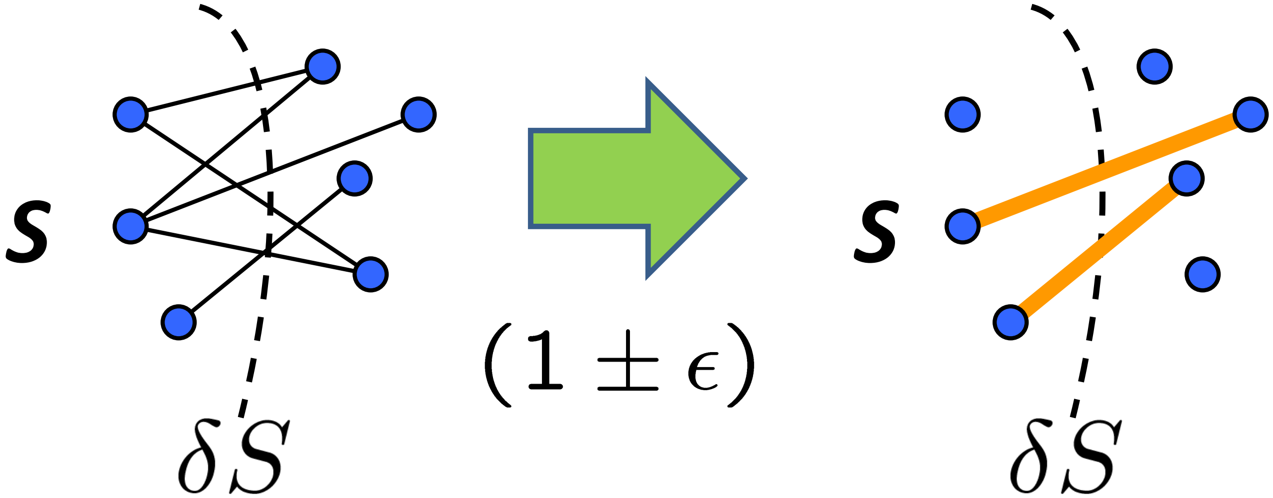
Cut Approximation [Benczur-Karger'96]

H approximates ***G*** if

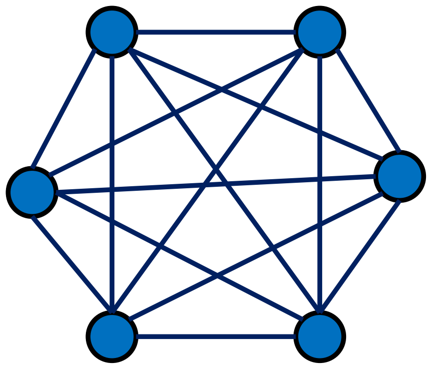
for every subset $S \subset V$

sum of weights of edges leaving ***S*** is preserved



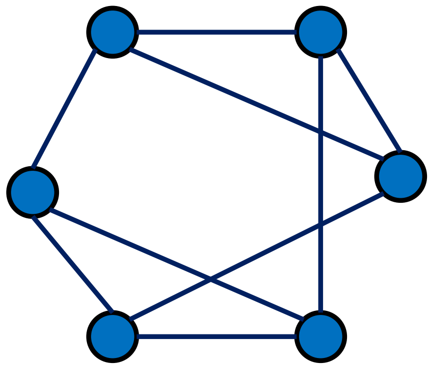
Example: The Complete Graph

G=K$_n$



$|E_G| = O(n^2)$

H = random d-regular



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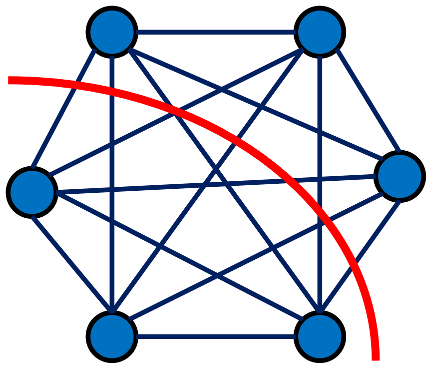

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$$wt_G(\delta S) = |S| \cdot |\overline{S}|$$

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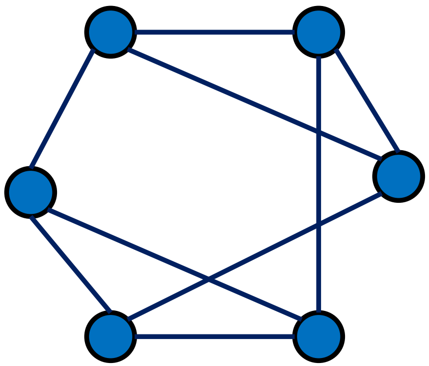

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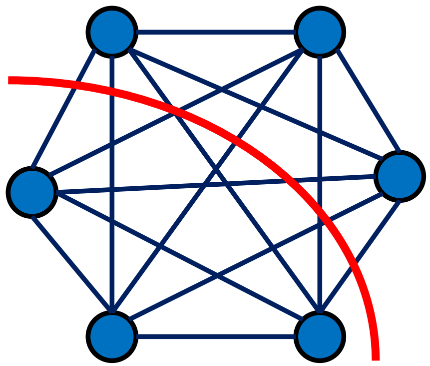

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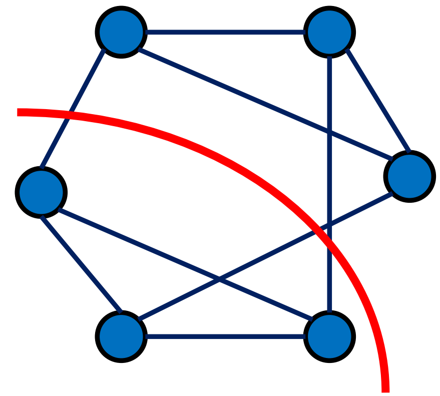

$|E_H| = O(dn)$

$$\mathbb{E} wt_H(\delta S) = (d/n)|S| \cdot |\overline{S}|$$

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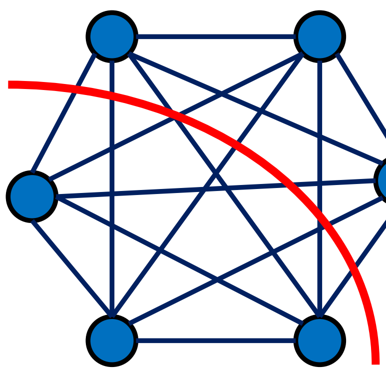

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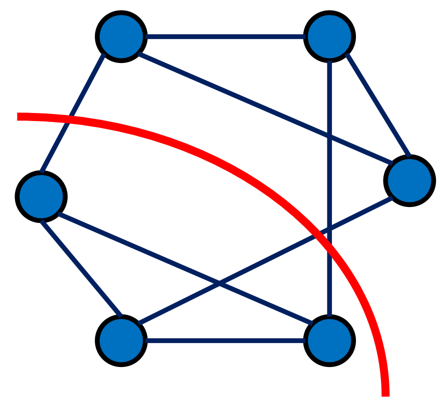


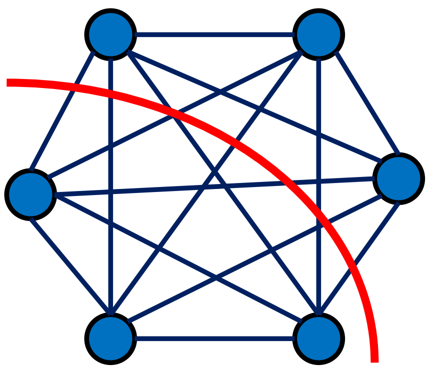
$|E_H| = O(dn)$

$$wt_H(\delta S) \simeq (d/n)|S| \cdot |\overline{S}|$$

with high probability

Example: The Complete Graph

$G=K_n$



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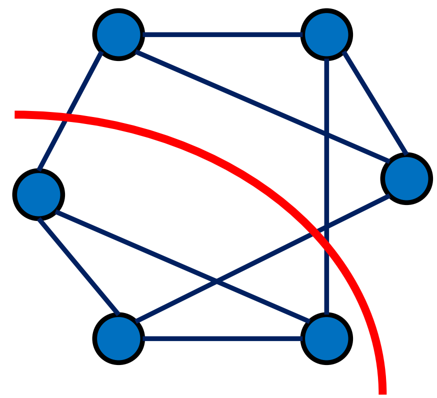


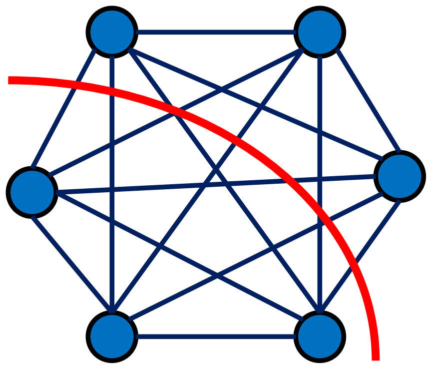
$|E_H| = O(dn)$

$$\forall S \subset V, \quad \frac{wt_G(\delta S)}{wt_H(\delta S)} \simeq (n/d)$$

whp

Example: The Complete Graph

G=K$_n$ **H** = random d-regular x **(n/d)**



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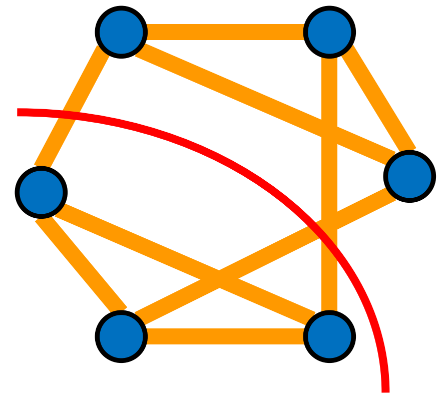

$|E_H| = O(dn)$

$$\forall S \subset V, \quad \frac{wt_G(\delta S)}{wt_{(n/d)H}(\delta S)} \simeq 1$$

whp

Cut Approximation [Benczur-Karger'96]

H approximates ***G*** if

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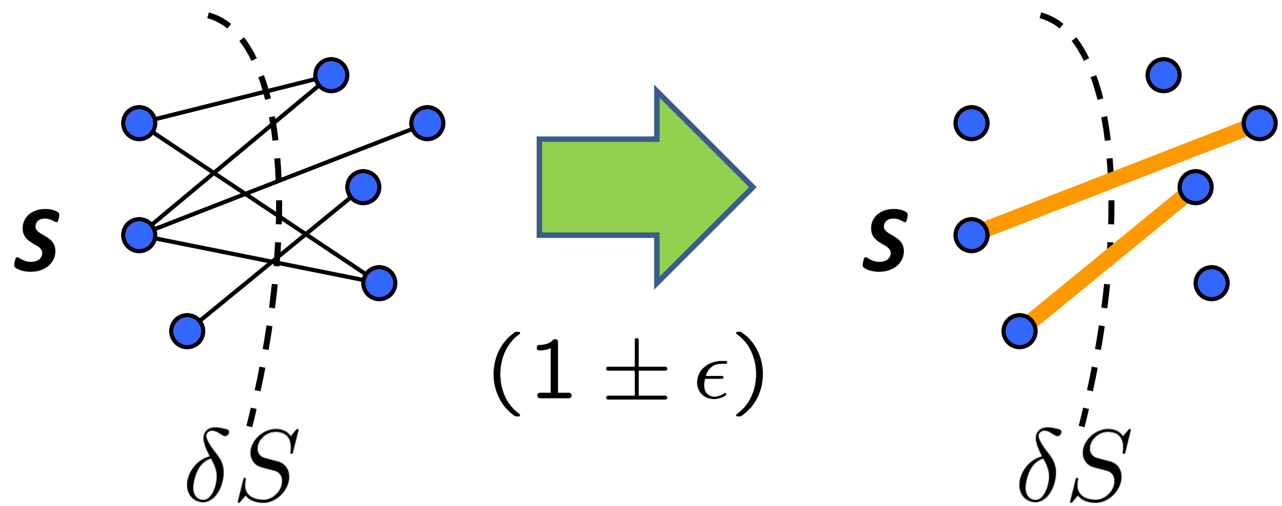

[Benczur-Karger'96]*:* For every ***G*** can quickly find ***H*** with **O(nlogn/ε²)** edges.

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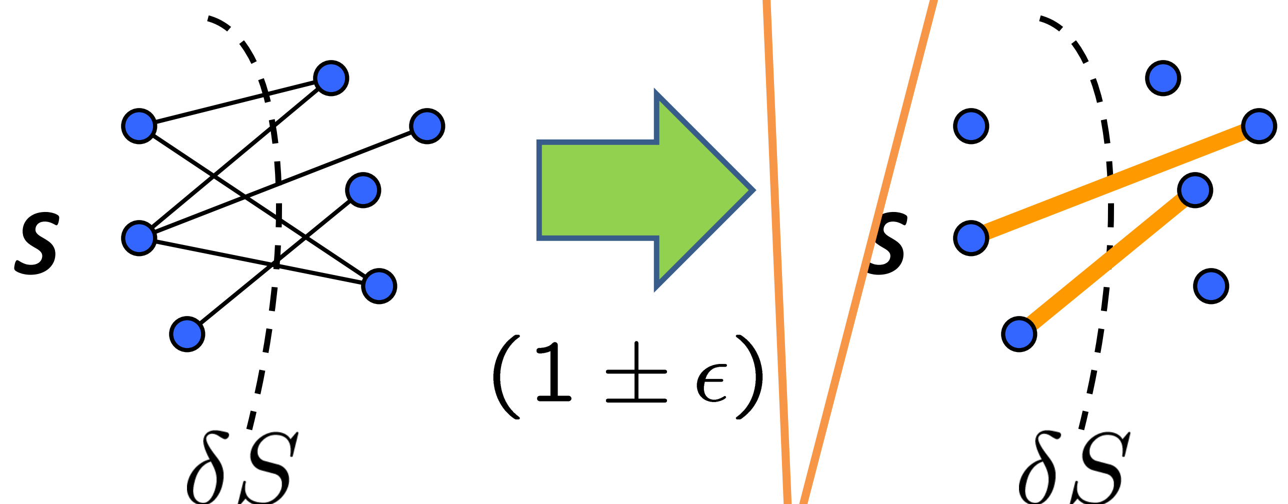
G and H are essentially the same for min cut, sparsest cut, etc.



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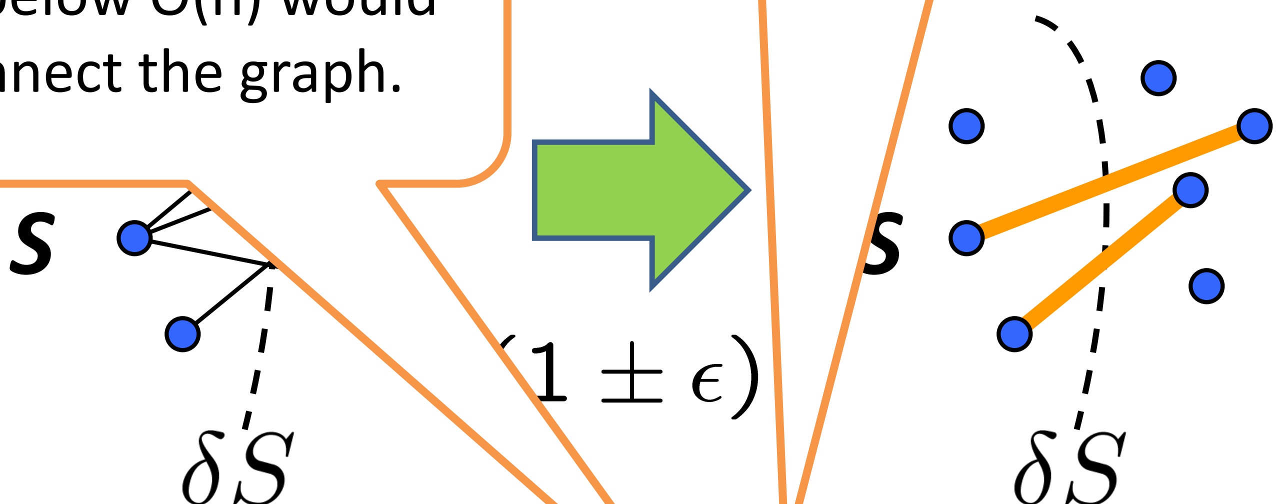
Going below O(n) would disconnect the graph.



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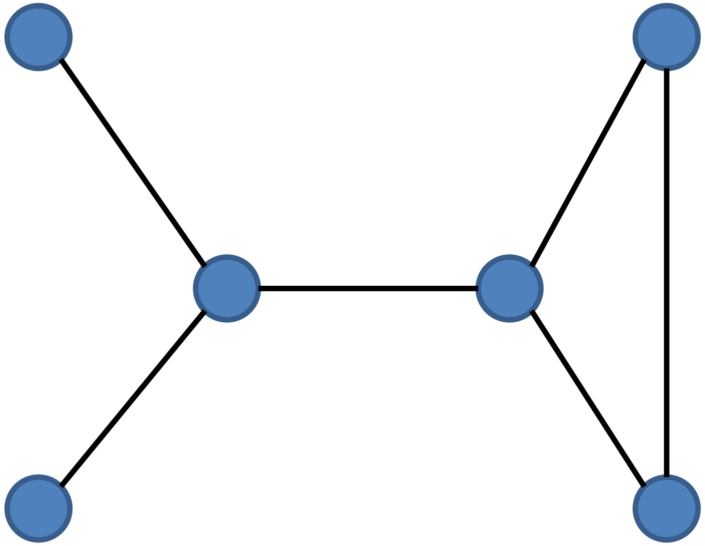
Physical Approximation [Spielman-Teng'04]

(i.e., spectral approximation)

Resistor Network Metaphor



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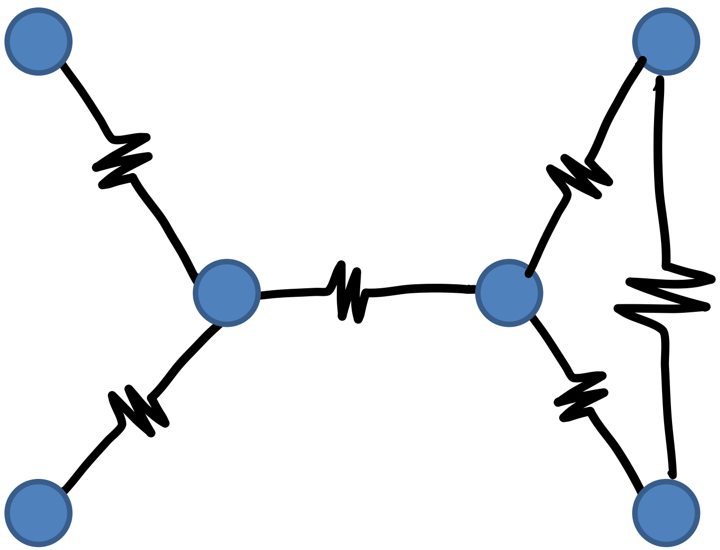

edge = 1Ω resistor

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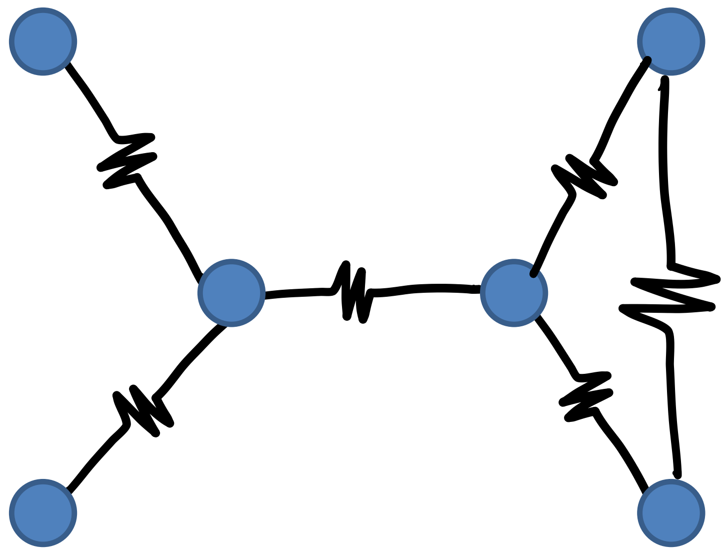

potentials $x: V \to \mathbb{R}$

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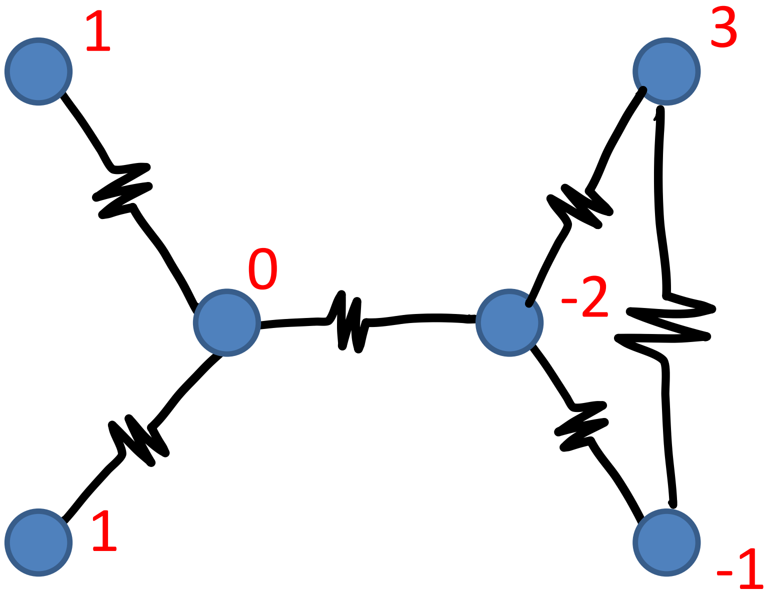

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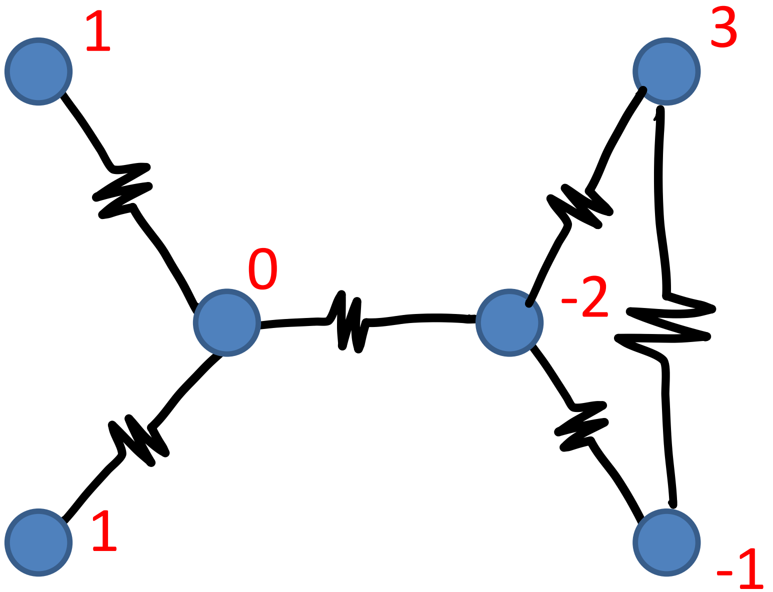

energy $\mathcal{E}_G(x) = \sum_{ij \in E} \left(x_i - x_j\right)^2$

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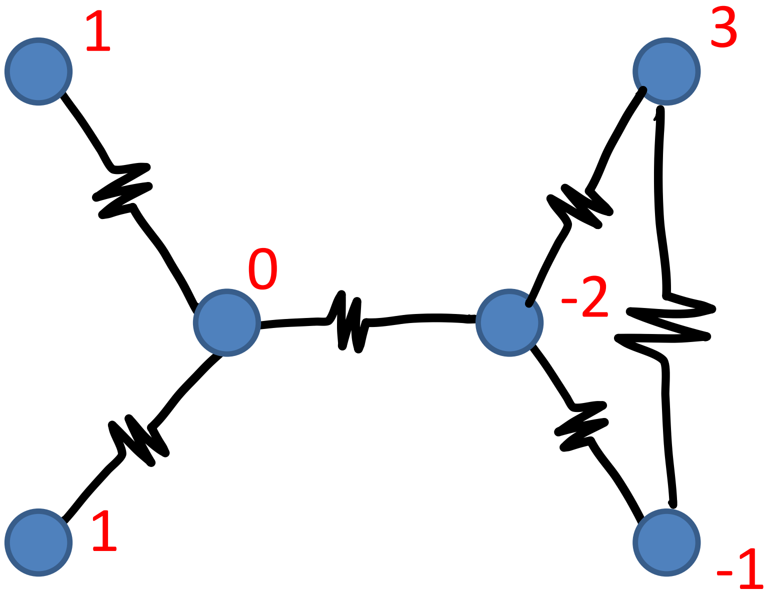

energy $\mathcal{E}_G(x) = \sum_{ij \in E} \left(x_i - x_j\right)^2$

$= 1^2 + 1^2 + 2^2 + 5^2 + 1^2 + 2^2 = 36$

Physical Approximation [ST'04]

Definition. $H = (V, F, u)$ is a κ −approximation of $G = (V, E, w)$ if for all potentials $x: V \to \mathbb{R}$:

$$\mathcal{E}_H(x) \leq \mathcal{E}_G(x) \leq \kappa \cdot \mathcal{E}_H(x)$$

"Electrically Equivalent"

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Laplacian matrix $x^T L_G x$ $x^T L_H x$

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is the **Laplacian** matrix of ***G***.

