On the hardness of the NTRU problem

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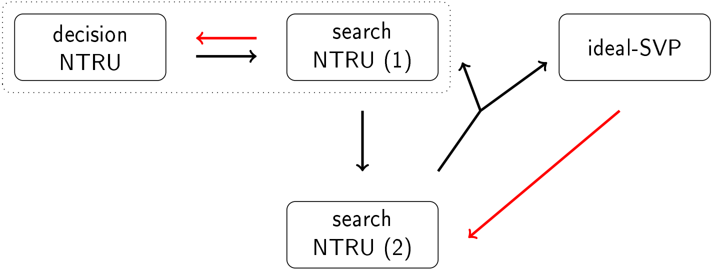
Lattices: Algorithms, Complexity, and Cryptography
reunion workshop

What is this talk about



Outline of the talk

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1. The different NTRU problems
2. What we know about NTRU
3. Techniques

NTRU instances

$$R = \mathbb{Z}[X]/(X^n+1), \quad K = \mathbb{Q}[X]/(X^n+1), \quad n = 2^k, \quad R_q = R/(qR)$$

NTRU instance

A (γ, q)-NTRU instance is $h \in R_q$ s.t.

- $h = f/g \bmod q$ (or $gh = f \bmod q$)
- $\|f\|, \|g\| \leq \frac{\sqrt{q}}{\gamma}$ (if $y = \sum_{i=0}^{n-1} y_i X^i \in R$, then $\|y\| := \sqrt{\sum_i y_i^2}$)

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The pair (f, g) is a trapdoor for h.

Claim: if (f, g) and (f', g') are two trapdoors for the same h,

$$\frac{f'}{g'} = \frac{f}{g} =: h_K \in K \quad \text{(division performed in K)}$$

Decisional NTRU problem

dNTRU

The (γ, q)-decisional NTRU problem ((γ, q)-dNTRU) asks, given $h \in R_q$, to decide whether

- $h \leftarrow \mathcal{D}$ where $\mathcal{D}$ is a distribution over (γ, q)-NTRU instances
- $h \leftarrow \mathcal{U}(R_q)$

Search NTRU problems

$\mathrm{NTRU}_{\mathrm{vec}}$

The (γ, γ', q)-search NTRU vector problem ((γ, γ', q)-$\mathrm{NTRU}_{\mathrm{vec}}$) asks, given a (γ, q)-NTRU instance h, to recover $(f, g) \in R^2$ s.t.

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$\mathrm{NTRU}_{\mathrm{mod}}$

The (γ, q)-search NTRU module problem ($(\gamma, q)\text{-}\mathrm{NTRU}_{\mathrm{mod}}$) asks, given a (γ, q)-NTRU instance h, to recover h_K.
(Recall $h_K = f/g \in K$ for any trapdoor (f, g))

(The two problems exist in worst-case and average-case variants)

NTRU is a (module) lattice problem

NTRU lattice

The NTRU (module) lattice associated to an NTRU instance h is

$$\Lambda(h) = \{(g, f)^T \in R^2 \mid gh = f \bmod q\}.$$

Fact: $\Lambda(h)$ has basis $B_h = \begin{pmatrix} 1 & 0 \\ h & q \end{pmatrix}$ (in columns)

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  - $\mathrm{NTRU}_{\mathrm{vec}}$ asks to recover (a short multiple of) the short vector
- $\Lambda(h)$ has an unexpectedly dense sub-lattice (sub-module) of rank n
  - $\mathrm{NTRU}_{\mathrm{mod}}$ asks to recover the dense sub-lattice (sub-module)

Outline of the talk

Previous works

Reductions:

[SS11, WW18] If $f, g \leftarrow D_{R,\sigma}$ with $\sigma \geq \mathrm{poly}(n) \cdot \sqrt{q}$
then $f/g \bmod q \approx \mathcal{U}(R_q)$ (cyclotomic fields)

- dNTRU is provably hard when $\gamma \leq \frac{1}{\mathrm{poly}(n)}$

[SS11] Stehlé and Steinfeld. Making NTRU as secure as worst-case problems over ideal lattices. Eurocrypt.
[WW18] Wang and Wang. Provably secure NTRUEncrypt over any cyclotomic field. SAC.

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[Pei16] Peikert. A decade of lattice cryptography. Foundations and Trends in TCS.