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$$\begin{matrix} & i & j \\ i & 1 & -1 \\ j & -1 & 1 \end{matrix}$$

is the **Laplacian** matrix of ***G***.

Properties of the Laplacian

$$L_G = \sum_{ij \in E} w_{ij} \left(\delta_i - \delta_j\right)\left(\delta_i - \delta_j\right)^T = \sum_{ij \in E} w_{ij} L_{ij}$$

$x^T L_G x \geq 0$ so **positive semidefinite** $L_G \succcurlyeq 0$.

$A \succcurlyeq B$ means $x^T A x \geq x^T B x$

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nullspace = $\text{span}\{(1,1, \ldots, 1)\}$ for connected G.

$\sum_{ij \in E} w_{ij} \left(x_i - x_j\right)^2 = 0$ iff $x_i = x_j$ for every $ij \in E$

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Can talk about **square root** $L_G^{-1/2}$ because $L_G^{-1} \succcurlyeq 0$.

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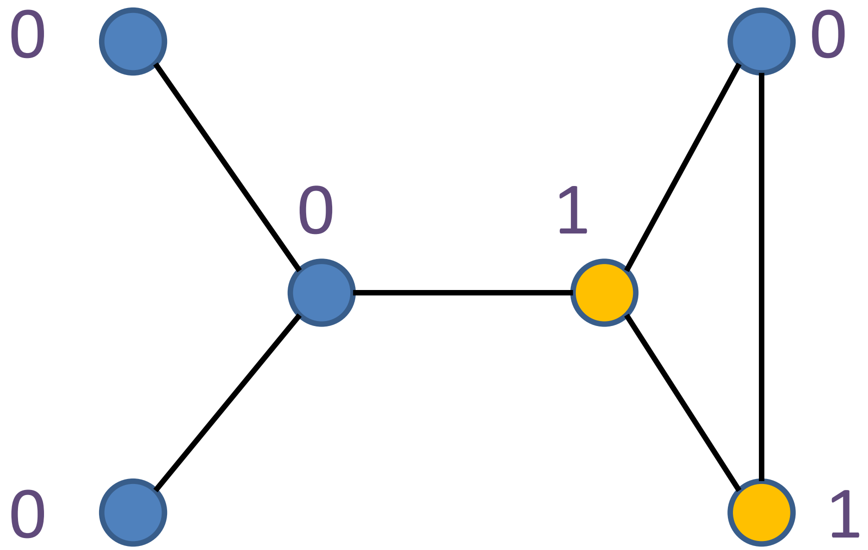
Why?

0. Energy Encodes Cuts

$$x: V \to \{0,1\}$$



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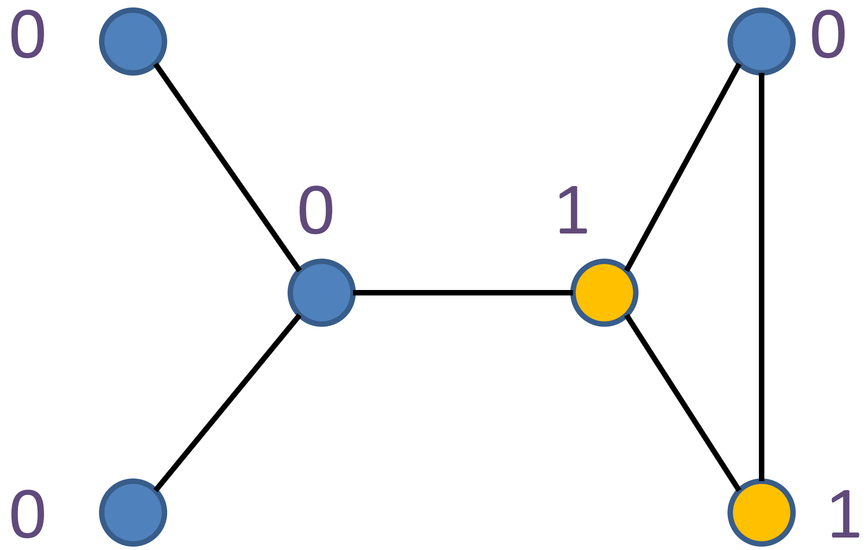

$$\mathcal{E}_G(x) = 1^2 + 1^2 + 1^2 = 3$$

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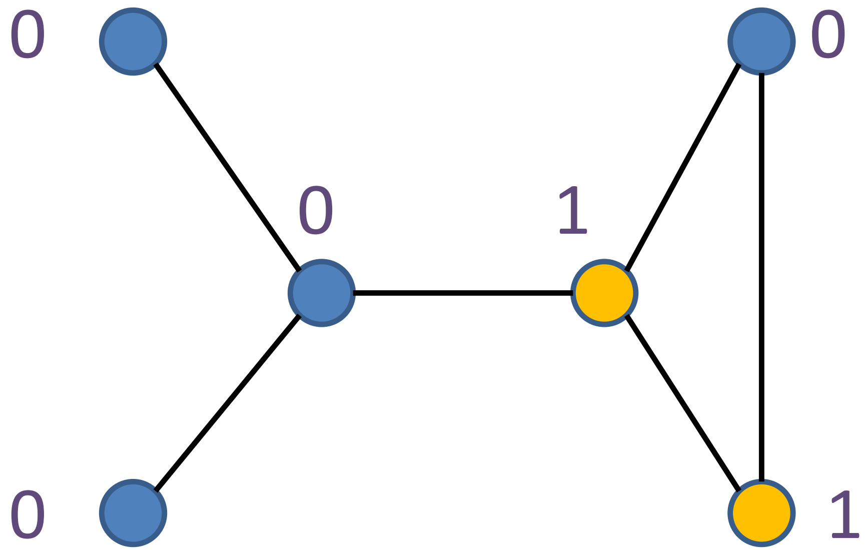
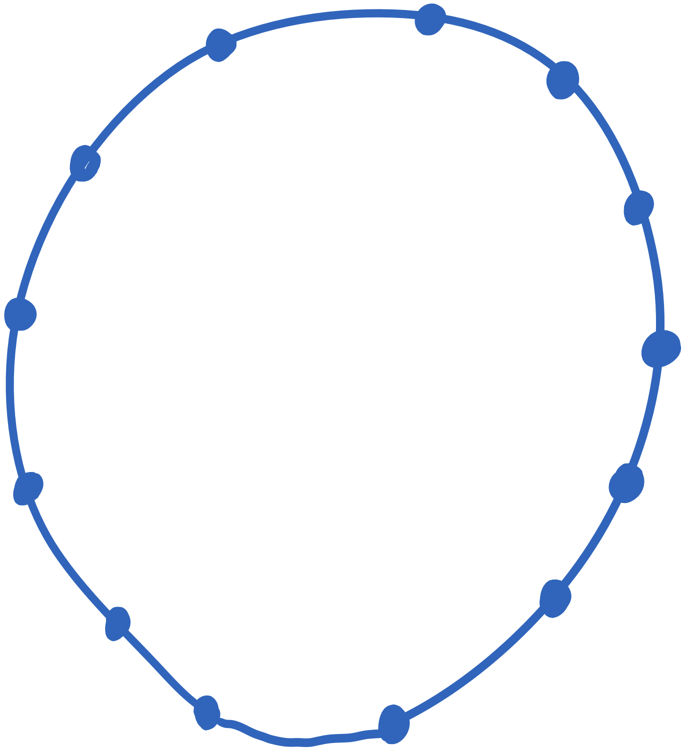


Physical approx. implies cut approx.

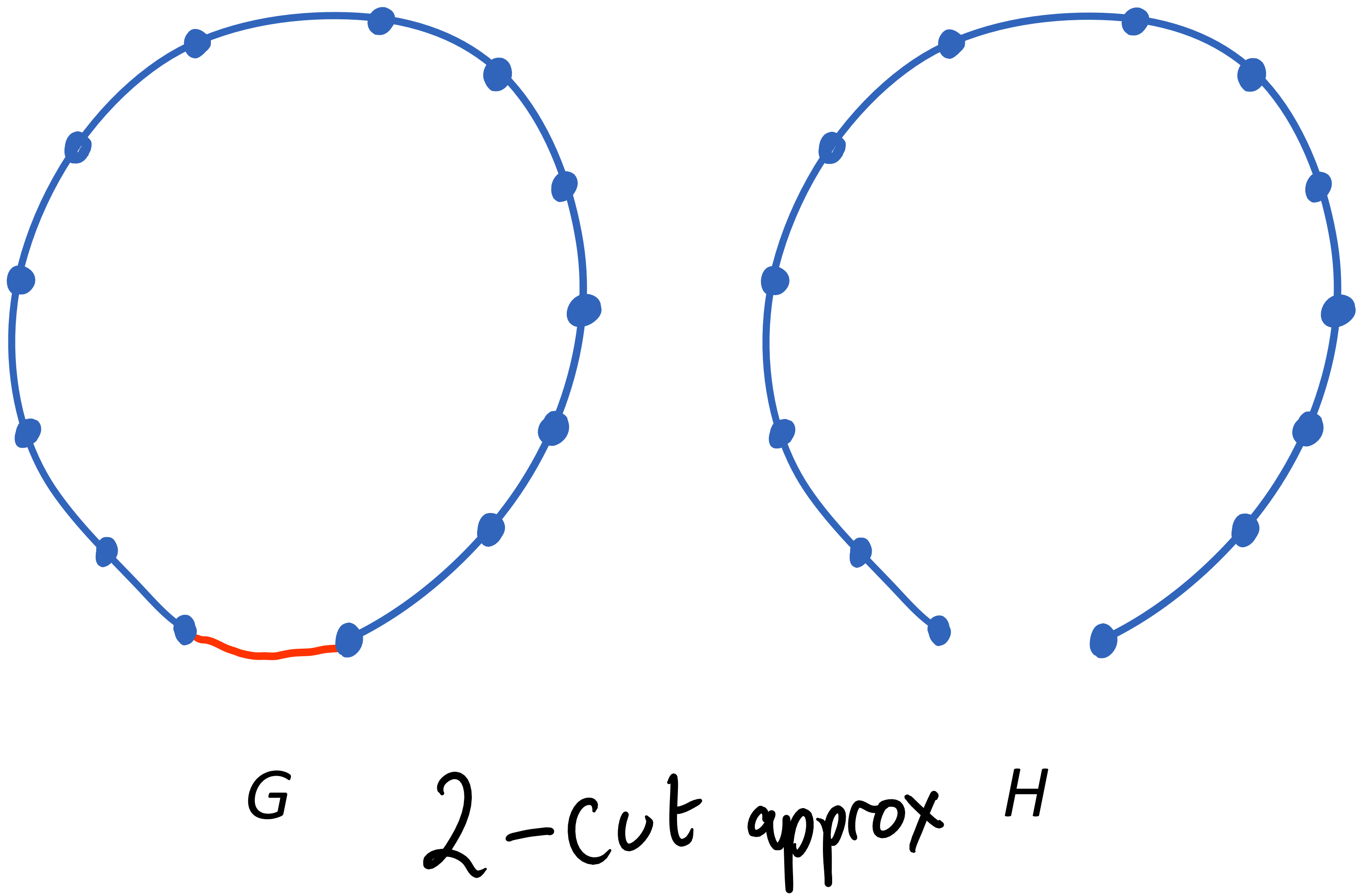
$\mathcal{E}$ is stronger than cut approx



G = cycle

Min cut = 2

$\mathcal{E}$ is stronger than cut approx



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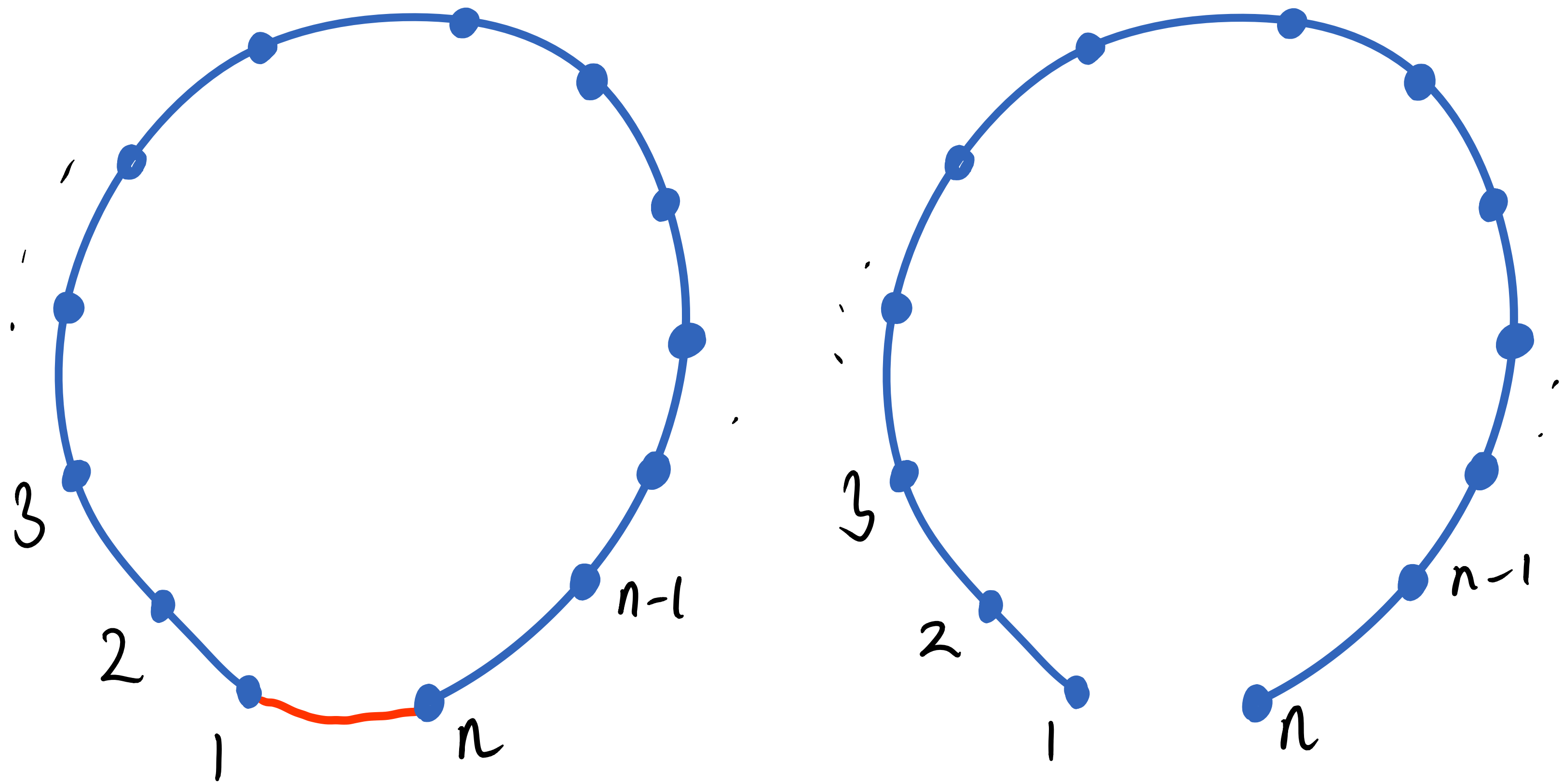

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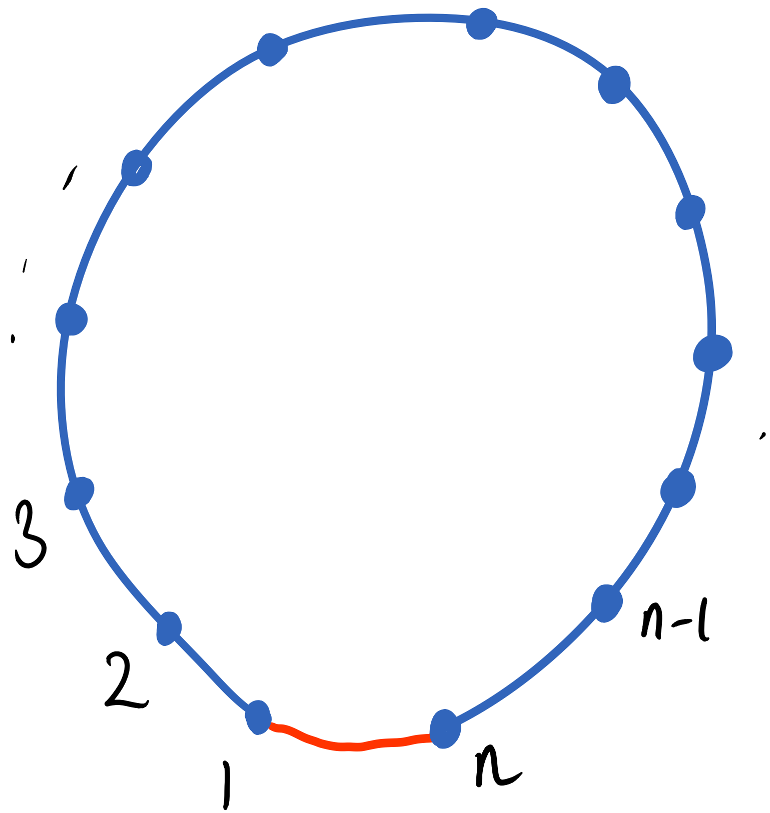

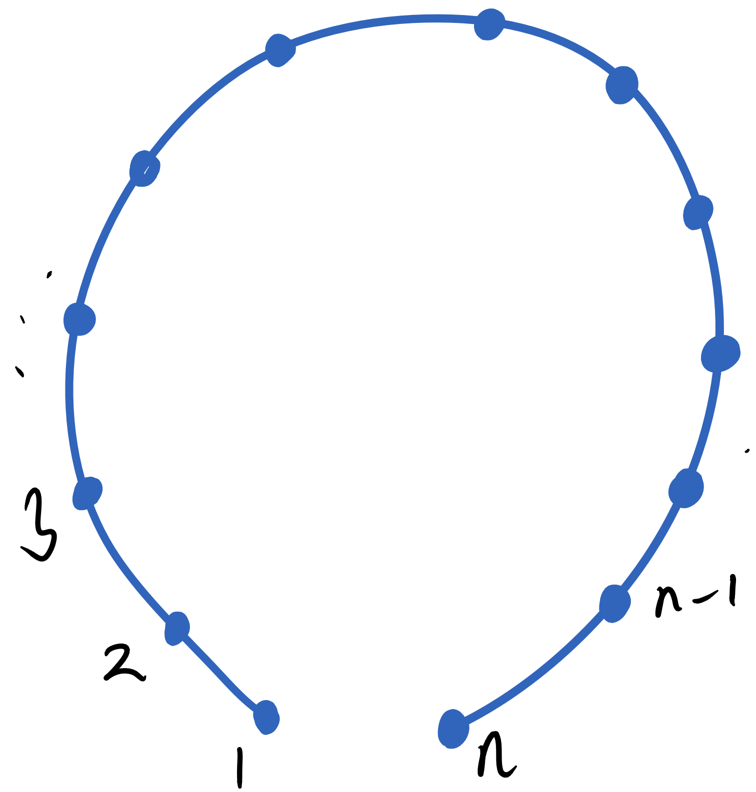

$$\mathcal{E}_q = n-1+(n-1)^2 \gg \mathcal{E}_H = n-1$$

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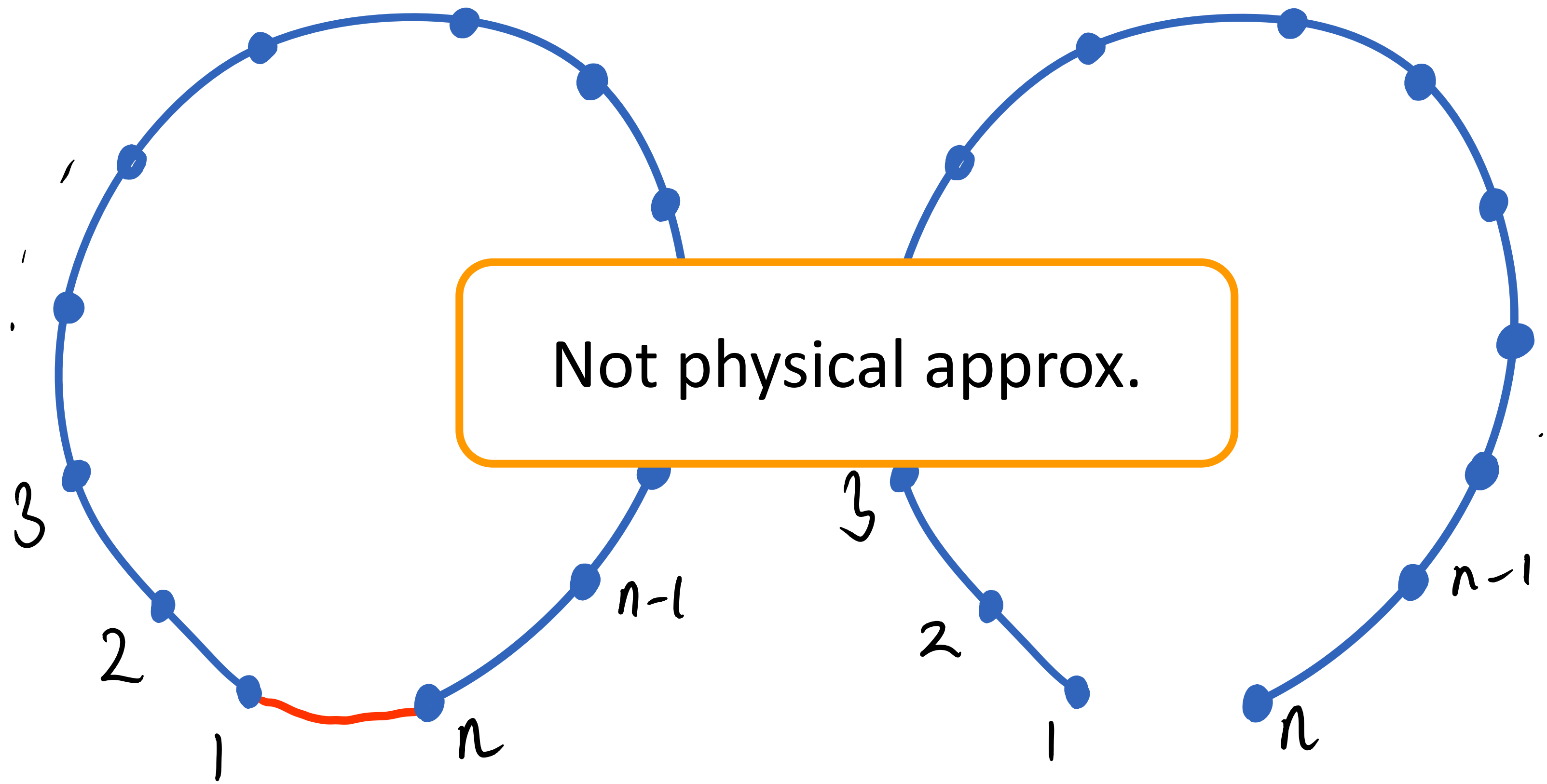

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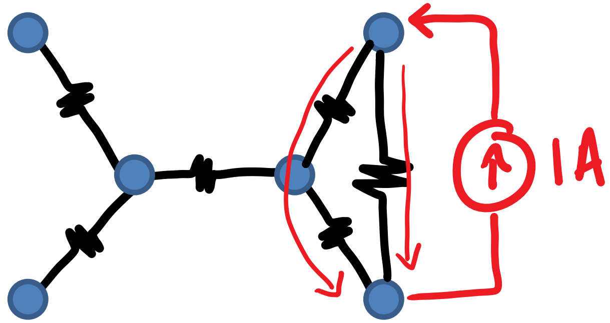
1. Energy controls physical processes

Electrical Flow:

minimizes energy



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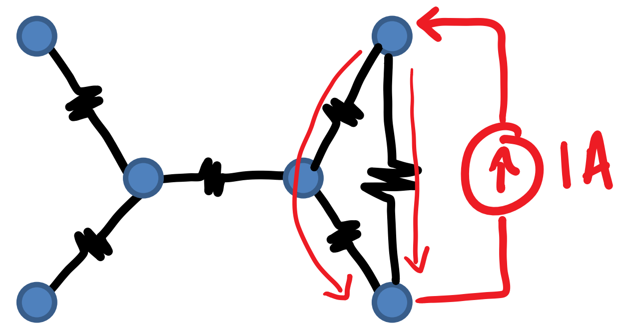
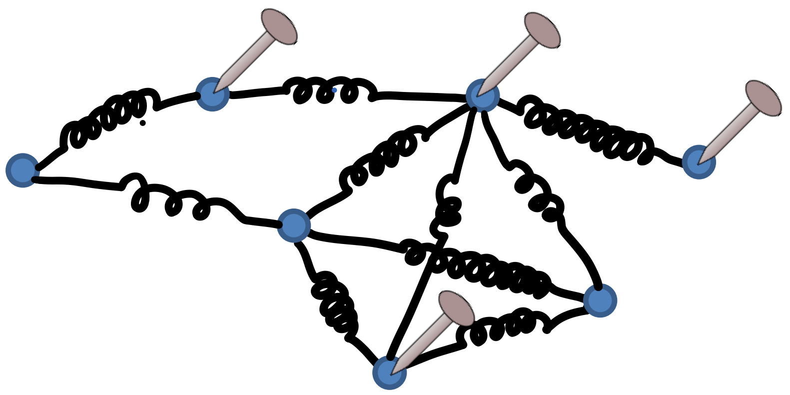


Spring Network:

settles at min. energy



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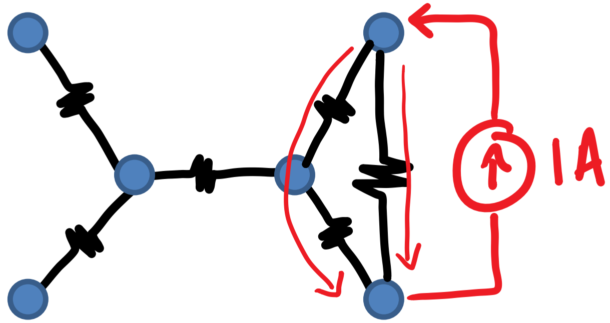

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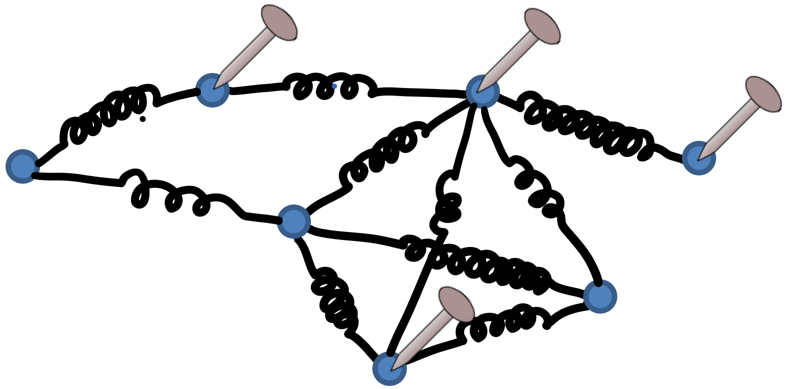
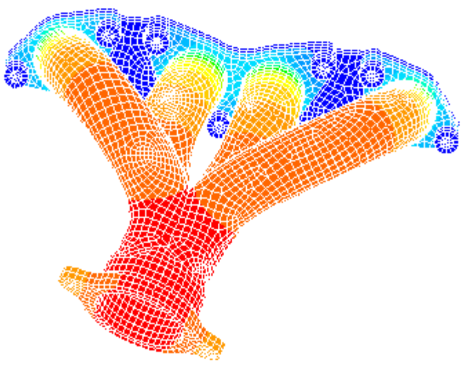


Heat Flow:



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Electrical Flow:

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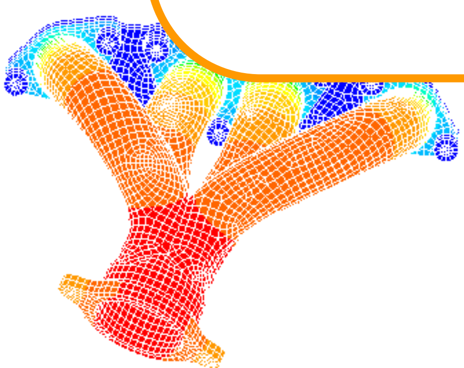
Spring Network:

settles at min

Heat Flow:



Solving any of these reduces to solving a Laplacian linear system

$$Lx = b$$

1. Solving $Lx = b$ fast [ST'04]

$\boldsymbol{x^T L_G x \sim x^T L_H x}$: can solve systems

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ST'04	$O(m\log^{30} n)$
KMP'10	$O(m\log n)$

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ST'04	$O(m\log^{30} n)$
KMP'10	$O(m\log n)$

Thm [ST'o4] $\forall G$ can find H with $O(n \log^8 n)$ edges.

1. Solving $Lx = b$ fast [ST'04]

$\boldsymbol{x^T L_G x \sim x^T L_H x}$: can solve systems in $\mathbf{L_G}$ by solving systems in $\mathbf{L_H}$.

$Lx = b$

Electrical Flow

Heat Flow

Spring Network

1. Solving $Lx = b$ fast [ST'04]

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Electrical Flow

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Spring Network

Max Flow [CKMST11, LRS13, M13]
Random Spanning Tree [KM08]
Resistance Distance [SS08]

Graph Partitioning [OSV11]
Regression on Graphs [ZGL03]
…

2. Spectral Graph Theory

Courant-Fischer Thm: $\boldsymbol{x^T L_G x}$ determines $\lambda_i(L_G)$

$$\lambda_{max}(L) = \max \frac{x^T L x}{x^T x} \qquad \lambda_{min}(L) = \min \frac{x^T L x}{x^T x}$$

Thus for physical approx. **H** of **G**:

$$(1-\epsilon)\lambda_i(G) \leq \lambda_i(H) \leq (1+\epsilon)\lambda_i(G)$$

Now **H** inherits many combinatorial properties:

random walks, colorings, spanning trees, etc.

3. Natural Setting

Spectral formulation more tractable:

$\boldsymbol{x^T L x}$ better behaved over $\mathbf{R}^n$ than $\{0,1\}^n$.

Cuts are discrete objects.

Quadratic forms are continuous objects,

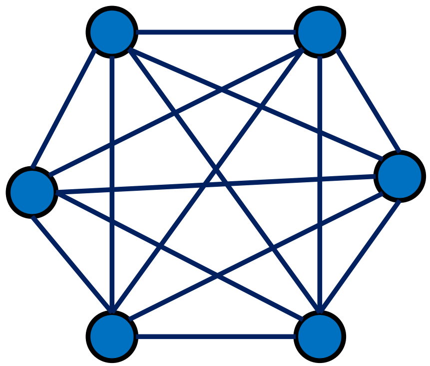
with a richer set of global transformations.

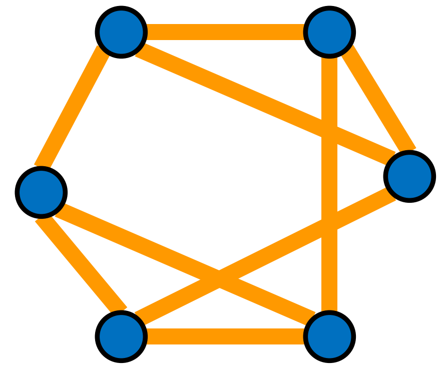
Examples

Example: The Complete Graph

$\mathbf{G=K_n}$

H = random d-regular x **(n/d)**





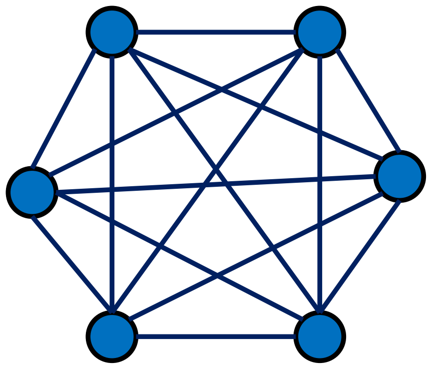
$|E_G| = O(n^2)$

$|E_H| = O(dn)$

$$\forall S \subset V, \qquad \frac{wt_G(\delta S)}{wt_{(n/d)H}(\delta S)} \simeq 1 \pm \epsilon$$

Example: The Complete Graph

G=K$_n$ **H** = random d-regular x **(n/d)**



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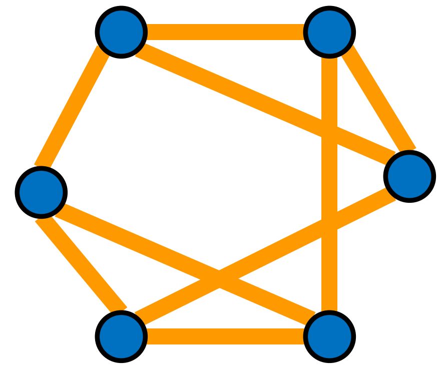
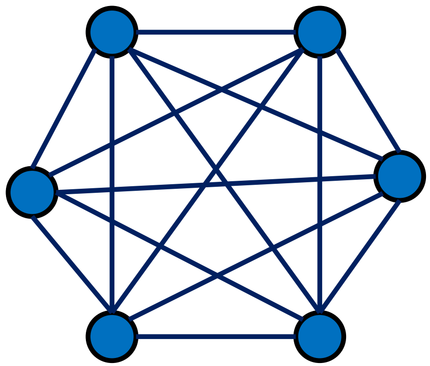


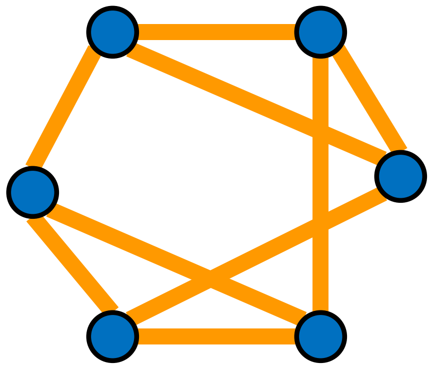
$|E_H| = O(dn)$

$$\forall x, \qquad \frac{x^T L_G x}{x^T L_H x} \simeq 1 \pm \epsilon$$

Example: The Complete Graph

G=K$_n$ **H** = random d-regular x **(n/d)**



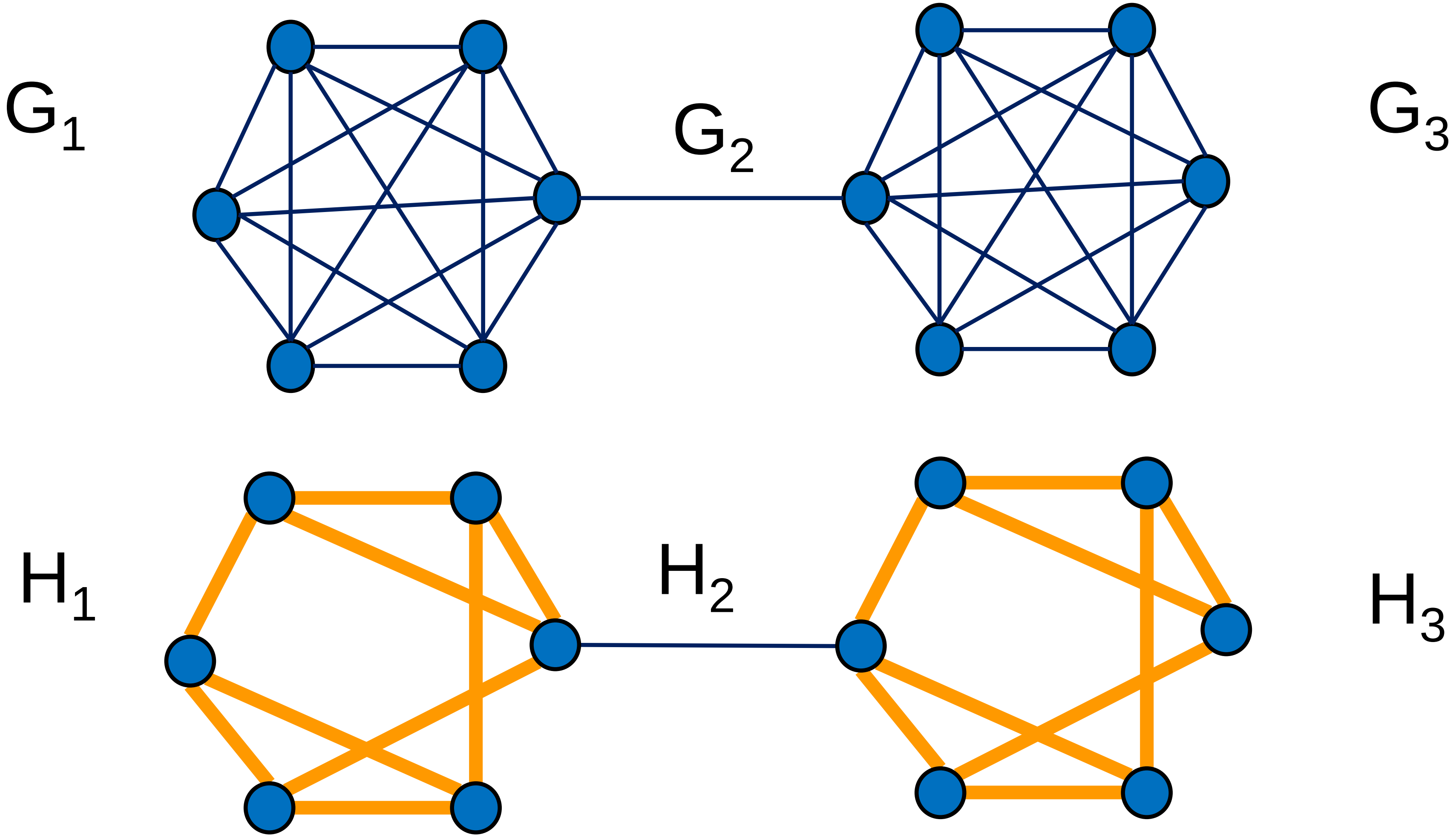


$|E_G| = O(n^2)$ $|E_H| = O(dn)$

$$d = 2/\epsilon^2$$

$$\forall x, \quad \frac{x^T L_G x}{x^T L_H x} \simeq 1 \pm \epsilon$$

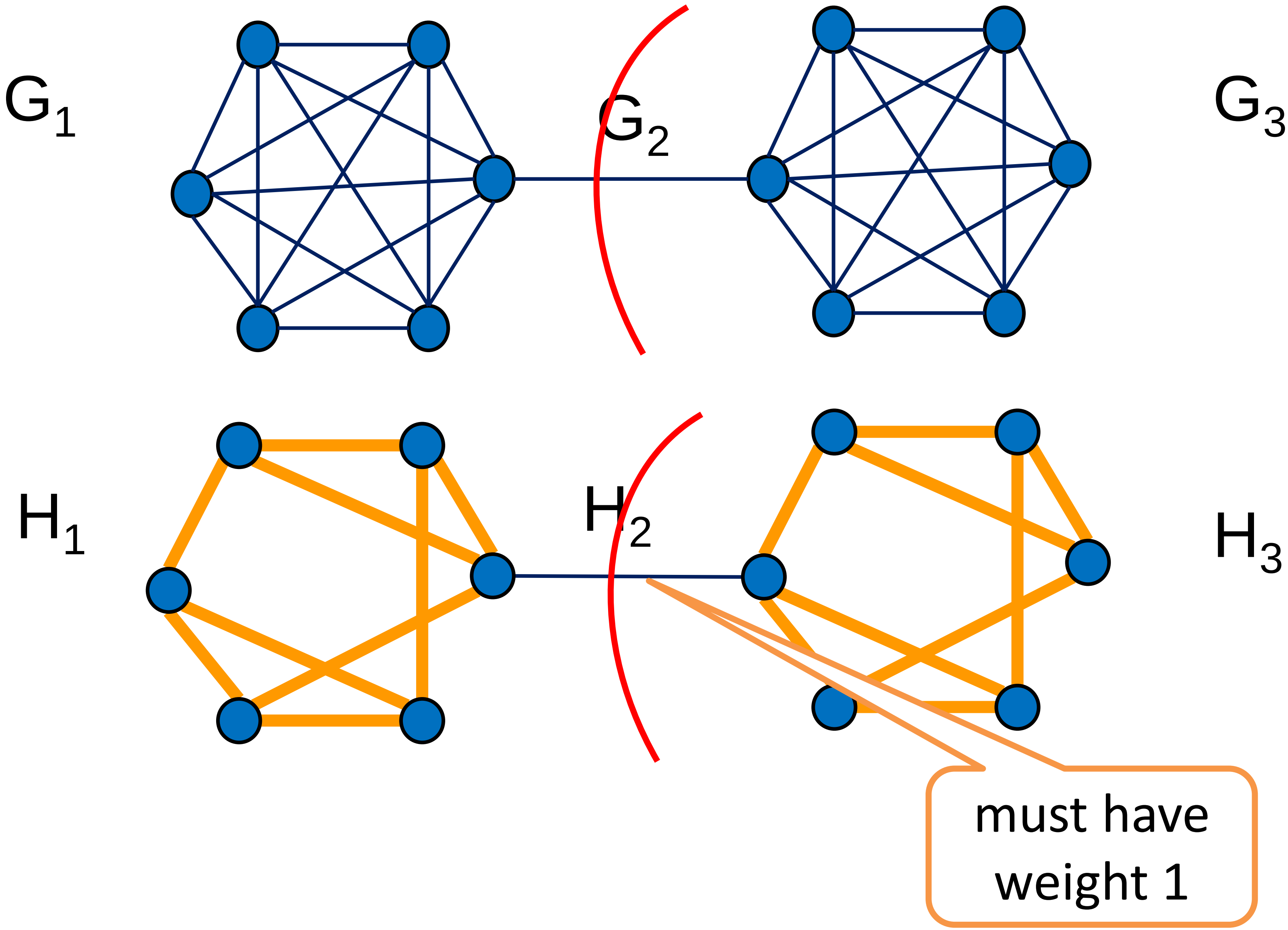
Example: Dumbell



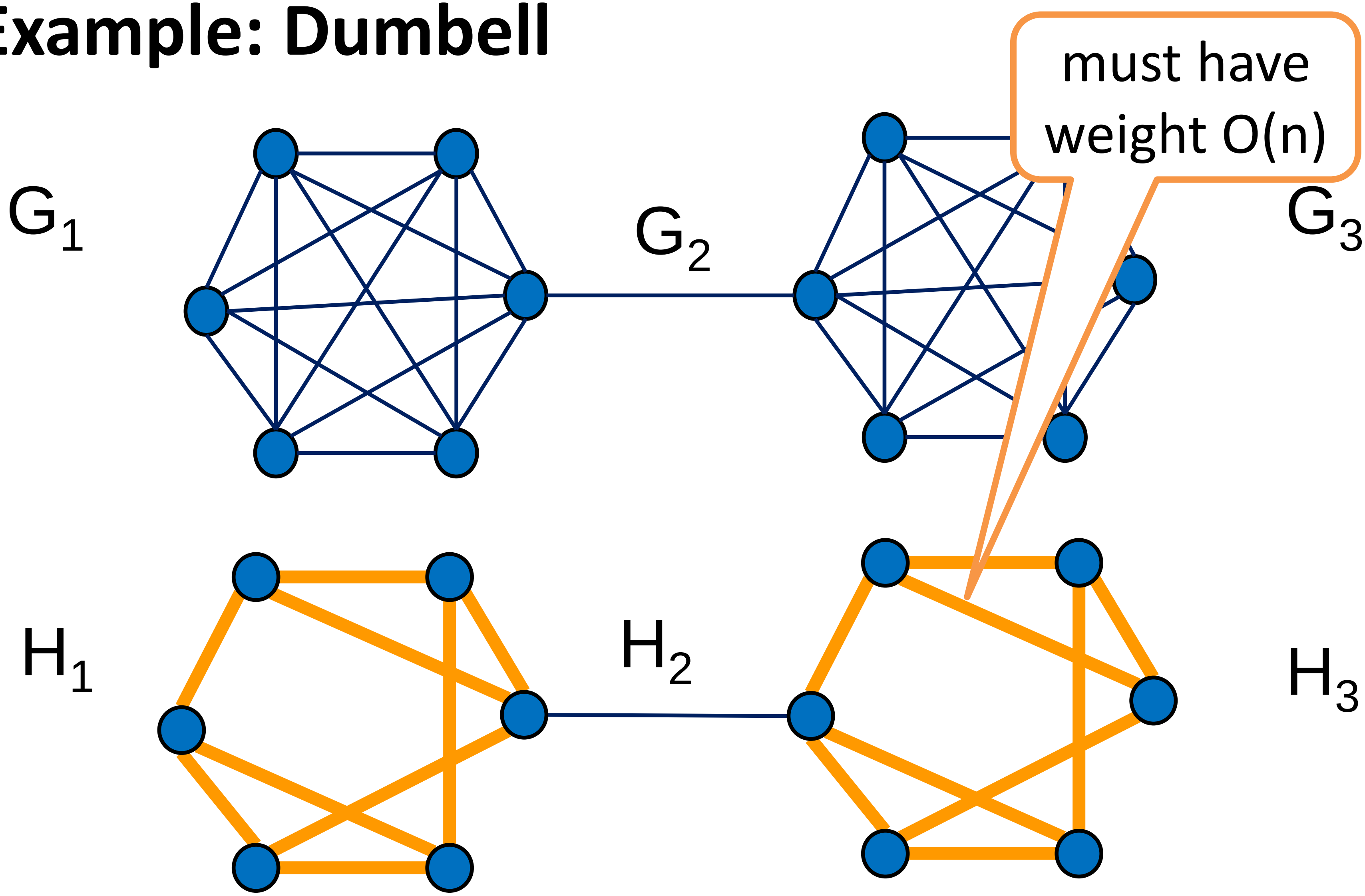
$$G = G_1 + G_2 + G_3$$

$$x^T G x = x^T G_1 x + x^T G_2 x + x^T G_3 x$$

Example: Dumbell



Example: Dumbell



Will show how to do this
for every graph…

Theorem. Every weighted graph **G** has a weighted subgraph **H** with at most $9n\log n/\epsilon^2$ edges s.t.

$$L_G \preccurlyeq L_H \preccurlyeq (1+\epsilon)L_G.$$

Moreover, H can be found in time $O^{\sim}(m/\epsilon^2)$.

Basic idea: **Random Sampling**

Choose each edge e with some probability p_e.

take k independent samples.

If included, add to H with weight $1/kp_e$.

$$\mathbb{E}[L_H] = \mathbb{E}[L_e] = \sum_{e \in G} p_e \cdot \frac{b_e b_e^T}{p_e} = L_G.$$

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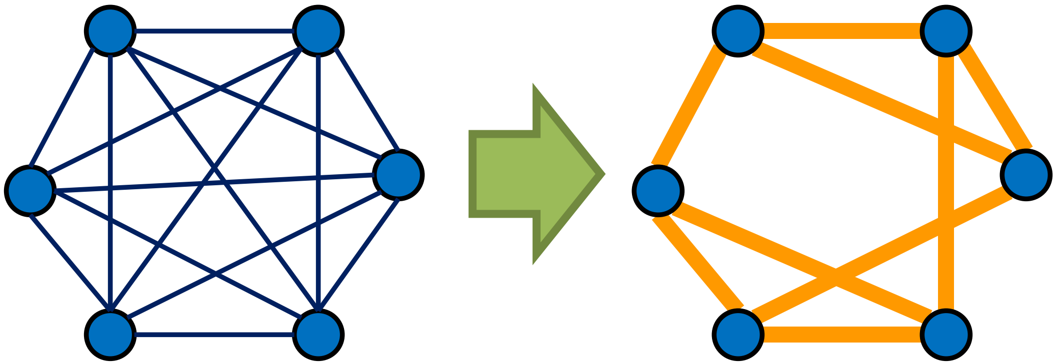
Law of large numbers: as $k \to \infty$,

$$L_H \to L_G$$

Question: how fast does this happen?