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Attacks: (polynomial time)

[LLL82] dNTRU, $\mathrm{NTRU}_{\mathrm{mod}}$ broken if $\gamma \geq 2^n$
$\mathrm{NTRU}_{\mathrm{vec}}$ broken if $\gamma \geq 2^n \cdot \gamma'$

[LLL82] Lenstra, Lenstra, Lovász. Factoring polynomials with rational coefficients. Mathematische Annalen.

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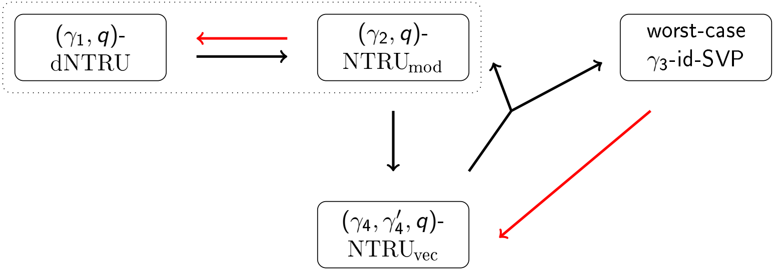
[ABD16, CLJ16] [KF17] $\mathrm{dNTRU}, \mathrm{NTRU}_{\mathrm{mod}}$ broken if $(\log q)^2 \geq n \cdot \log \frac{\sqrt{q}}{\gamma}$
(e.g., $q \approx 2^{\sqrt{n}}$ and $\gamma = \sqrt{q}/\mathrm{poly}(n)$)

[ABD16] Albrecht, Bai, and Ducas. A subfield lattice attack on overstretched NTRU assumptions. Crypto.

[CJL16] Cheon, Jeong, and Lee. An algorithm for NTRU problems. LMS J Comput Math.

[KF17] Kirchner and Fouque. Revisiting lattice attacks on overstretched NTRU parameters. Eurocrypt

Our results



Worst-case γ-id-SVP: given any ideal lattice $I \subset R$ (for instance $I = \{gr \mid r \in R\}$), find $v \in I \setminus \{0\}$ such that $\|v\| \leq \gamma \cdot \min_{w \in I \setminus \{0\}} \|w\|$.

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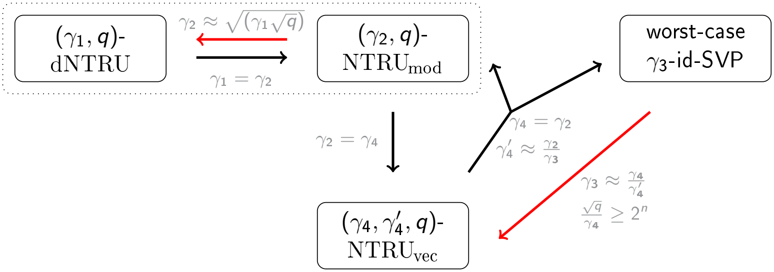

Remarks

- $a \approx b \Leftrightarrow a = \mathrm{poly}(n) \cdot b$ (cyclotomic/NTRUPrime fields)
- the reductions only work for certain distributions of NTRU instances
- the constraint $\frac{\sqrt{q}}{\gamma_4} \geq 2^n$ can be relaxed if the run time is increased

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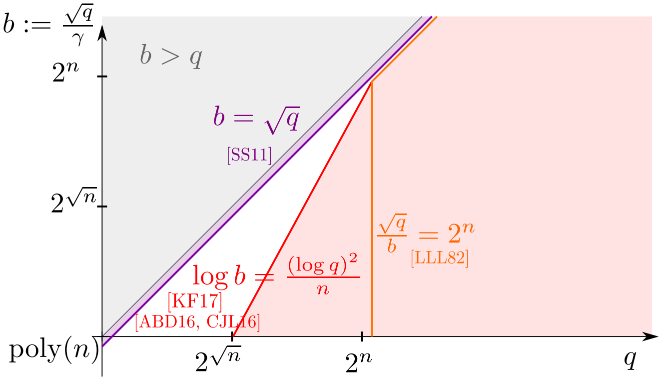
One big picture: poly time attacks and reductions (cyclotomics)