Attempt: Uniform Sampling

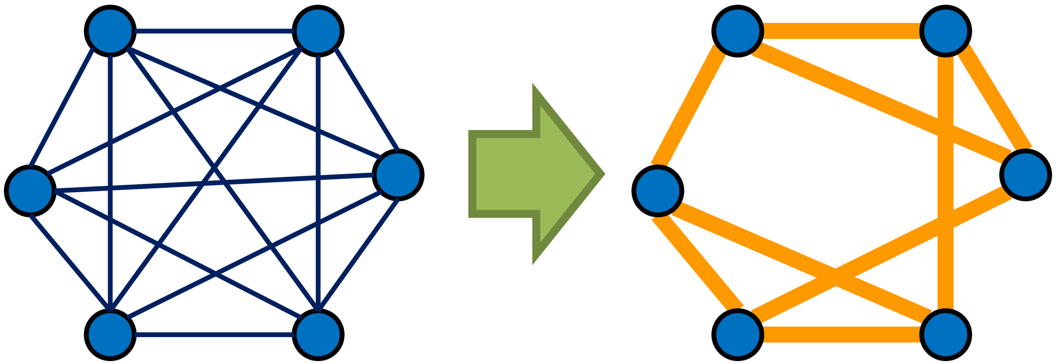
Works for K_n:



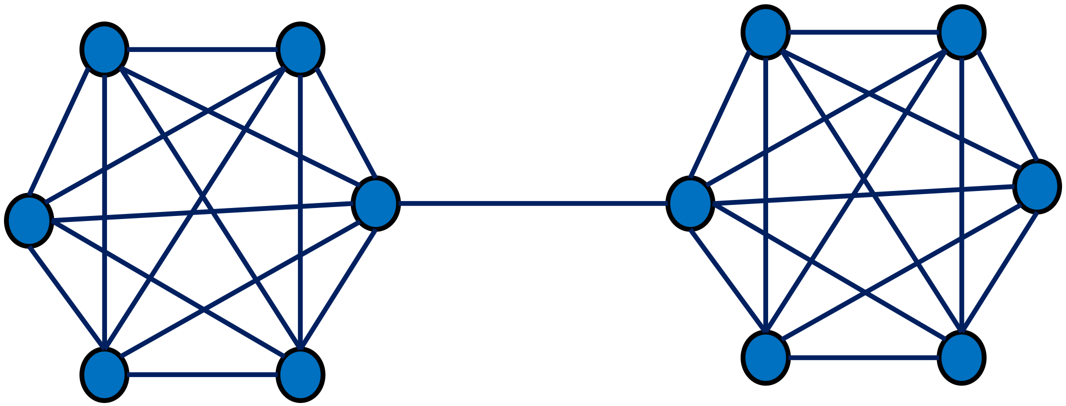
**O(nlogn) samples for i.i.d. edges*

Attempt: Uniform Sampling

Works for K_n:



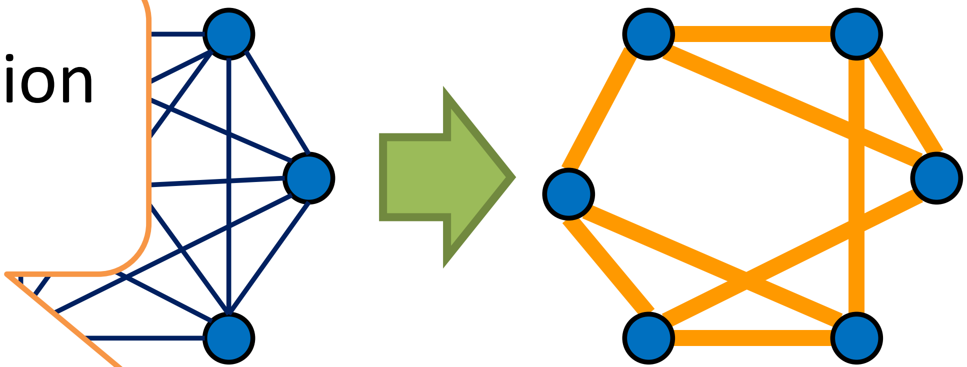
Bad for dumbbell:



Need $\Omega(m)$ samples to catch the bridge edge.

Attempt: Uniform Sampling

Need to bias distribution towards this edge



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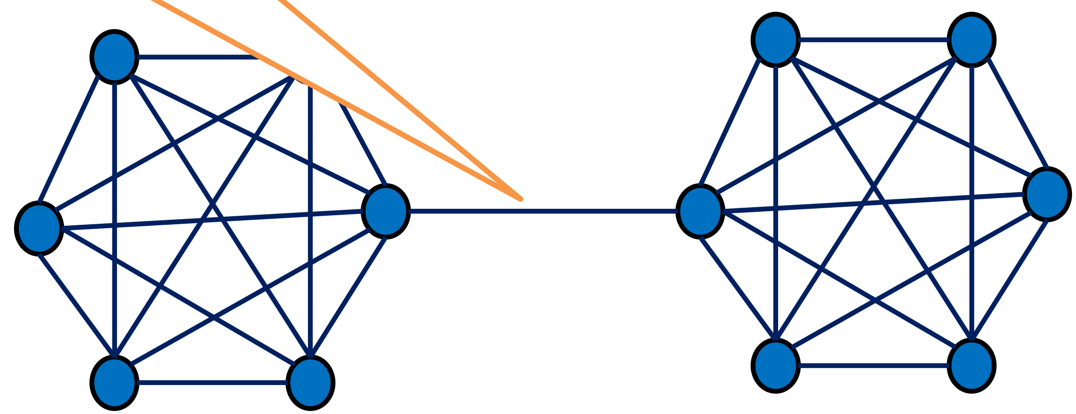

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Effective Resistance

$\mathbf{Reff}(e)$ **=** energy dissipation when a unit current is injected/removed across ends of e**.**

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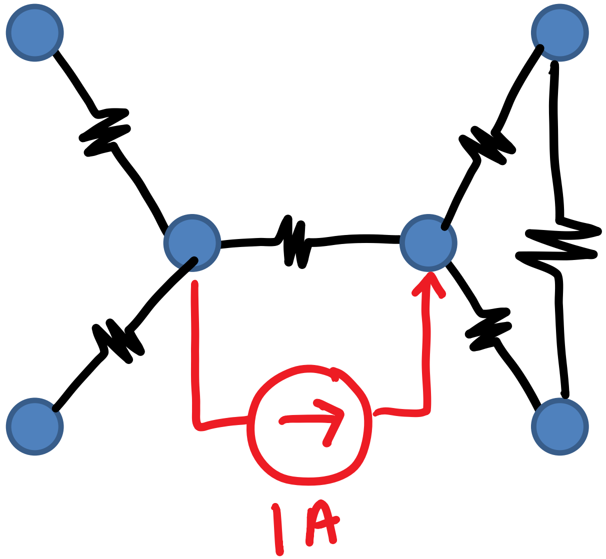

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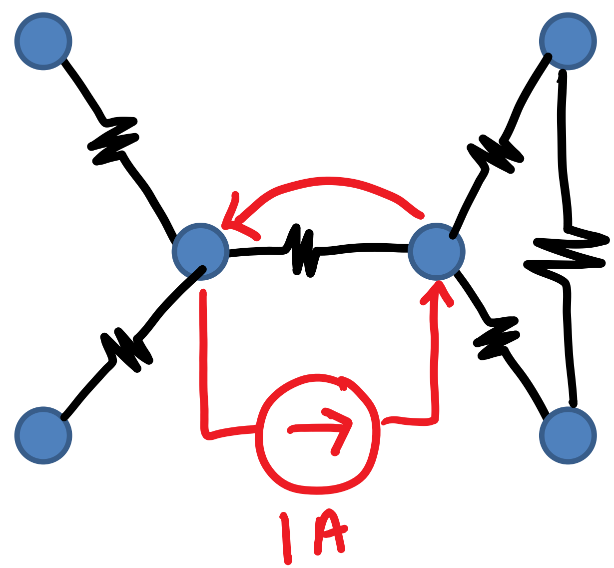

electrical flow minimizes energy

Effective Resistance

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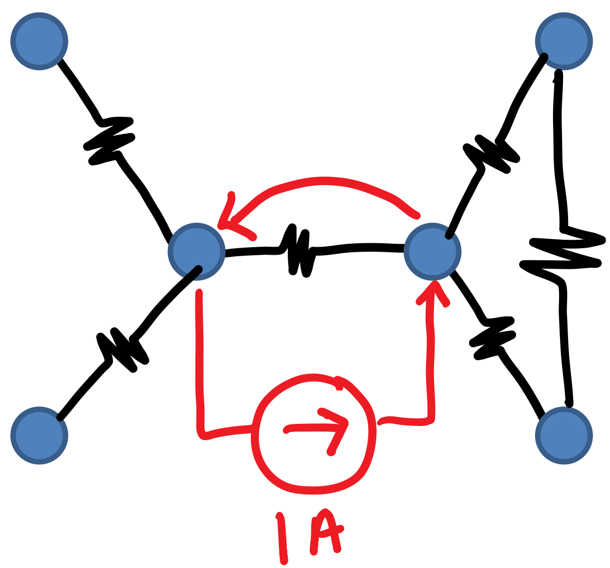
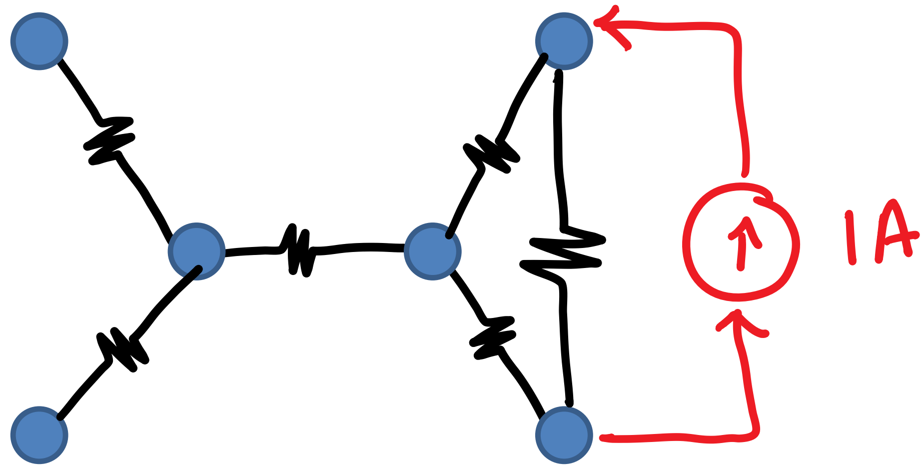


$$\mathbf{Reff}(e) = 1^2 = 1$$

Effective Resistance

Reff(e) = energy dissipation when a unit current is injected/removed across ends of e.



Effective Resistance

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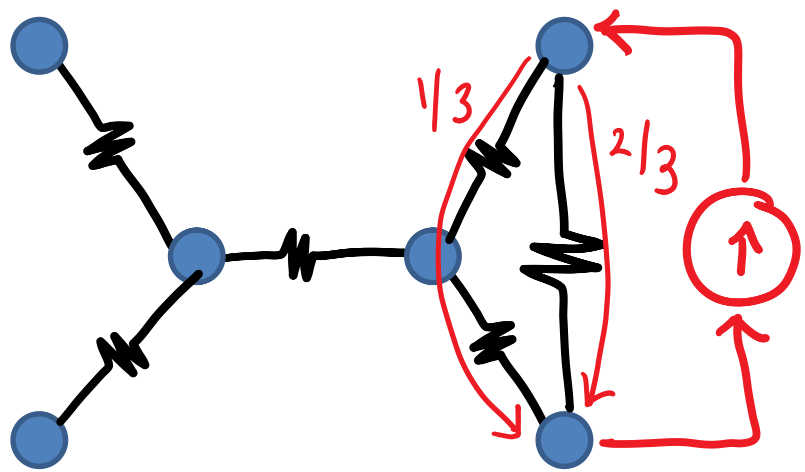
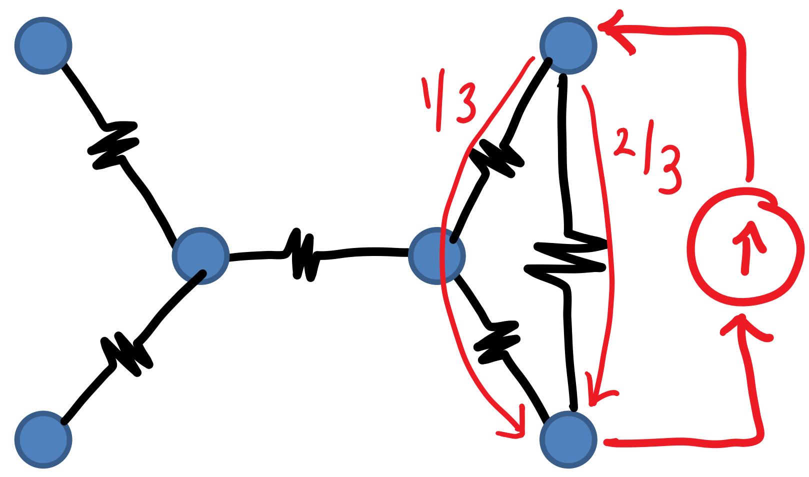


$$\mathbf{Reff}(e) = (2/3)^2 + (1/3)^2 + (1/3)^2 = 2/3$$

Effective Resistance

Reff(e) **=** energy dissipation when a unit current is injected/removed across ends of e**.**



many alternate paths = lower resistance
= **electrically "redundant"**

Effective Resistance

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few alternate paths = high resistance
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Effective Resistance

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= **electrically “redundant”**

Idea: sample edges according to effective resistances.

Theorem. Every weighted graph **G** has a weighted subgraph **H** with at most $9n \log n / \epsilon^2$ edges s.t.

$$L_G \preccurlyeq L_H \preccurlyeq (1+\epsilon) L_G.$$

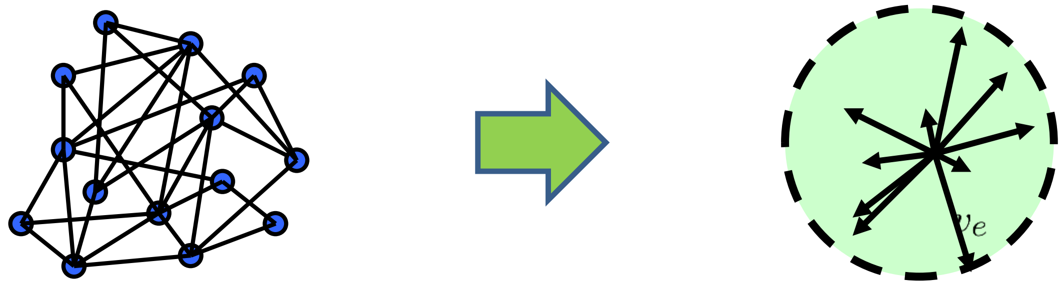
Moreover, H can be found in time $O^{\sim}(m/\epsilon^2)$.

Algorithm: sample $9n \log n / \epsilon^2$ edges independently according to effective resistances.

3 Step Proof

1. Reduction to a linear algebra problem
2. Solution of linear algebra problem by random matrix theory.
3. Fast computation of sampling probabilities

[Spielman-S'08]



Part 1: Reduction to Linear Algebra

Original Goal

Given G

Find sparse H

satisfying $L_G \preceq L_H \preceq \kappa \cdot L_G$

Outer Product Expansion

Recall:

$$L_G = \sum_{ij \in E} (\delta_i - \delta_j)(\delta_i - \delta_j)^T = \sum_{e \in E} b_e b_e^T .$$

Outer Product Expansion

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$$L_G = \sum_{ij \in E} (\delta_i - \delta_j)(\delta_i - \delta_j)^T = \sum_{e \in E} b_e b_e^T.$$

For a weighted subgraph ***H***:

$$L_H = \sum_{e \in E} s_e b_e b_e^T$$

where s_e=wt(e) in H.

Original Goal

Given G

Find **sparse** H

satisfying

$$L_G \preceq L_H \preceq \kappa \cdot L_G$$

Original Goal

Given $$L_G = \sum_{e \in G} b_e b_e^T \qquad b_{ij} = \delta_i - \delta_j$$

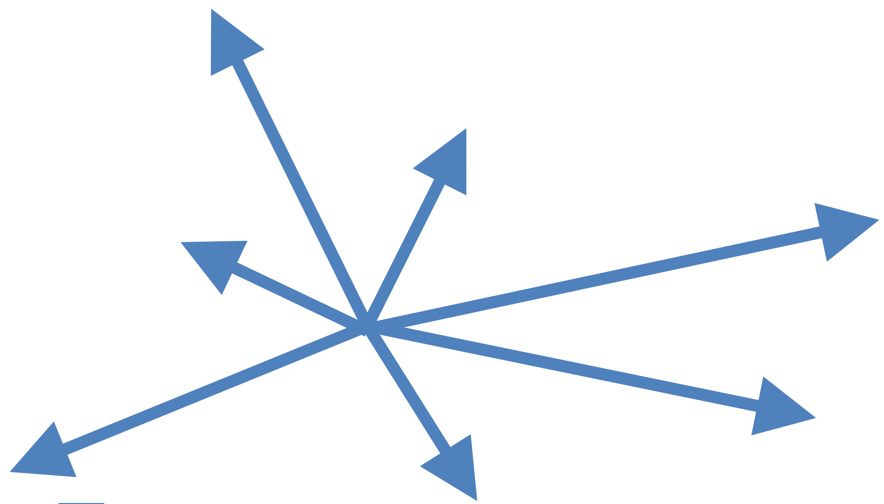
Find **sparse**

$$s_e \geq 0$$

satisfying

$$L_G \preceq L_H = \sum_{e \in G} s_e b_e b_e^T \preceq \kappa \cdot L_G$$

Quadratic Forms as Ellipsoids

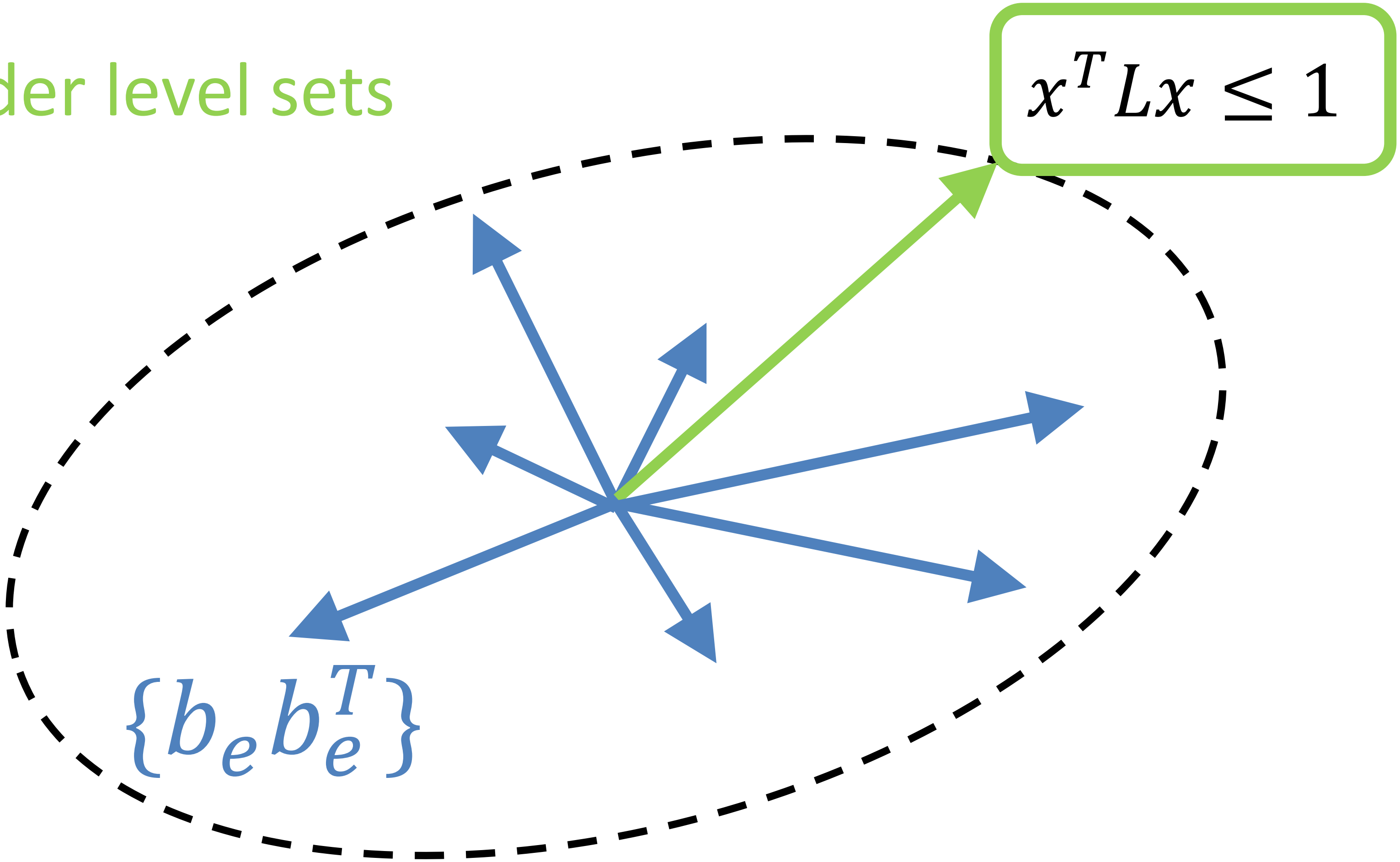


$\{b_e b_e^T\}$

$$L_G = \sum_{e \in G} b_e b_e^T \qquad b_{ij} = \delta_i - \delta_j$$

Quadratic Forms as Ellipsoids

Consider level sets



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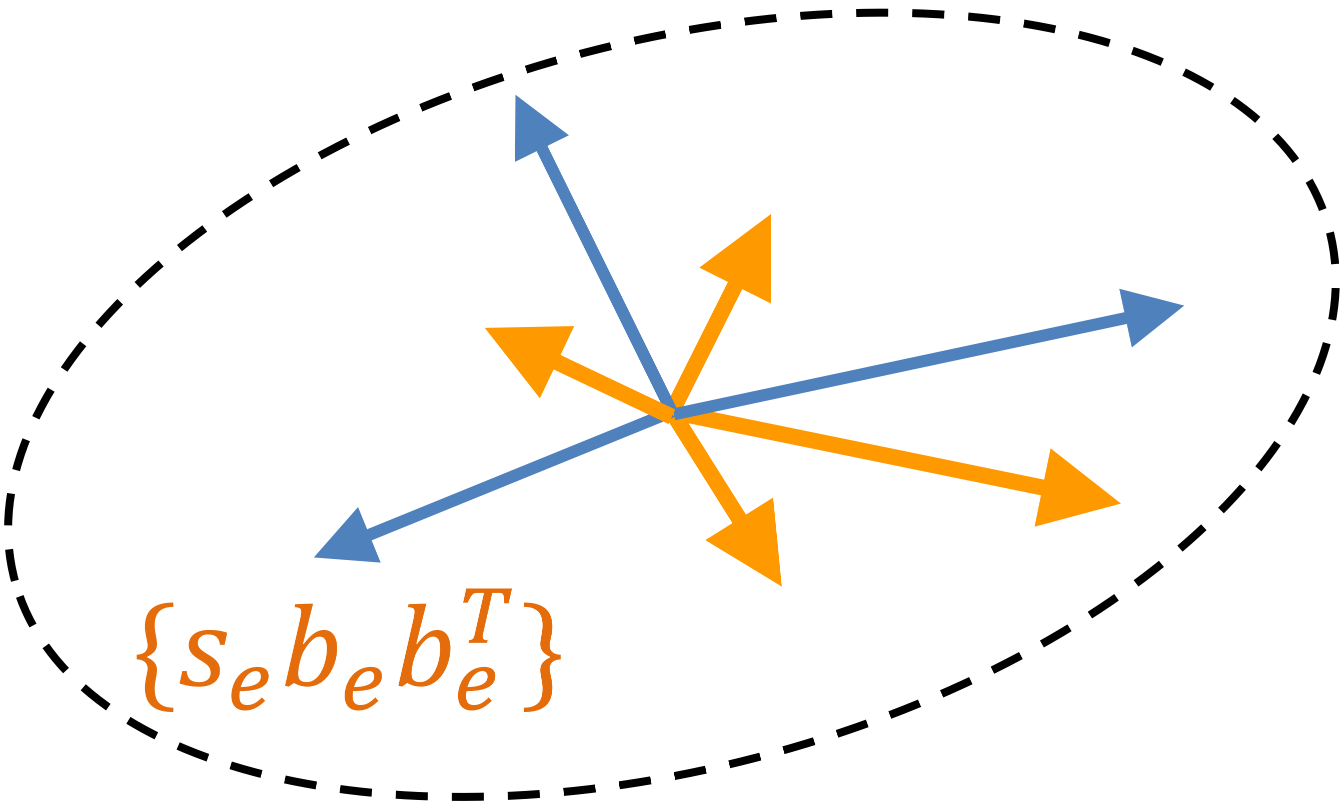
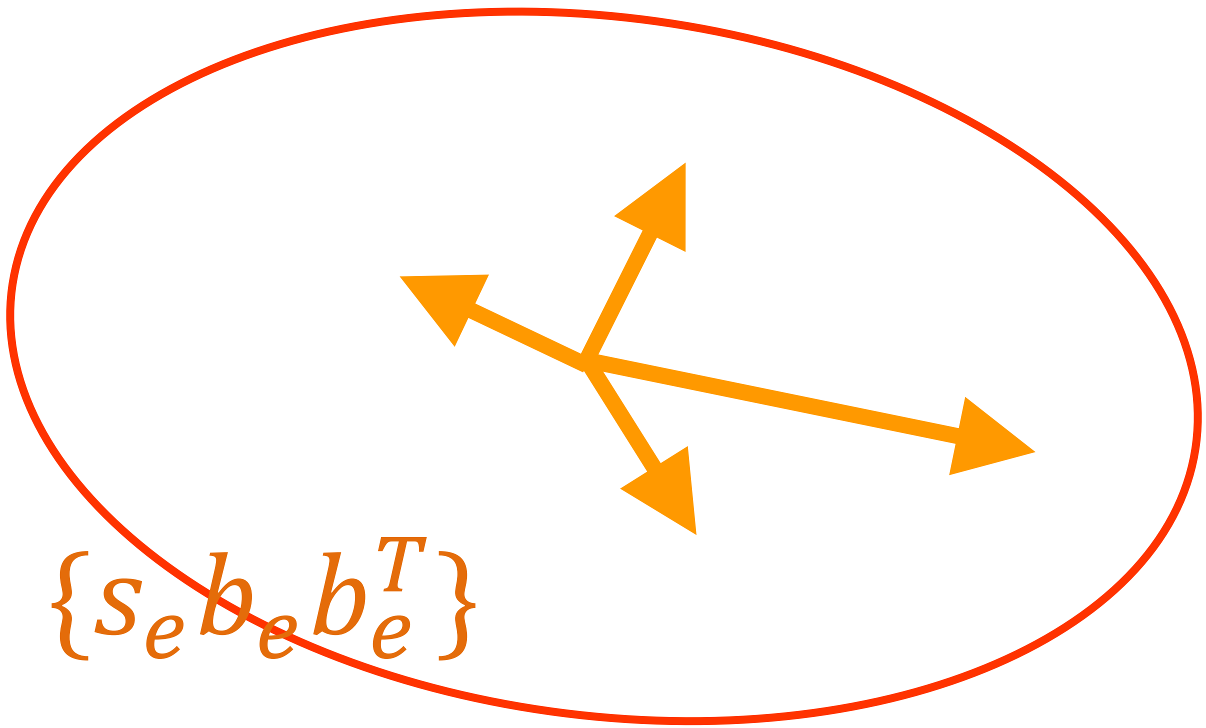
Quadratic Forms as Ellipsoids



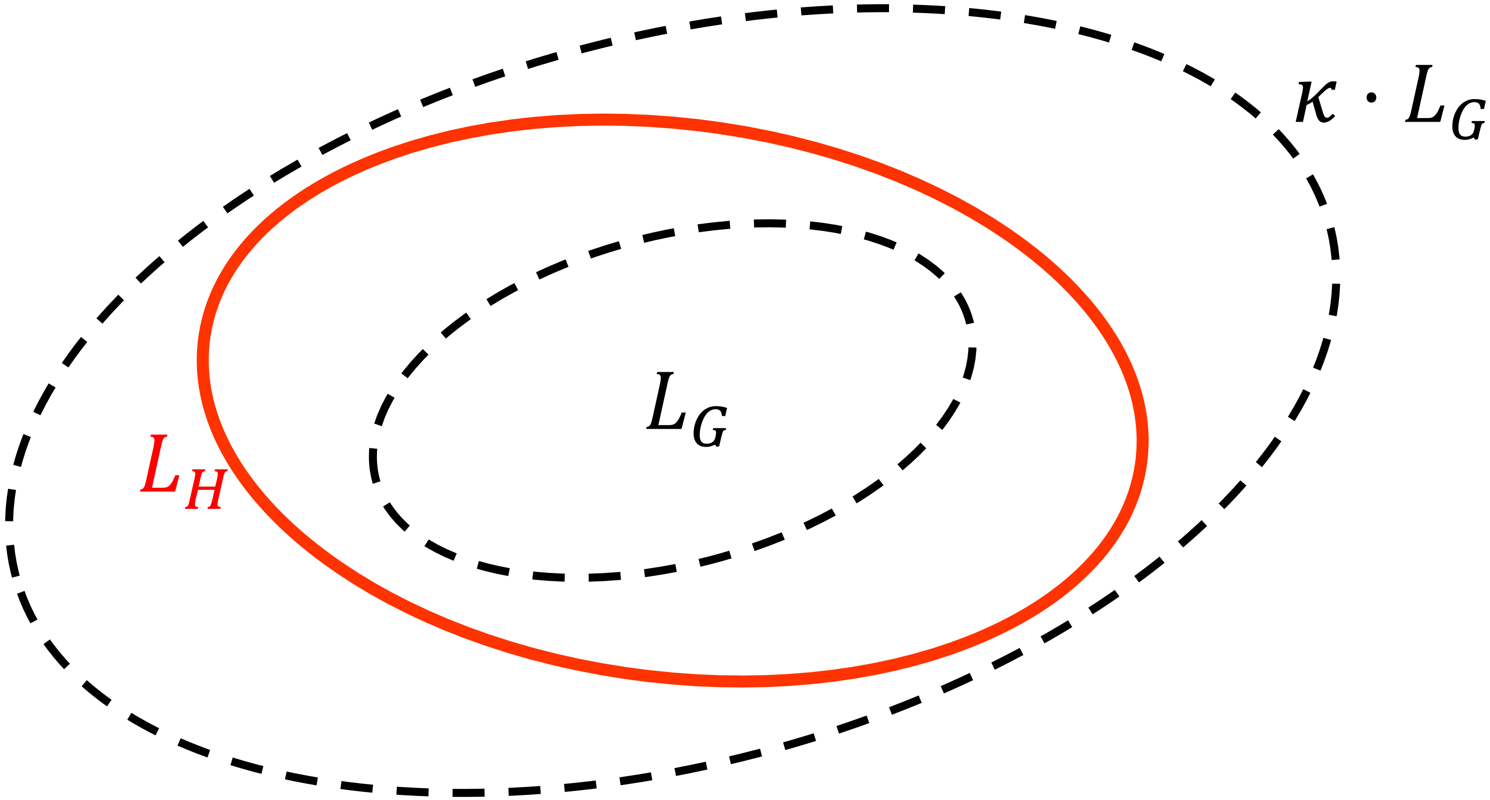
$$L_H = \sum_{e \in E} s_e b_e b_e^T$$

Quadratic Forms as Ellipsoids



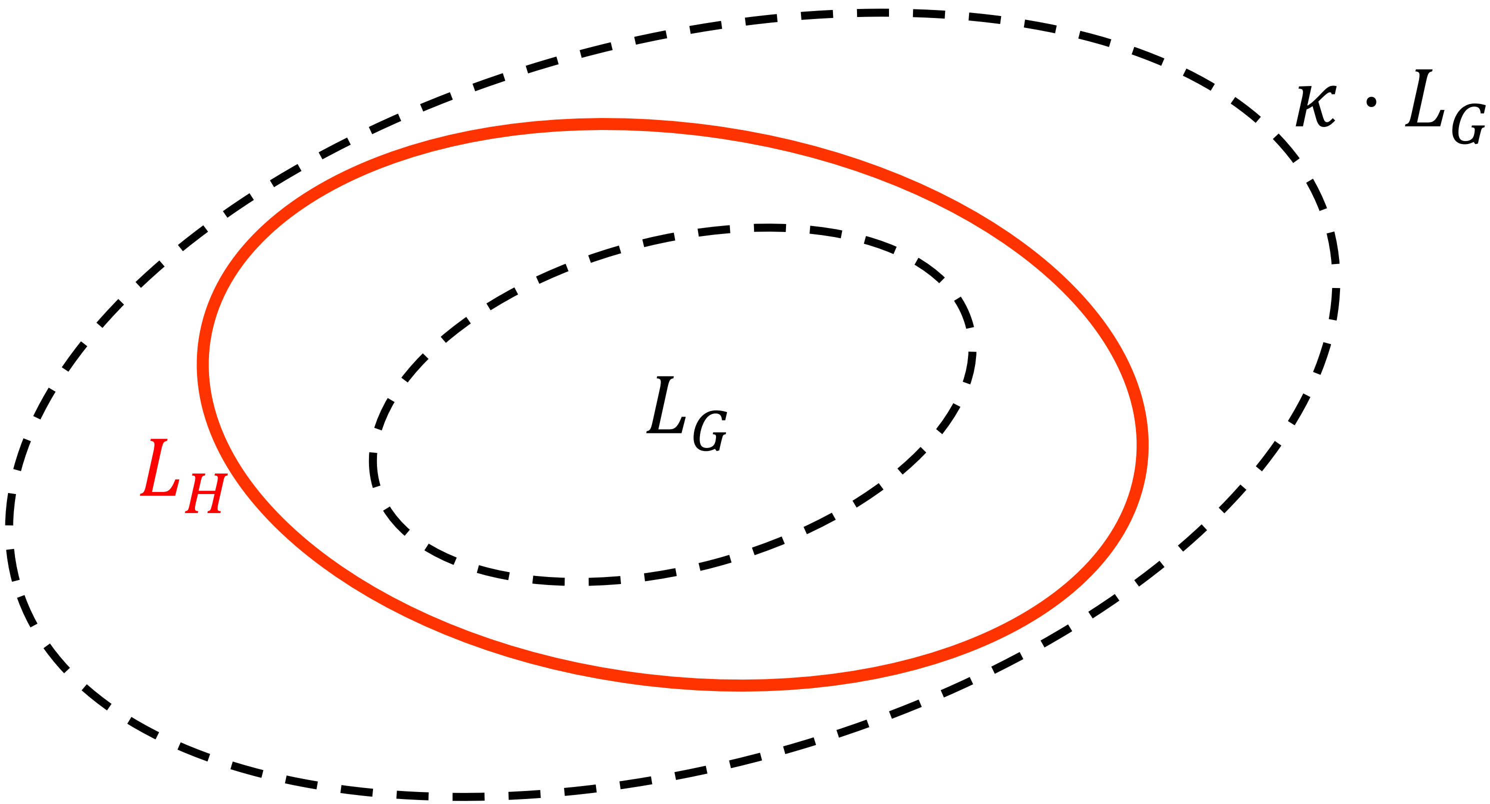
$$L_H = \sum_{e \in E} s_e b_e b_e^T$$

Containment of Ellipsoids



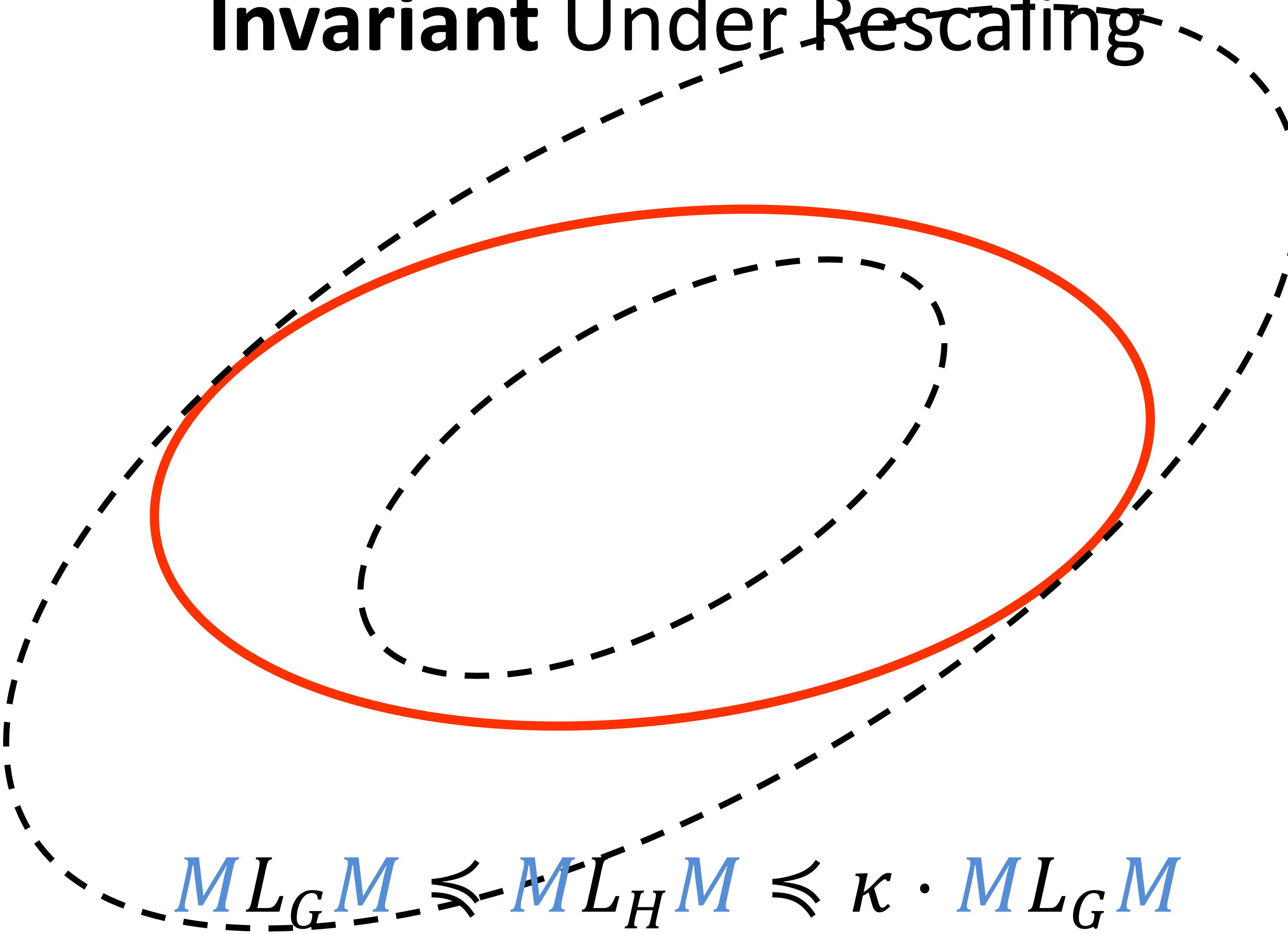
$$L_G \preccurlyeq L_H \preccurlyeq \kappa \cdot L_G$$

Invariant Under Rescaling



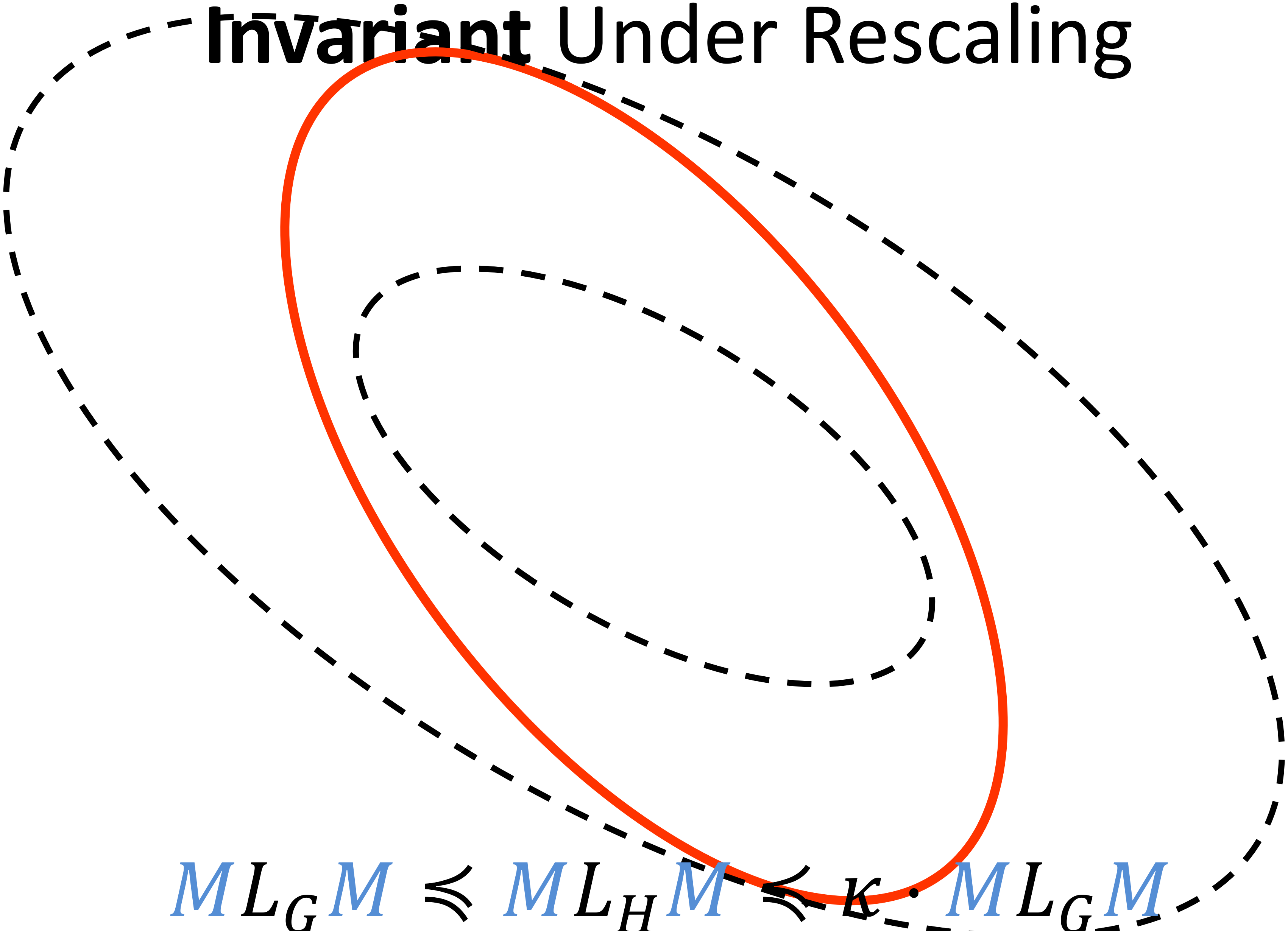
$$M L_G M \preccurlyeq M L_H M \preccurlyeq \kappa \cdot M L_G M$$

Invariant Under Rescaling



$$ML_GM \preccurlyeq ML_HM \preccurlyeq \kappa \cdot ML_GM$$

Invariant Under Rescaling



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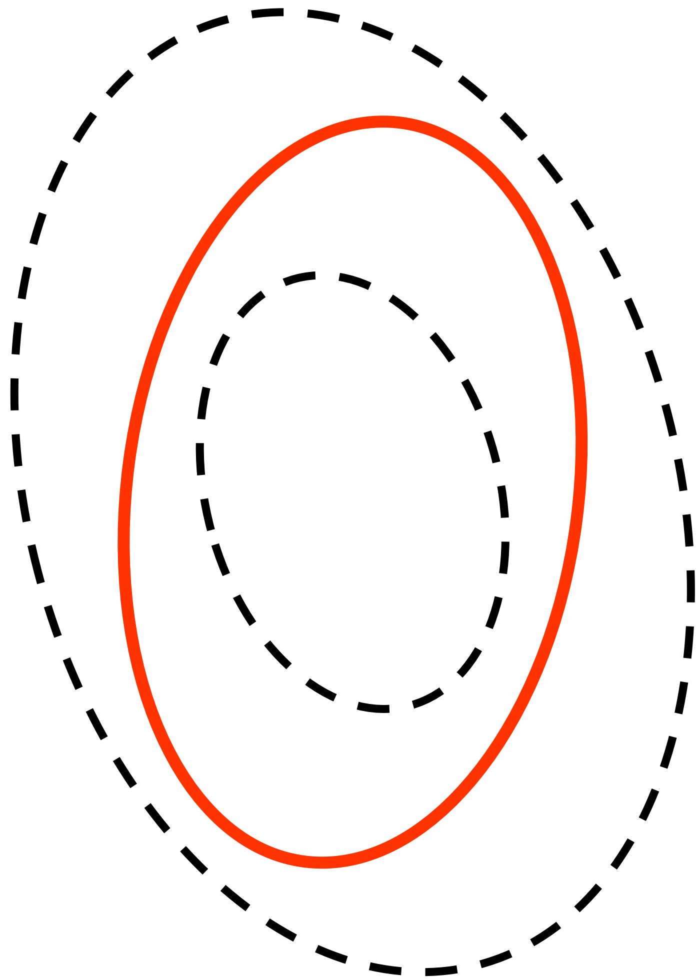
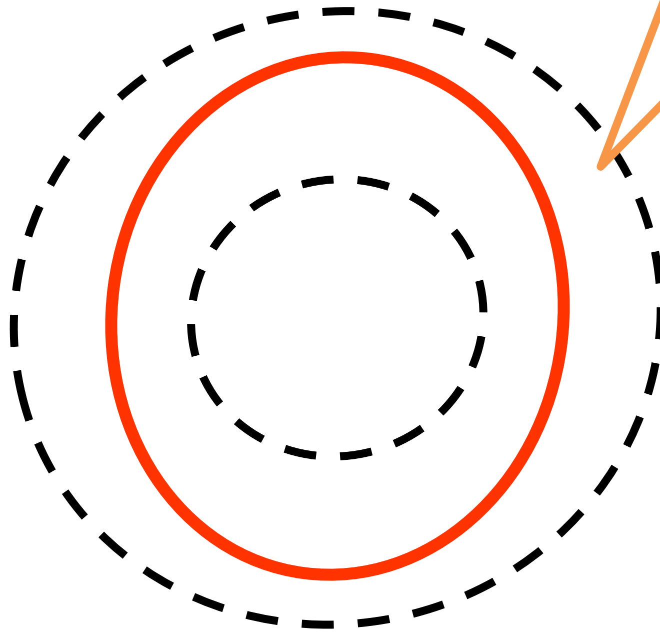


$$ML_G M \preccurlyeq ML_H M \preccurlyeq \kappa \cdot ML_G M$$

Invariant Under Re

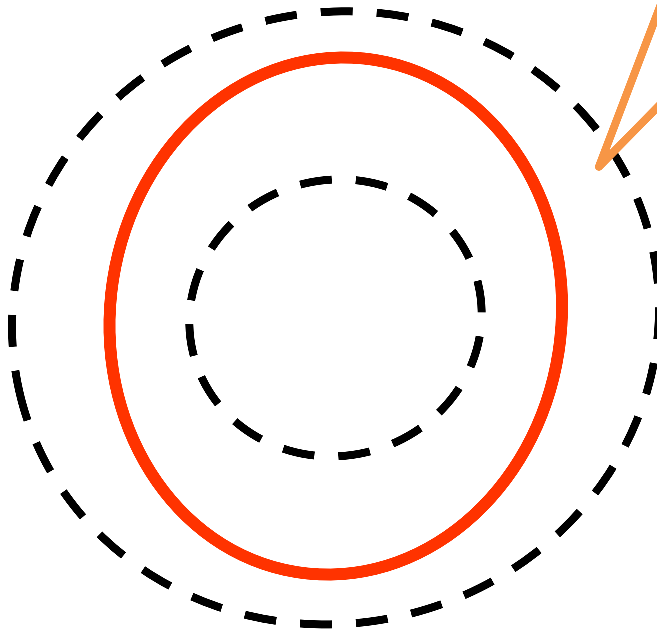
Choose $M = L_G^{-1/2}$



$$L_G^{-1/2} L_G L_G^{-1/2} \preccurlyeq L_G^{-1/2} L_H L_G^{-1/2} \preccurlyeq \kappa \cdot L_G^{-1/2} L_G L_G^{-1/2}$$

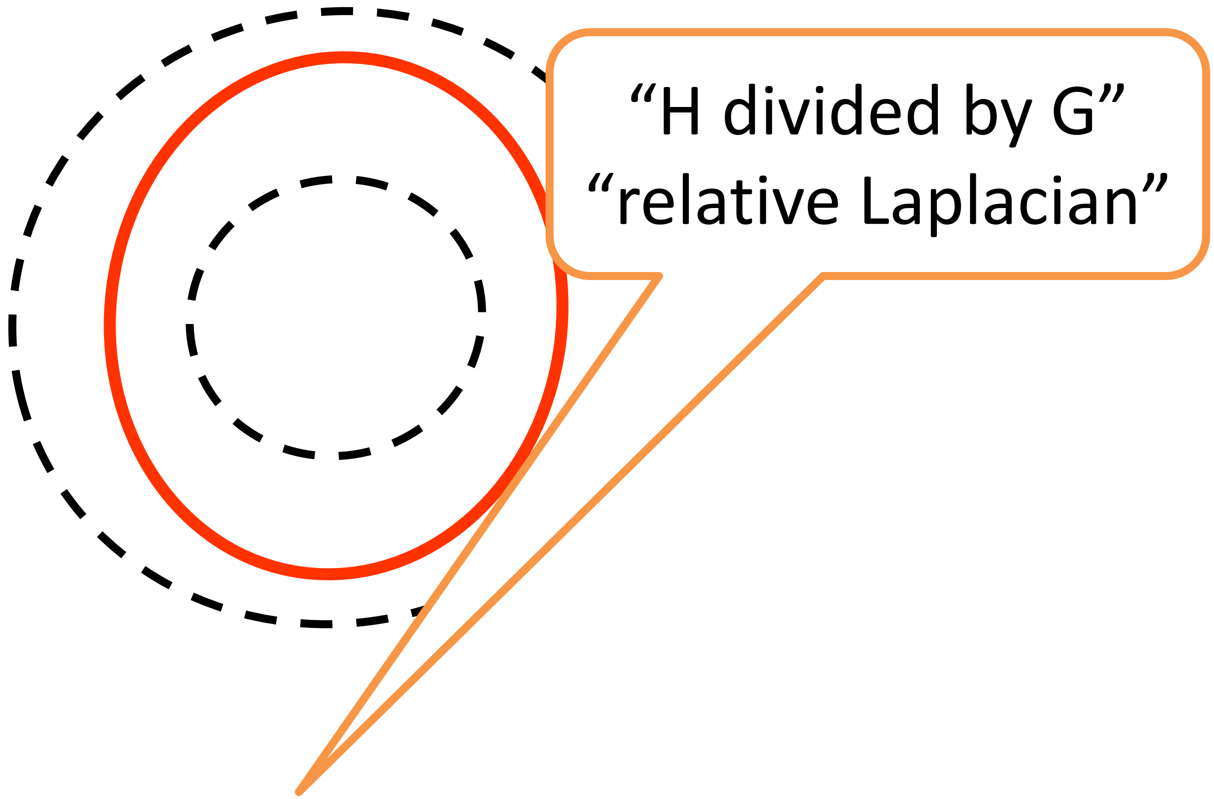
Invariant Under Re

Choose $M = L_G^{-1/2}$



$$I \preccurlyeq L_G^{-1/2} L_H L_G^{-1/2} \preccurlyeq \kappa \cdot I$$

Invariant Under Rescaling



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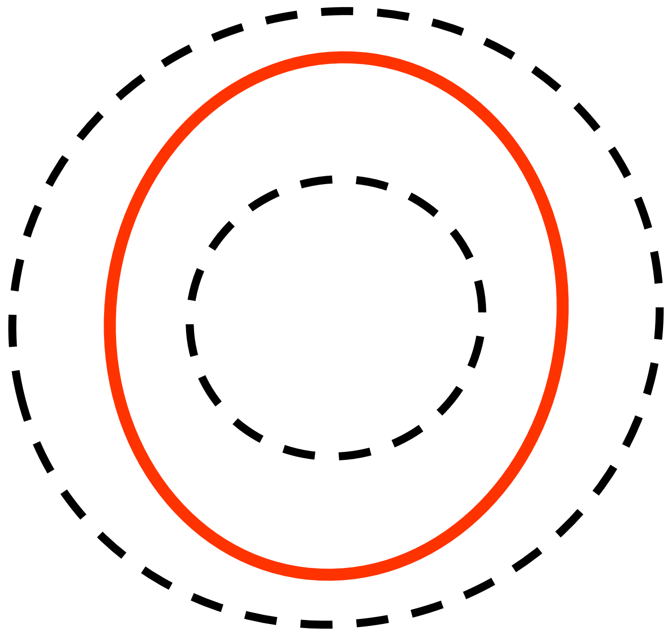
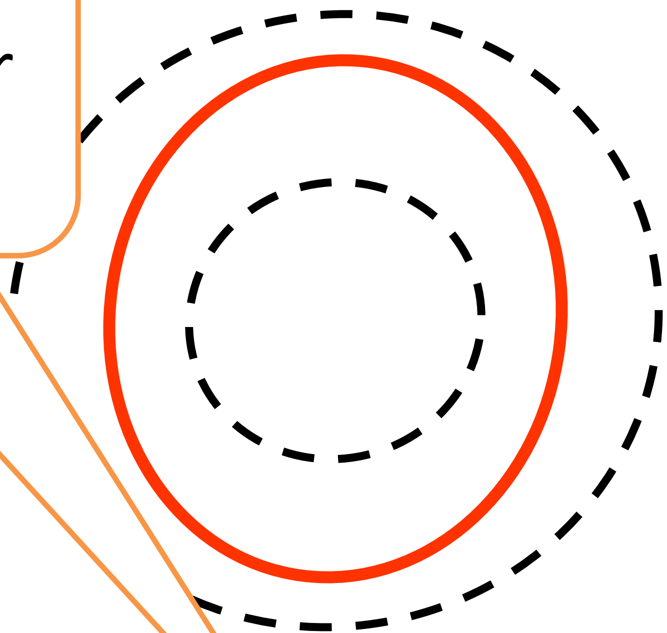
Invariant Under Rescaling



$$I \preccurlyeq \sum_{e} s_e L_G^{-1/2} b_e b_e^T L_G^{-1/2} \preccurlyeq \kappa \cdot I$$

Invariant Under Rescaling

Rescaled incidence vector
$v_e := L_G^{-1/2} b_e$

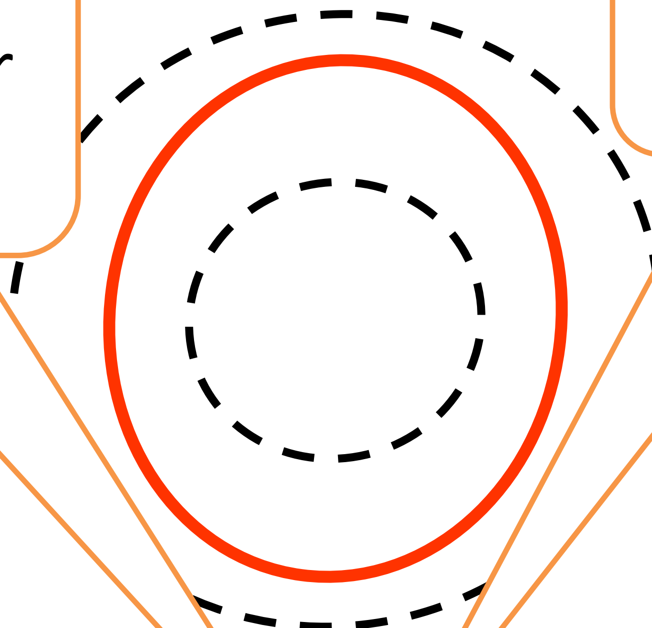


$$I \preccurlyeq \sum_e s_e v_e v_e^T \preccurlyeq \kappa \cdot I$$

Invariant Under Rescaling

Rescaled incidence vector $v_e := L_G^{-1/2} b_e$

$$\sum_e v_e v_e^T = I$$



$$I \preccurlyeq \sum_e s_e v_e v_e^T \preccurlyeq \kappa \cdot I$$

Equivalent Problem

Given $$I = \sum_e v_e v_e^T$$

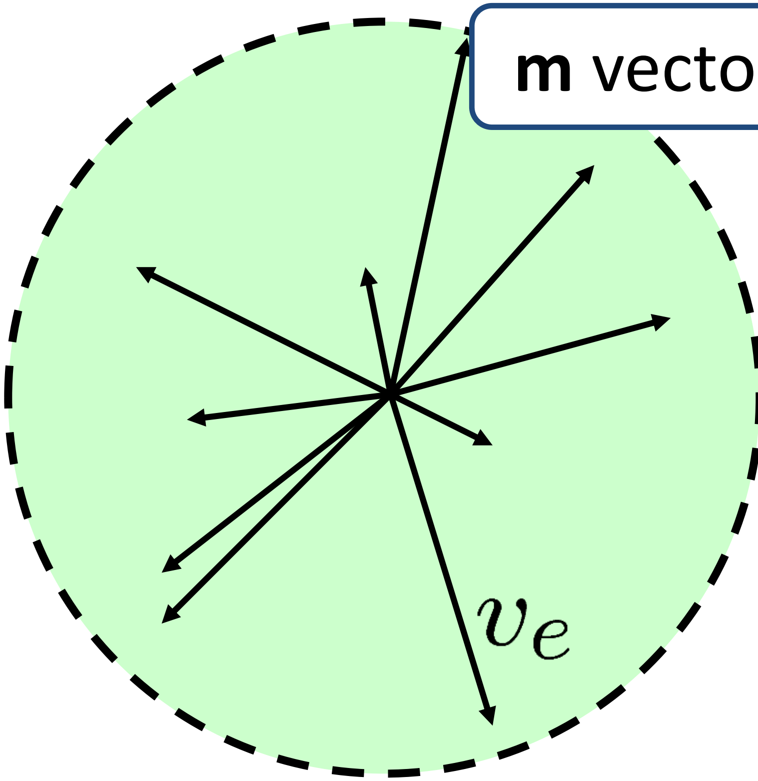
Find **sparse**

$$s_e \geq 0$$

satisfying

$$I \preceq \sum_{e \in G} s_e v_e v_e^T \preceq \kappa \cdot I$$

Core Problem

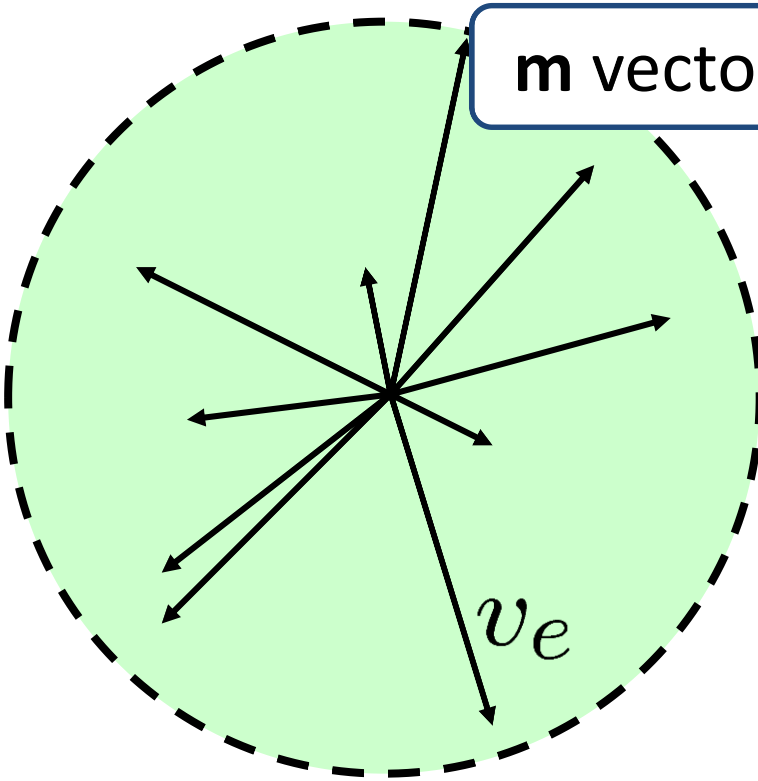


m vectors in $\mathbf{R}^n$

$$I = \sum_e v_e v_e^T$$

Core Problem

m vectors in $\mathbf{R}^n$

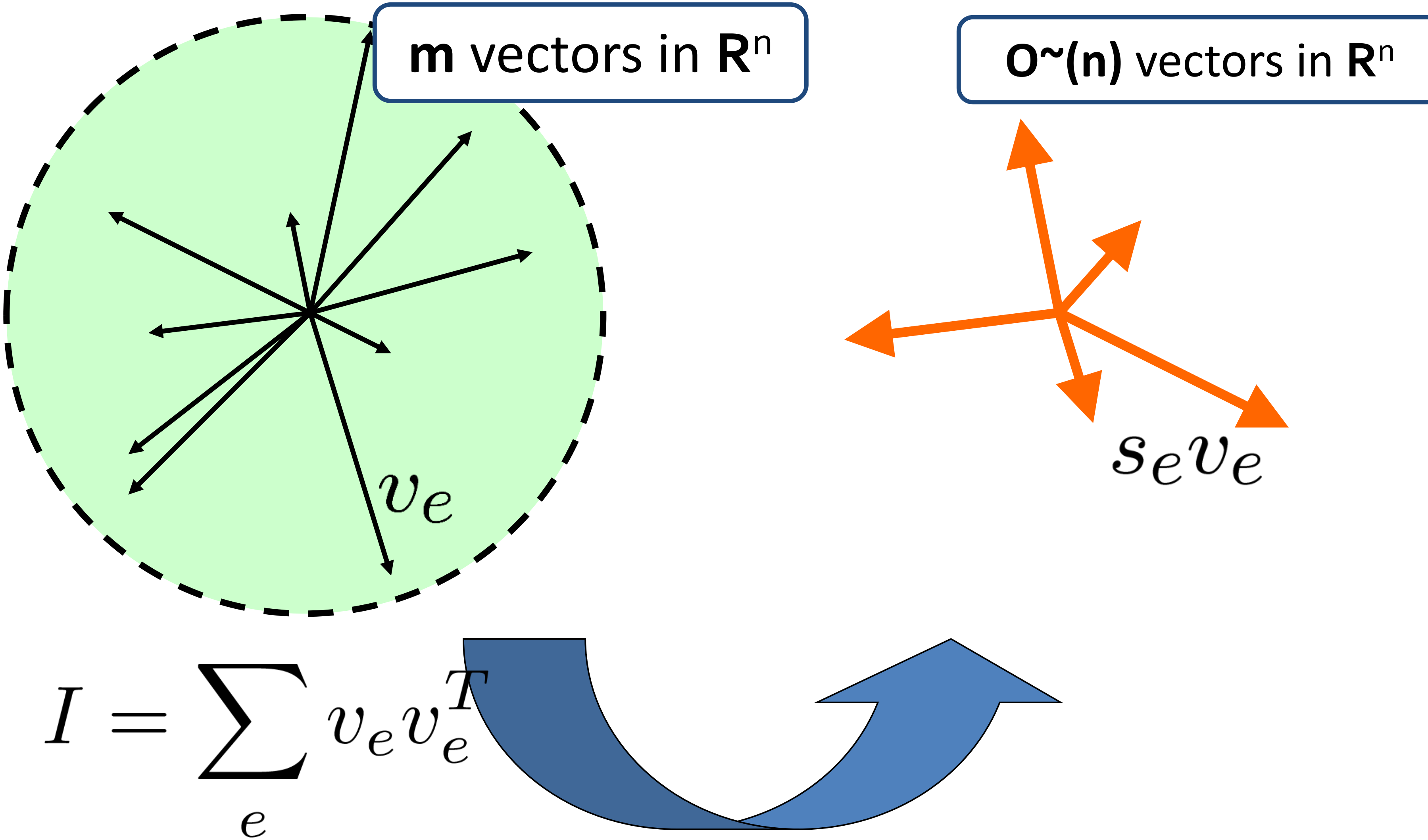


$$\forall u \quad \sum_e \langle u, v_e \rangle^2 = 1$$

variance is the same in every direction

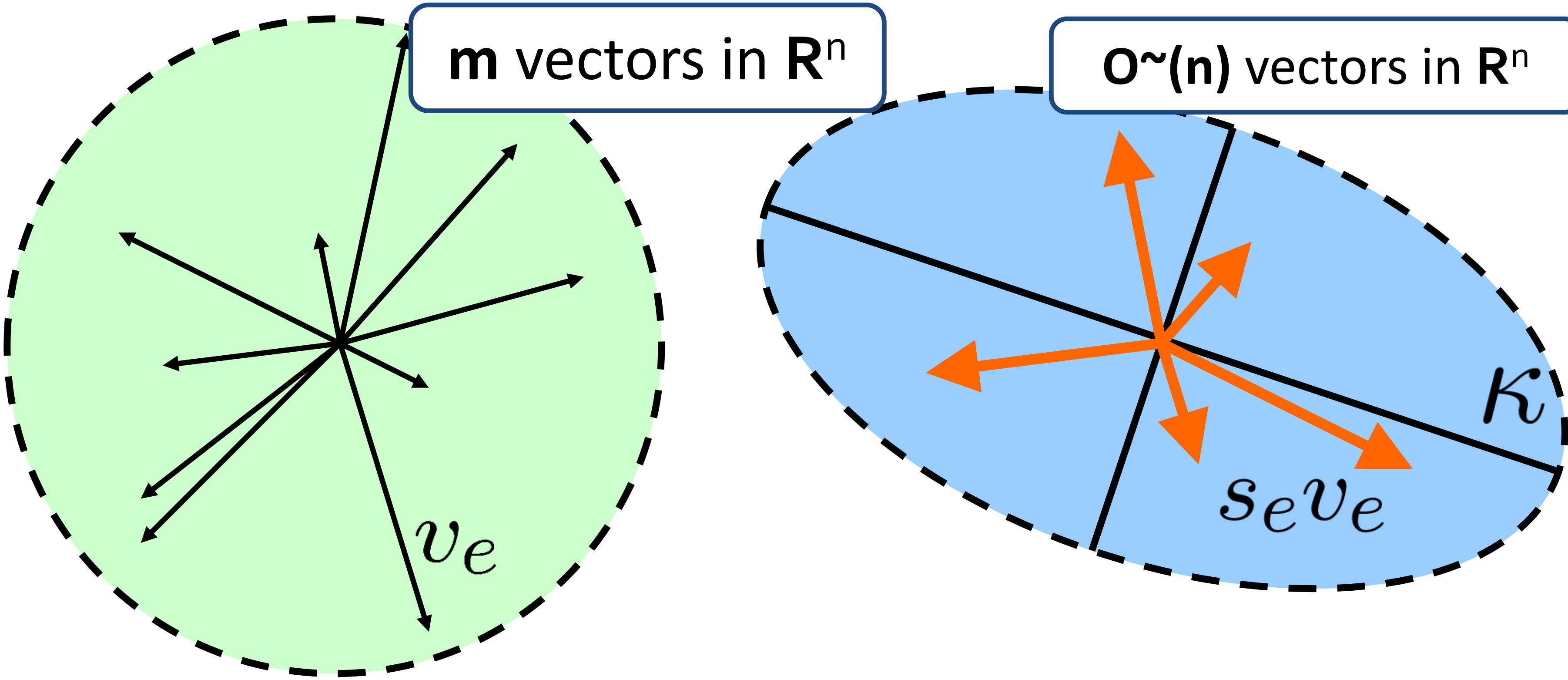
$$I = \sum_e v_e v_e^T$$

Core Problem



$$I = \sum_e v_e v_e^T$$

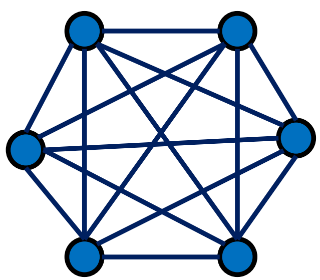
Core Problem



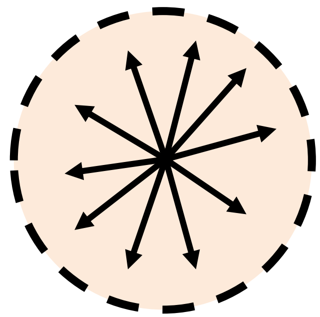
$$I = \sum_e v_e v_e^T \qquad\qquad I \preceq \sum_e s_e v_e v_e^T \preceq \kappa I$$

Examples of the Reduction

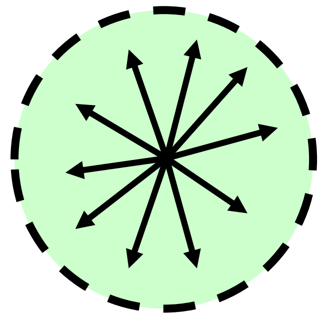
Graph



$$L_G = \sum_e b_e b_e^T$$



$$I = \sum_e v_e v_e^T$$

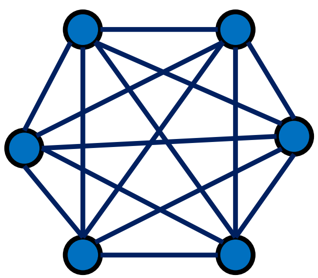


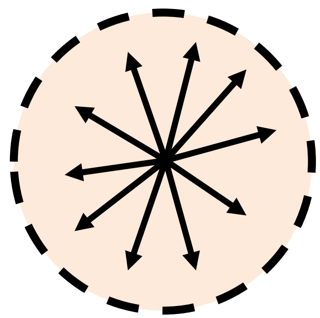
$$v_e = L_G^{-1/2} b_e$$

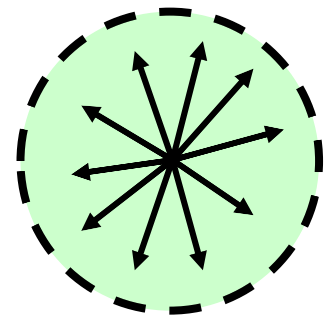
Examples of the Reduction

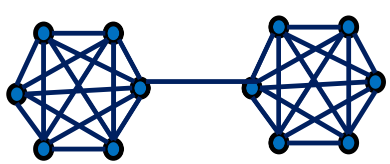
Graph

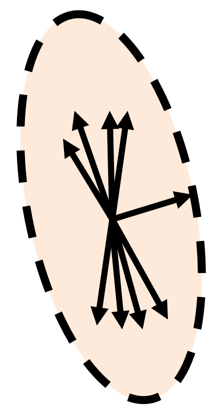
$$L_G = \sum_e b_e b_e^T \qquad I = \sum_e v_e v_e^T$$

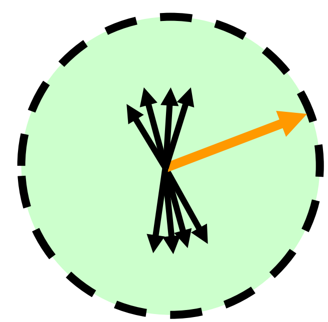












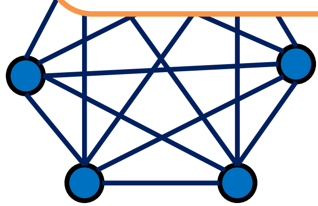
$$v_e = L_G^{-1/2} b_e$$

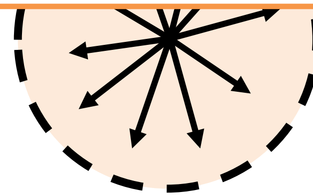
Examples of the Reduction

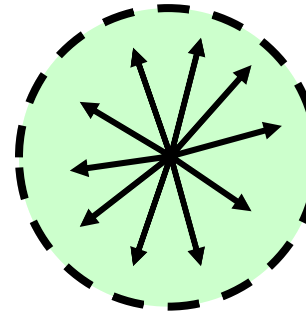
Q: Why rescale to identity?

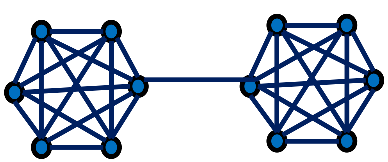
A: All test directions are equally important in multiplicative approximation.

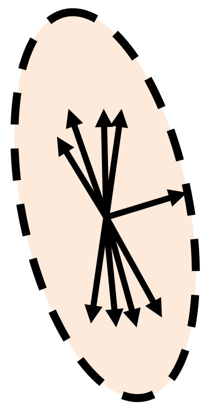
$$I = \sum_e v_e v_e^T$$

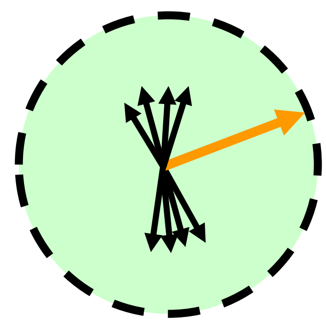












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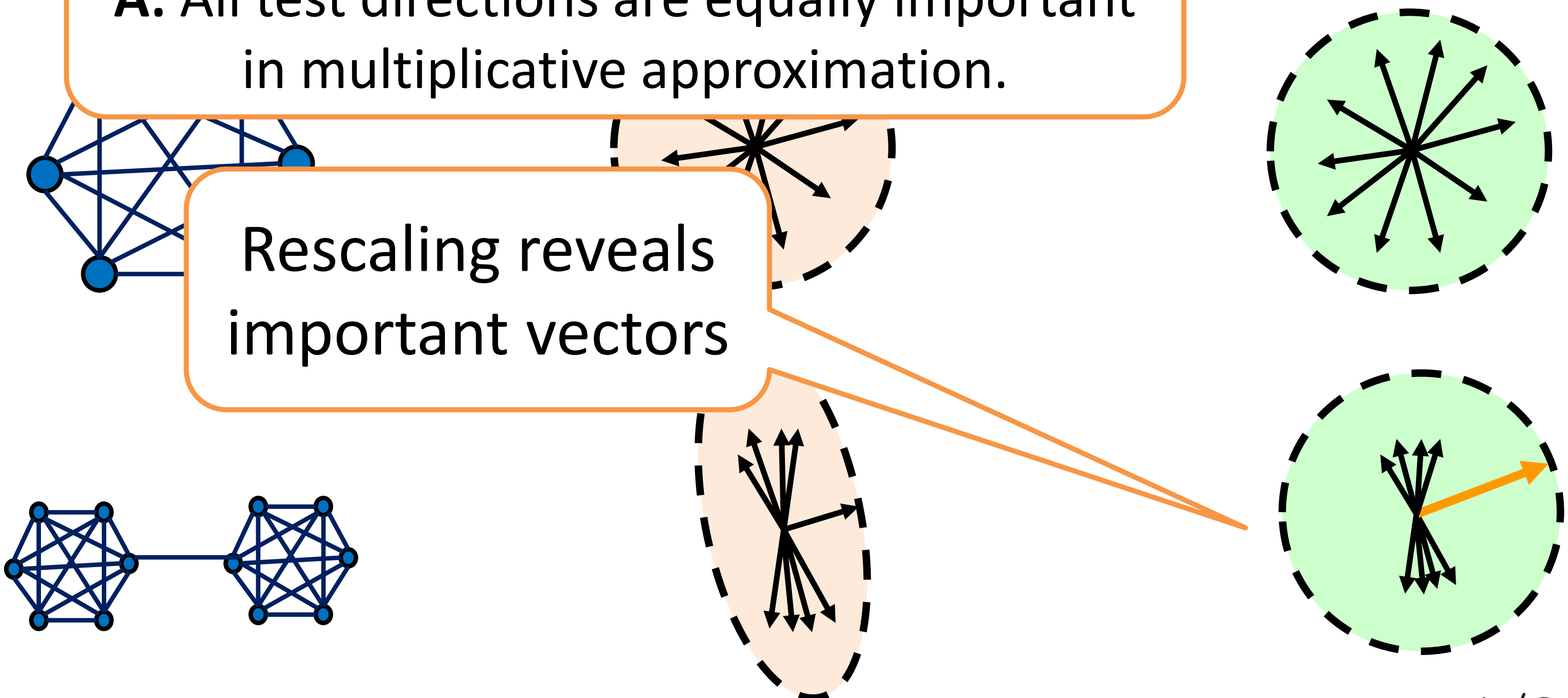

Rescaling reveals important vectors

$$v_e = L_G^{-1/2} b_e$$

Effective Resistance View

For a graph ***G***, the vectors are $v_e = L_G^{-1/2} b_e$

Lengths of vectors are:

$$\|v_e\|^2 = \|L_G^{-1/2} b_e\|^2 = b_e^T L_G^{-1} b_e$$

Effective Resistance View

For a graph ***G***, the vectors are $v_e = L_G^{-1/2} b_e$

Lengths of vectors are:

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Electrical Flow minimizes energy:

minimize	$\frac{1}{2} x^T L_G x - (x_i - x_j)$
Subject to	$x \perp 1$

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Electrical Flow minimizes energy:

minimize	$\frac{1}{2} x^T L_G x - (x_i - x_j)$
Subject to	$x \perp 1$

Optimality conditions:	$L_G x = (\delta_i - \delta_j) = b_{ij}$
Optimal energy:	$b_{ij}^T L_G^{-1} b_{ij}$

Effective Resistance View

For a graph ***G***, the vectors are $v_e = L_G^{-1/2} b_e$

Lengths of vectors are:

$$\|v_e\|^2 = \|L_G^{-1/2} b_e\|^2 = b_e^T L_G^{-1} b_e = \mathbf{Reff}_G(e)$$

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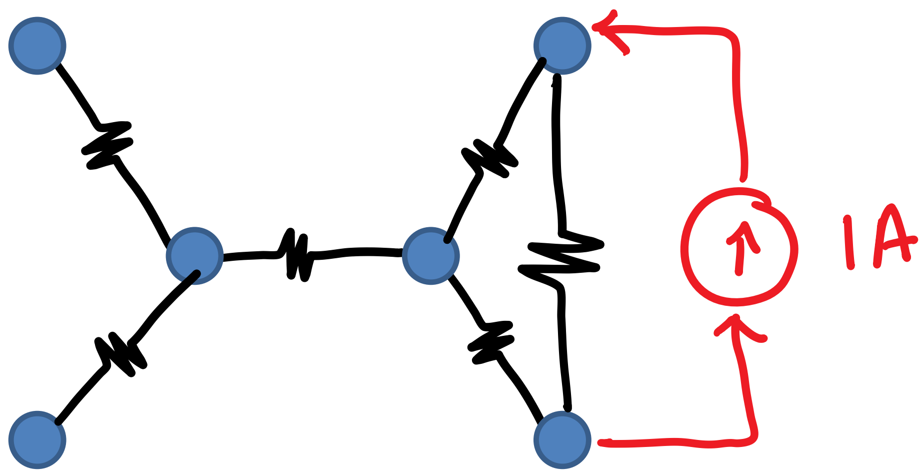
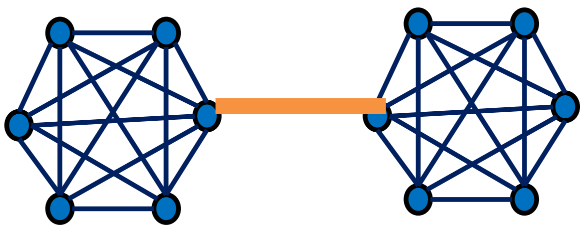
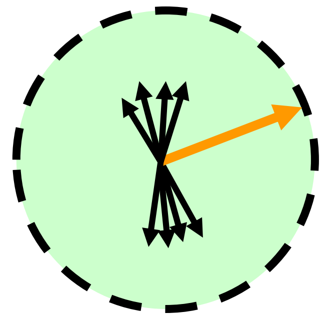


Confirmation of Electrical Intuition

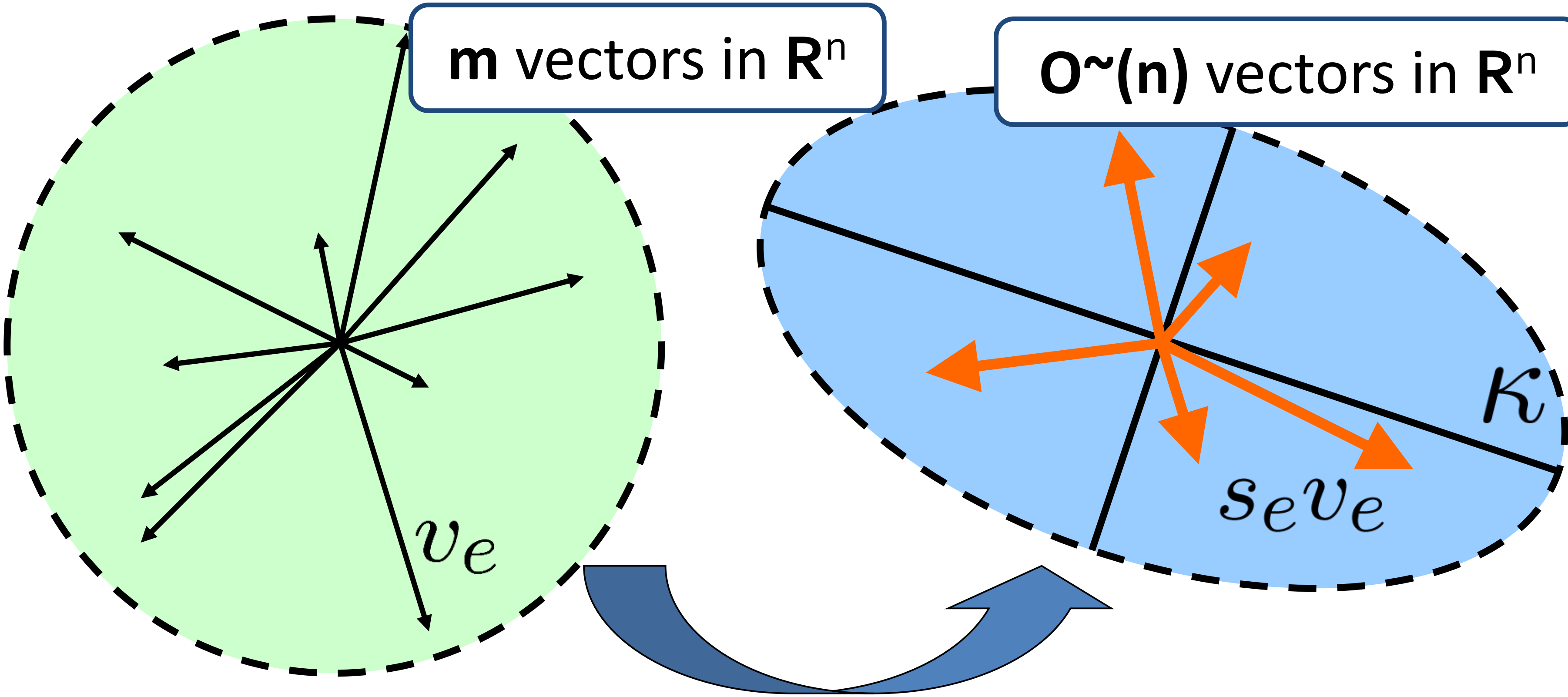
- Want ***G*** an ***H*** to be electrically equivalent
- Edges with higher **Reff** are more electrically significant = have higher norm after rescaling



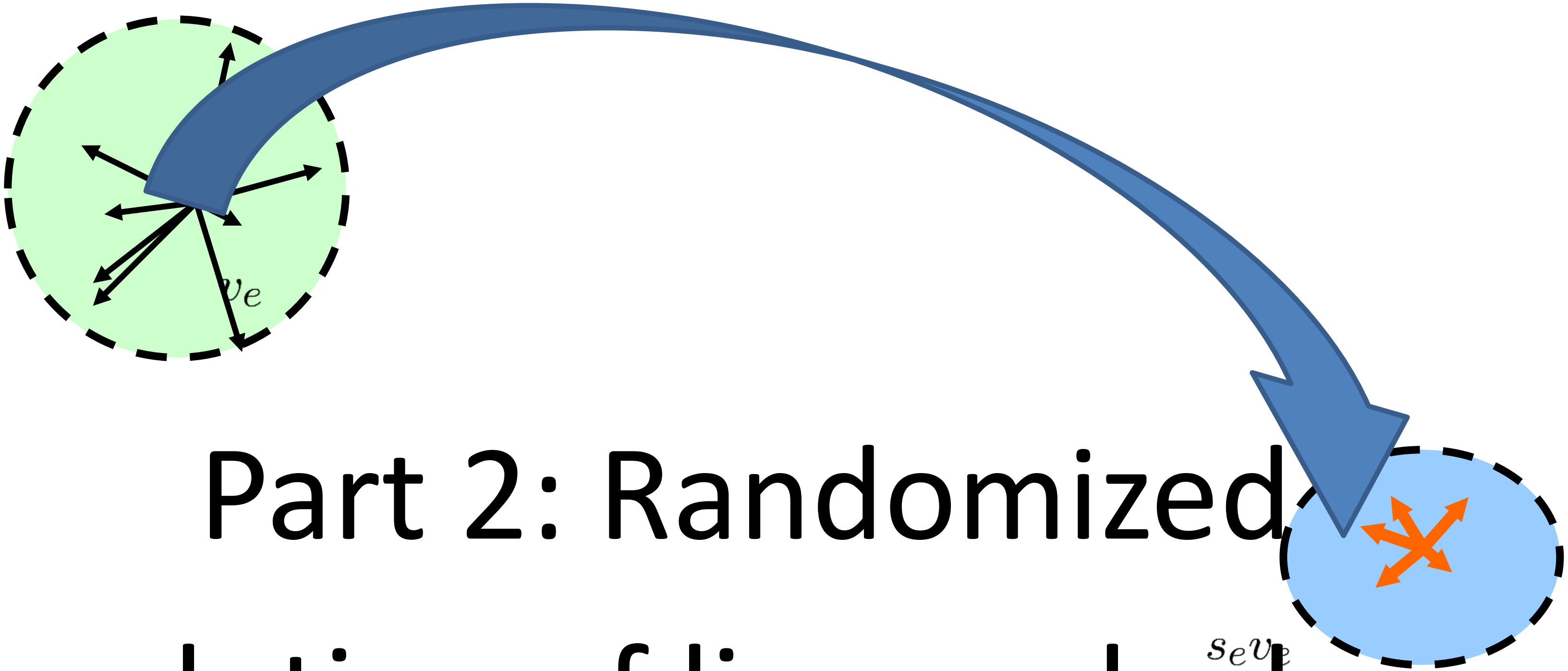


$$v_e = L_G^{-1/2} b_e$$

Core Problem

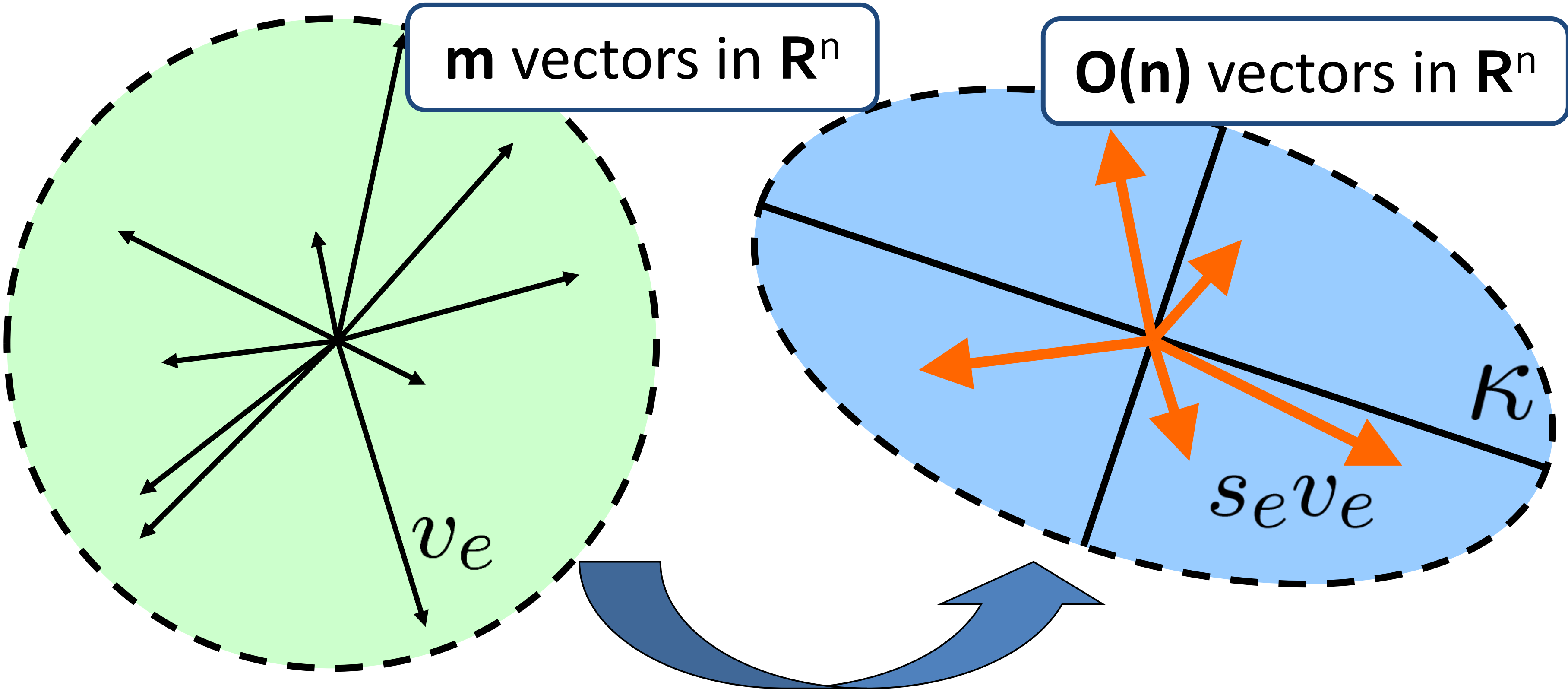


$$I = \sum_{i \leq m} v_i v_i^T \qquad I \preceq \sum_i s_i v_i v_i^T \preceq \kappa I$$



Part 2: Randomized solution of linear algebra problem

Core Problem



$$I = \sum_{i \leq m} v_i v_i^T \qquad I \preceq \sum_i s_i v_i v_i^T \preceq \kappa I$$

Approximating the Identity

Given $\sum_i v_i v_i^T = I$, consider the random matrix

$$X = \frac{v_i v_i^T}{p_i} \qquad \text{with probability } \; p_i$$

Then $\mathbb{E}X = \sum_i v_i v_i^T = I$.

Take k i.i.d. samples $X_1, \dots, X_k$. Would like

$$(1-\epsilon)I \preccurlyeq \frac{1}{k}\sum_i X_i \preccurlyeq (1+\epsilon)I$$

The Chernoff Bound

Suppose $X_1, \dots, X_k$ are i.i.d. random variables with

$0 \le X_i \le M$ and $\mathbb{E}X_i = 1.$

Then

$$\mathbb{P}\left[\left|\frac{1}{k}\sum_i X_i - 1\right| \ge \epsilon\right] \le 2\exp\left(-\frac{k\epsilon^2}{4M}\right)$$

The Chernoff Bound

$k = 4M/\epsilon^2$ samples give

$$\frac{1}{k}\sum_i X_i \approx_\epsilon 1$$

with constant probability.

dom variables with

$\mathbb{E}X_i = 1.$

Then

$$\mathbb{P}\left[\left|\frac{1}{k}\sum_i X_i - 1\right| \geq \epsilon\right] \leq 2\exp\left(-\frac{k\epsilon^2}{4M}\right)$$

The Chernoff Bound

Suppose $X_1, \dots, X_k$ are i.i.d. random variables with

$$0 \le X_i \le M \qquad \text{and} \qquad \mathbb{E}X_i = 1.$$

Then

$$\mathbb{P}\left[\left|\frac{1}{k}\sum_i X_i - 1\right| \ge \epsilon\right] \le 2\exp\left(-\frac{k\epsilon^2}{4M}\right)$$

The **Matrix** Chernoff Bound

[Rudelson'99, AW'02, Tropp'11]

Suppose $X_1, \dots, X_k$ are i.i.d. random $d \times d$ **matrices** with

$0 \preccurlyeq X_i \preccurlyeq M \cdot I$ and $\mathbb{E} X_i = I$.

Then

$$\mathbb{P}\left[\left\|\frac{1}{k}\sum_i X_i - I\right\| \geq \epsilon\right] \leq 2\boldsymbol{d}\exp\left(-\frac{k\epsilon^2}{4M}\right)$$

The **Matrix** Chernoff Bound

[Rudelson'99, AW'02, Tropp'11]

$k = 4M \log d / \epsilon^2$ samples give

$$\frac{1}{k}\sum_i X_i \approx_\epsilon I$$

with constant probability.

dom $d \times d$

$\mathbb{E}X_i = I$.

Then

$$\mathbb{P}\left[\left\|\frac{1}{k}\sum_i X_i - I\right\| \geq \epsilon\right] \leq 2\boldsymbol{d}\exp\left(-\frac{k\epsilon^2}{4M}\right)$$

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Then

$$\mathbb{P}\left[\left\|\frac{1}{k}\sum_i X_i - I\right\| \geq \epsilon\right] \leq 2\boldsymbol{d}\exp\left(-\frac{k\epsilon^2}{4M}\right)$$

In our case

$$X = \frac{v_i v_i^T}{p_i} \quad \text{with prob. } p_i, \qquad \mathbb{E}X = I.$$

Want to minimize $M = \max_i \left\| \frac{v_i v_i^T}{p_i} \right\| = \max_i \frac{||v_i||^2}{p_i}$

To make this this tight for all v_i set $p_i = \frac{||v_i||^2}{M}$.

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But $\sum_i p_i = \sum_i \frac{||v_i||^2}{M}$

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But $\sum_i p_i = \sum_i \frac{||v_i||^2}{M} = \sum_i \frac{Tr\left(v_i v_i^T\right)}{M}$

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But $\sum_i p_i = \sum_i \frac{||v_i||^2}{M} = \sum_i \frac{Tr(v_i v_i^T)}{M} = \frac{Tr(\sum_i v_i v_i^T)}{M}$

In our case

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In our case

$$X = \frac{v_i v_i^T}{p_i} \quad \text{with prob. } p_i, \qquad \mathbb{E}X = I.$$

Must have $M = n$

Thm: $4n \log n / \epsilon^2$ samples suffice.

$$\Big\| \Big\| = \max_i \frac{||v_i||^2}{p_i}$$

To make this this tigh... set $p_i = \frac{||v_i||^2}{M}$.

But $\sum_i p_i = \sum_i \frac{||v_i||^2}{M} = \sum_i \frac{Tr\left(v_i v_i^T\right)}{M} = \frac{Tr\left(\sum_i v_i v_i^T\right)}{M} = \frac{n}{M}$

How to Approximate the Identity

Given $\sum_i v_i v_i^T = I$

Sample $n \log n / \epsilon^2$ vectors randomly with replacement, by $p_i \propto \|v_i\|^2$.

Set $s_i = 1/p_i$ for chosen vectors.

Rudelson'99: This works whp:

$$1 - \epsilon \preceq \sum_i s_i v_i v_i^T \preceq 1 + \epsilon$$

How to Approximate the Identity

For a graph, $p_e \propto \boldsymbol{Reff_G(e)}$

Given $\sum_i v_i v_i^T = I$

Sample $n \log n / \epsilon^2$ vectors randomly with replacement, by $p_i \propto \|v_i\|^2$.

Set $s_i = 1/p_i$ for chosen vectors.

Rudelson'99: This works whp:

$$1 - \epsilon \preceq \sum_i s_i v_i v_i^T \preceq 1 + \epsilon$$

How to Approximate any Matrix

Given $\sum_i v_i v_i^T = V$

Sample $n \log n / \epsilon^2$ vectors randomly with replacement, by $p_i \propto \|V^{-1/2} v_i\|^2.$

Set $s_i = 1/p_i$ for chosen vectors.

Rudelson'99: This works whp:

$$1 - \epsilon \preceq \sum_i s_i v_i v_i^T \preceq 1 + \epsilon$$

Theorem. Every weighted graph **G** has a weighted subgraph **H** with at most $4n \log n / \epsilon^2$ edges s.t.

$$L_G \preccurlyeq L_H \preccurlyeq (1+\epsilon) L_G.$$

Algorithm: sample $4n \log n / \epsilon^2$ edges independently according to effective resistances.

Theorem. Every weighted graph **G** has a weighted subgraph **H** with at most $9n \log n / \epsilon^2$ edges s.t.

$$L_G \preccurlyeq L_H \preccurlyeq (1+\epsilon) L_G.$$

Moreover, H can be found in time $O^{\sim}(m/\epsilon^2)$.

Algorithm: sample $9n \log n / \epsilon^2$ edges independently according to **approximate** effective resistances.

[Spielman-S'08]

Part 3: Fast Calculation of Sampling Probabilities

Resistances are Distances

Outer product expansion:

$$L_G = \sum_e b_e b_e^T = B^T B \qquad \text{for rows}(B) = \{b_e^T\}$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 \\ & & \vdots & & \\ 1 & 0 & -1 & 0 & 0 \end{pmatrix}$$

Signed edge-vertex incidence matrix

Resistances are Distances

Outer product expansion:

$$L_G = \sum_e b_e b_e^T = B^T B \qquad \text{for rows}(B) = \{b_e^T\}$$

Sampling probabilities:

$$\begin{aligned} \|v_e\|^2 &= b_e^T L_G^{-1} b_e \\ &= b_e^T L_G^{-1} B^T B L_G^{-1} b_e \\ &= \|B L_G^{-1}(\delta_i - \delta_j)\|^2 \qquad \text{for } e = ij. \end{aligned}$$

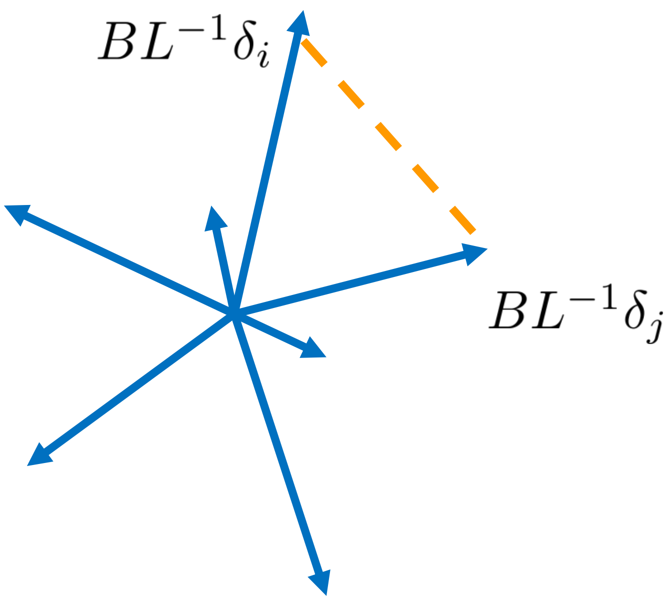
Nearly Linear Time

$$\mathbf{Reff}(ij) = \|BL^{-1}(\delta_i - \delta_j)\|^2$$

Nearly Linear Time

$$\mathbf{Reff}(ij) = \|BL^{-1}(\delta_i - \delta_j)\|^2$$

So care about distances between cols. of ***BL***$^{-1}$



Dimension Reduction

Johnson-Lindenstrauss Lemma [JL'84]:

Suppose $x_1, \ldots, x_n$ are points in $\mathbb{R}^d$.

Let $Q_{k\times n}$ be a random k −dimensional projection.

Then

$$||Qx_i - Qx_j||_2 = (1 \pm \epsilon)||x_i - x_j||_2$$

With high probability as long as

$$k \geq 10 \log n / \epsilon^2$$

Dimension Reduction

Johnson-Lindenstrauss Lemma [JL'84]:

Suppose $x_1, \dots, x_n$ are points in $\mathbb{R}^d$.

Let $Q_{k \times n}$ be a random Bernoulli matrix.

Then

$$||Qx_i - Qx_j||_2 \propto (1 \pm \epsilon)||x_i - x_j||_2$$

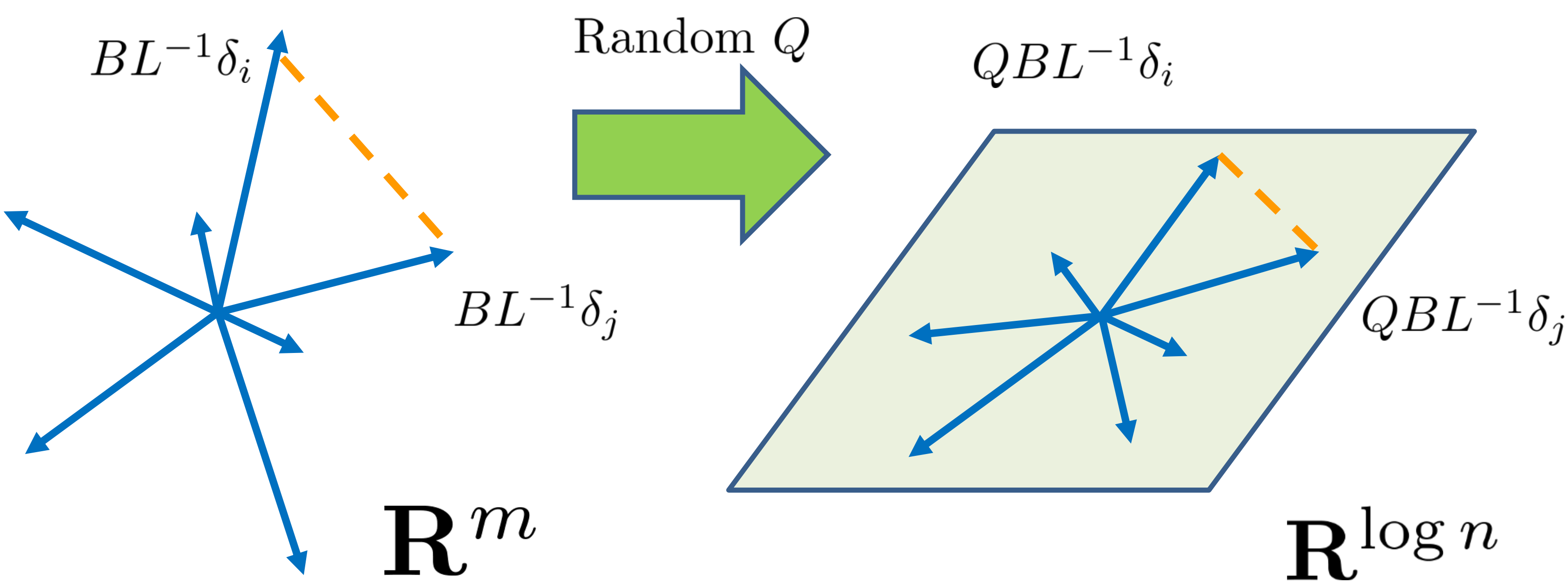
With high probability as long as

$$k \geq 10 \log n / \epsilon^2$$

Johnson-Lindenstrauss with $\epsilon = 1/2$

$$\mathbf{Reff}(ij) = \|BL^{-1}(\delta_i - \delta_j)\|^2$$

So care about distances between cols. of ***BL*$^{-1}$**



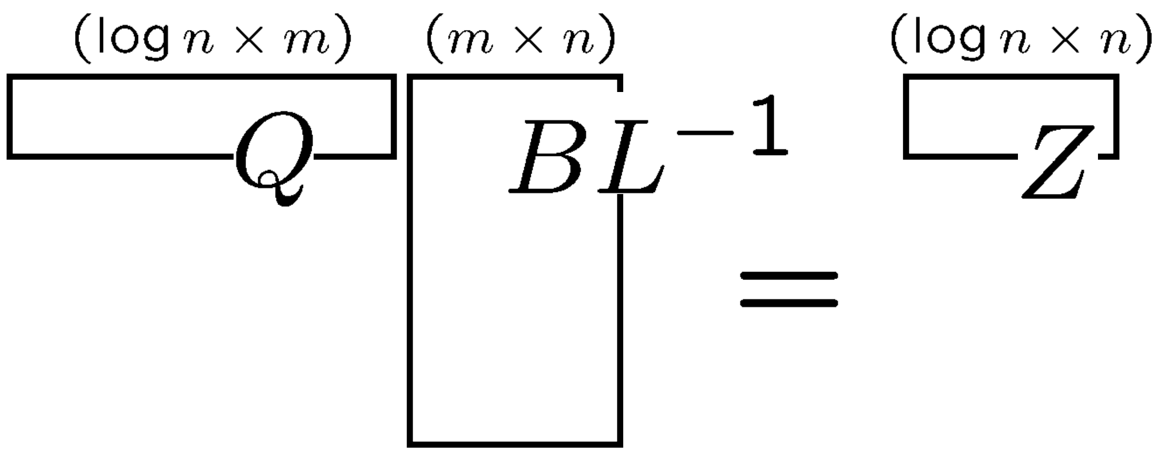
Nearly Linear Time

$$\mathbf{Reff}(ij) = \|BL^{-1}(\delta_i - \delta_j)\|^2$$

So care about distances between cols. of $\boldsymbol{BL^{-1}}$

Johnson-Lindenstrauss: Take random $\boldsymbol{Q_{\log n \times m}}$

Set $\boldsymbol{Z=QBL^{-1}}$

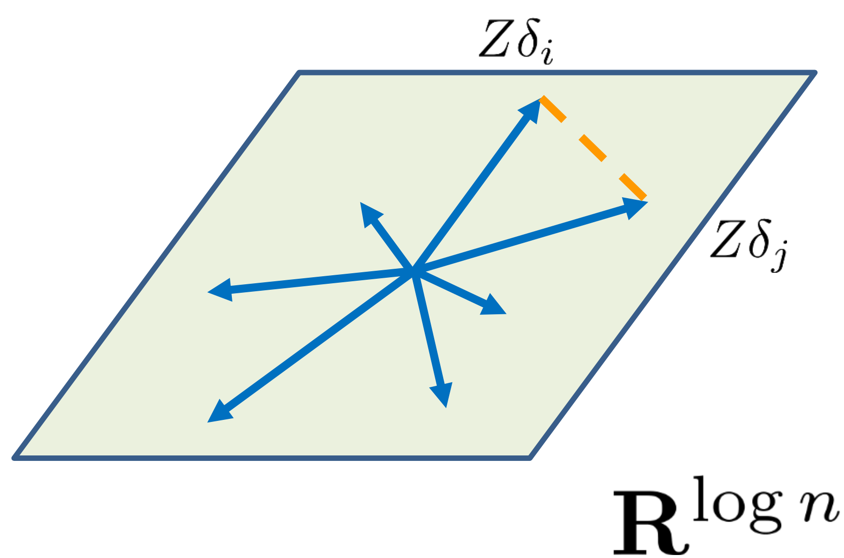


Nearly Linear Time

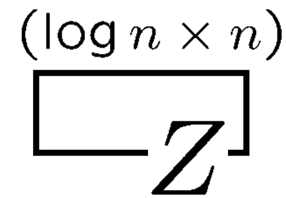
$(\log n \times n)$

Z

$$\mathbf{Reff}(ij) \sim \|Z(\delta_i - \delta_j)\|^2$$



Nearly Linear Time



Find **rows** of $\boldsymbol{Z_{\log n \times n}}$ by

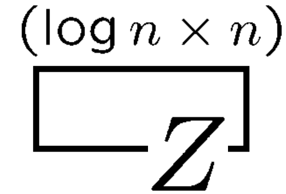
$$Z = QBL^{-1}$$

$$ZL = QB$$

$$z_i L = (QB)_i$$

$$\mathbf{Reff}(ij) \sim \|Z(\delta_i - \delta_j)\|^2$$

Nearly Linear Time



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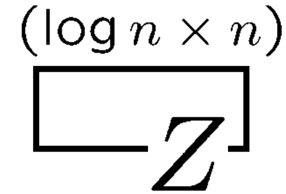
Solve $O(\log n)$ linear systems in L using fast Laplacian solver solver

learns all pairwise resistances by probing a few random electrical flows.

Nearly Linear Time



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$\boldsymbol{Z=QBL^{-1}}$

$\boldsymbol{ZL=QB}$

$\boldsymbol{z_iL=(QB)_i}$

$$\mathbf{Reff}(ij) \sim \|Z(\delta_i - \delta_j)\|^2$$

Solve ***O(logn)*** linear systems in ***L*** using

fast Laplacian solver solver

Can show approximate $\boldsymbol{R_{eff}}$ suffice.

(only change M by a constant factor)

□

Actual Algorithm

Input: undirected graph $G = (V, E, w)$

Output: subgraph **H** with $L_G \preccurlyeq L_H \preccurlyeq (1+\epsilon)L_G$

1. Let $Q_{\log n \times m}$ be a scaled random projection.

   Compute approximate resistance matrix

$$Z = QBL^{+}$$

   by solving $\log n$ Laplacian systems

2. Repeat the following $9nlogn/\epsilon^2$ times:

   choose edge $e = ij$ w.p. $p_e \propto ||Z(\delta_i - \delta_j)||^2$

   add e to H with weight $s_e = 1/p_e$

Actual Algorithm

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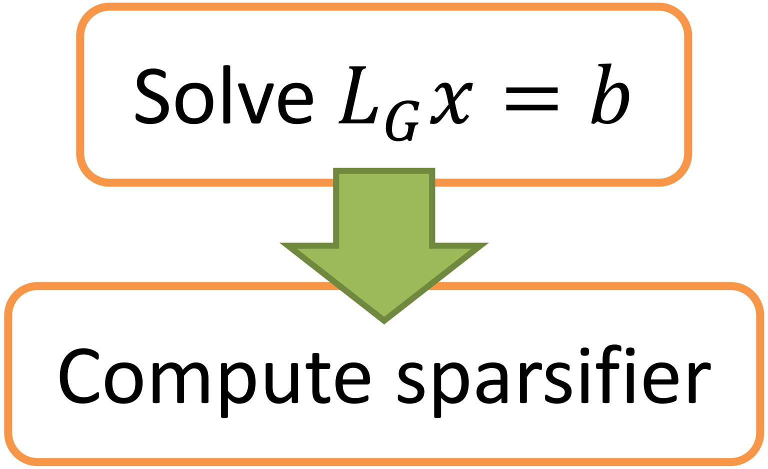
+improvements by [Koutis-Levin-Peng'12]

Chicken / Egg?

Solve $L_G x = b$

Chicken / Egg?



Chicken / Egg?

Solve $L_G x = b$

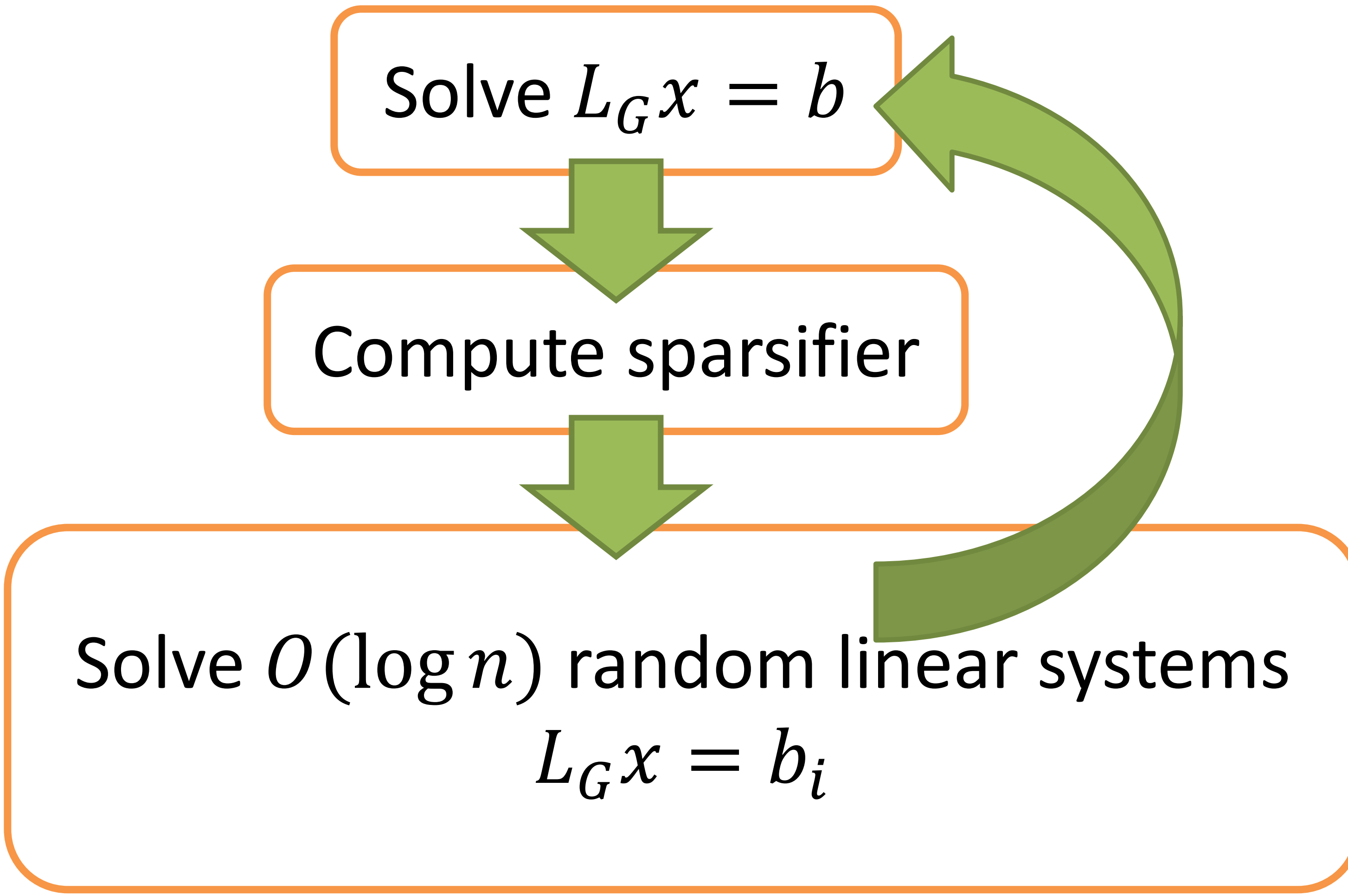
↓

Compute sparsifier

↓

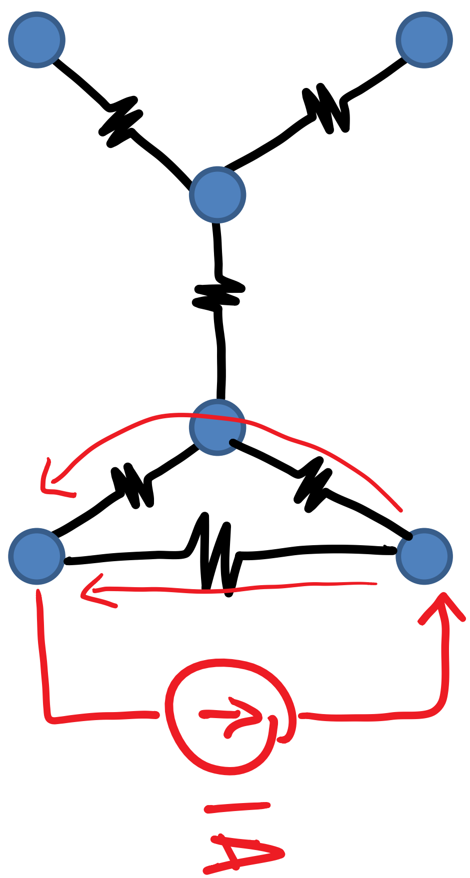
Solve $O(\log n)$ random linear systems
$L_G x = b_i$

[Koutis-Miller-Peng'10] resolve this

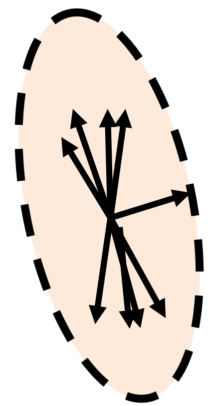


Two Useful Ways to view a Graph

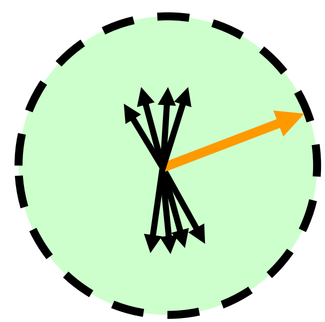
electrical network



bunch of vectors



$$L_G = \sum_e b_e b_e^T$$

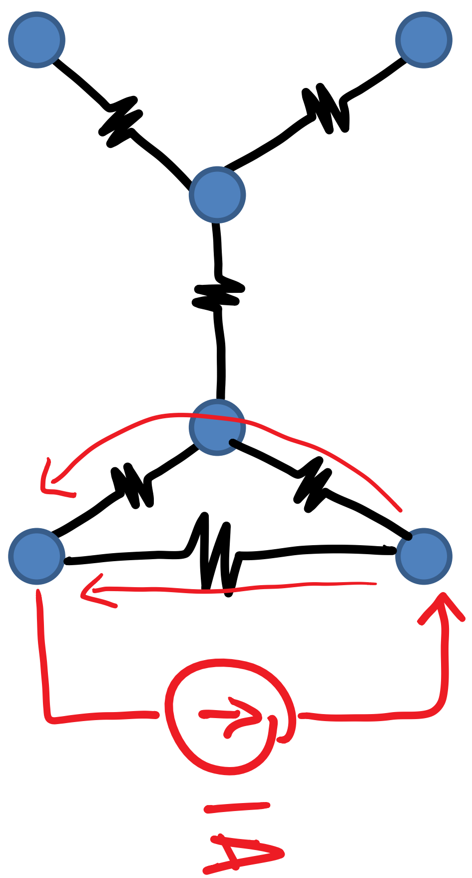


$$I = \sum_e v_e v_e^T$$

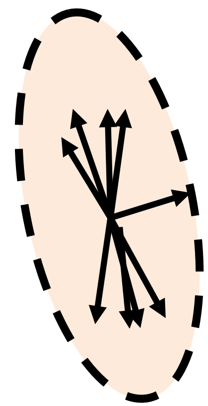
Tw ph

$$Reff(e) = ||L_G^{-1/2} b_e||^2 = ||v_e||^2$$

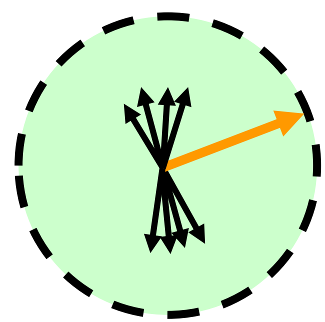
electrical network



bunch of vectors



$$L_G = \sum_e b_e b_e^T$$



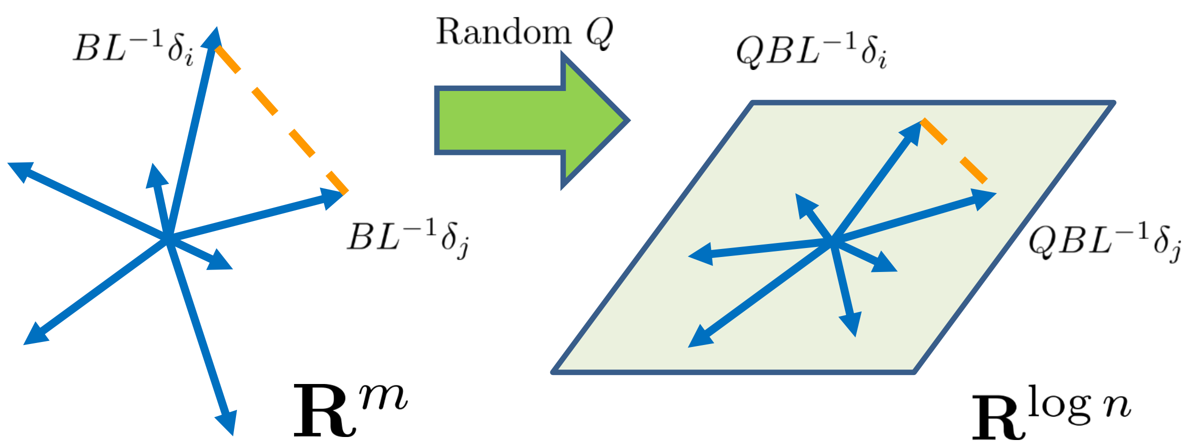
$$I = \sum_e v_e v_e^T$$

Two Useful Tools

Matrix Chernoff Bound

$$\mathbb{P}\left[\left\|\frac{1}{k}\sum_i X_i - I\right\| \geq \epsilon\right] \leq 2d\exp\left(-\frac{k\epsilon^2}{4M}\right)$$

Johnson-Lindenstrauss Lemma



Advantages over pure combinatorics

There is a global **rescaling** transformation:

$$L_G \approx L_H \quad \textbf{iff} \quad L_G^{-1/2} L_H L_G^{-1/2} \approx I$$

Powerful **random matrix** tools apply naturally:

1. Matrix Chernoff bound
2. Johnson-Lindenstrauss Lemma

Some Improvements

[Koutis-Levin-Peng'12] $O\left(\frac{m \log^2 n}{\epsilon^2}\right)$

[Kelner-Levin'11]   1-pass streaming algorithm

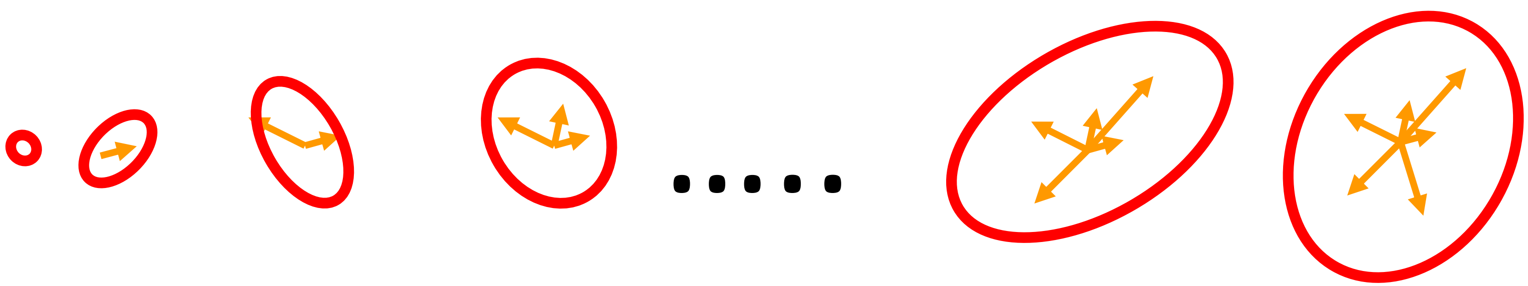
[Koutis'14] parallel algorithm

[Kapralov, Lee, Musco x2, Sidford'14]
1-pass dynamic streaming algorithm

Coming Up: A Slow Algorithm

Part II: Sparsifiers with $O\left({}^{n}/_{\epsilon^2}\right)$ edges.

Based on more delicate understanding of how eigenvalues of a matrix evolve on adding edges.



Two Open Questions

Faster approximation of effective resistances.

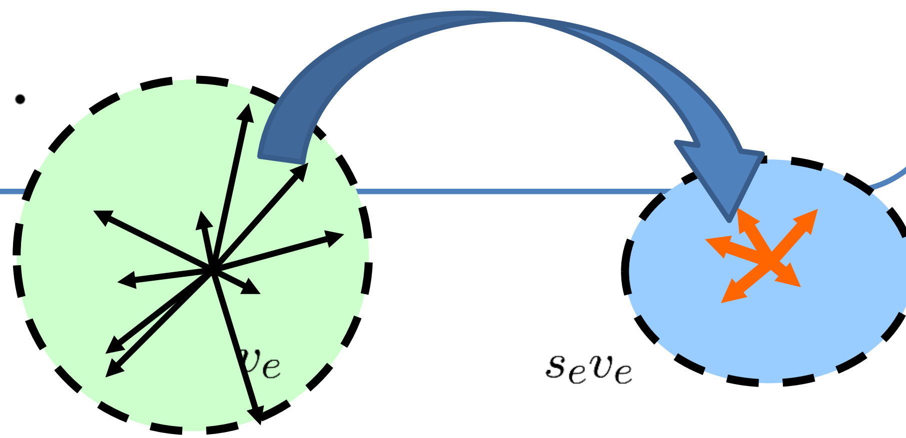
More physical processes on graphs.

Deterministic Solution [Batson-Spielman-S'09]

Spectral Sparsification Theorem:

Given $\sum_{i \leq m} v_i v_i^T = I_n$ there are $s_i \geq 0$ with:

- $(1-\epsilon)I \preceq \sum_i s_i v_i v_i^T \preceq (1+\epsilon)I$
- $\text{supp}(s) \leq 4n/\epsilon^2$.

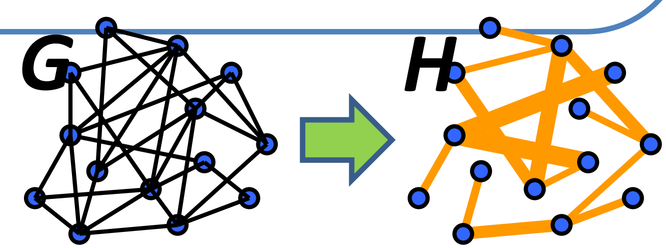


Deterministic Solution [BSS'09]

Spectral Sparsification Theorem:

Given $\sum_{i\leq m} b_e b_e^T = L_G$ there are $s_e \geq 0$ with:

- $(1-\epsilon)L_G \preceq \sum_i s_i v_i v_i^T \preceq (1+\epsilon)L_G$
- $\mathrm{supp}(s) \leq 4(n-1)/\epsilon^2$.



Deterministic Solution [BSS'09]

Spectral Sparsification Theorem:

Given $\sum_{i\leq m} b_e b_e^T = L_G$ there are $s_e \geq 0$ with:

- $(1-\epsilon)L_G \preceq \sum_i s_i v_i v_i^T \preceq (1+\epsilon)L_G$
- $\text{supp}(s) \leq 4(n-1)/\epsilon^2$.

Open: Fast Algorithm?