dNTRU
unconditionnally hard

dNTRU and NTRU_{mod}
easy

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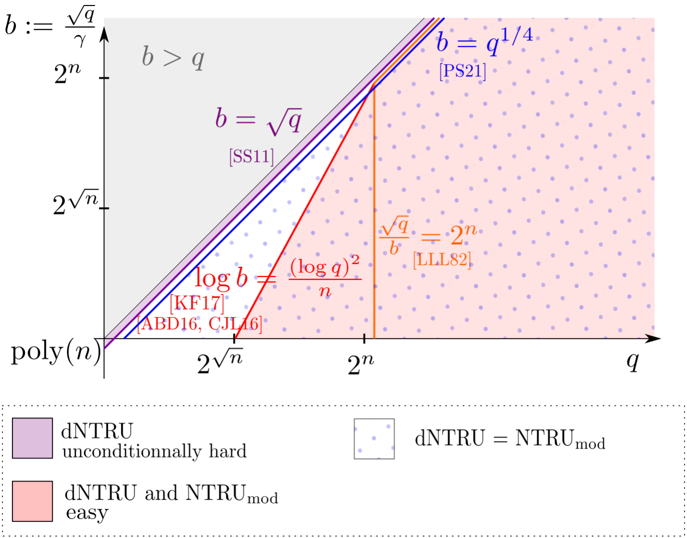

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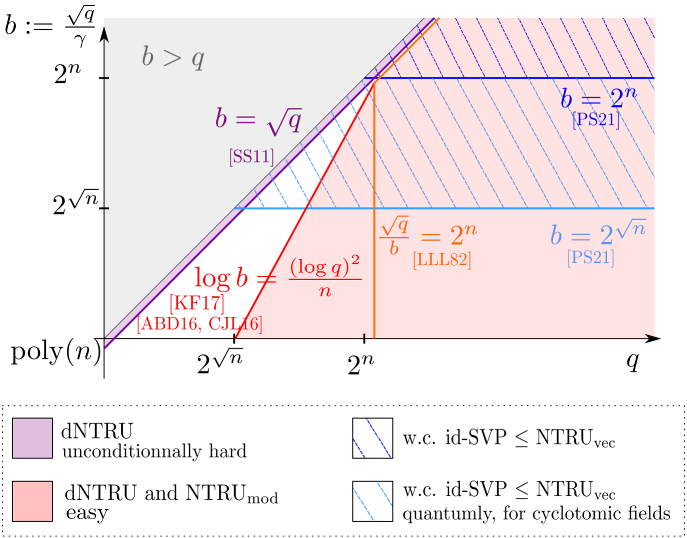

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Objective: Transform an ideal I into an NTRU instance h

- $I = \langle z \rangle = \{z \cdot r \,|\, r \in R\}$
- g short vector of I

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- ▶ $\|f\|, \|g\|$ small

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/!\ Not an NTRU instance ($h \in K$ is not in R_q)

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$$\{x\} = x - \lfloor x \rceil$$

  - $h = \lfloor q/z \rceil$, $f = -g\{q/z\}$
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From ideal-SVP to $\mathrm{NTRU}_{\mathrm{vec}}$ (2)

Summing up: If $I = \langle z \rangle = \{z \cdot r \mid r \in R\}$ and z known

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- for general ideals, $I = R \cap \langle z \rangle$ and z easily computed
  - everything still works with this z

From $\mathrm{NTRU}_{\mathrm{mod}}$ to dNTRU

Objective: given $h = f/g \bmod q$, recover $h_K = f/g \in K$ (division in K)

Can use an oracle: given $h \in R_q$, outputs

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Idea:

- take $x, y \in R$
- create $h' = x \cdot h + y = \frac{xf+yg}{g} \bmod q$
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$\Rightarrow$ we can choose x and y

$\Rightarrow$ we can modify the coordinates one by one

From $\mathrm{NTRU}_{\mathrm{mod}}$ to dNTRU (2)

Simplified problem

$f, g \in \mathbb{R}$ secret, $B \geq 0$ unknown.
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We obtain: $x_0 f + y_0 = x_1 f + y_1$, i.e., $f = \frac{y_1 - y_0}{x_0 - x_1}$

Some things I did not mention

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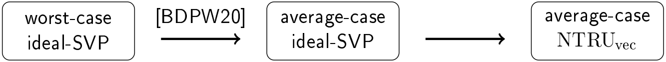

[BDPW20] de Boer, Ducas, Pellet-Mary, and Wesolowski. Random Self-reducibility of Ideal-SVP via Arakelov Random Walks. Crypto.

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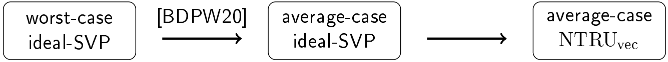


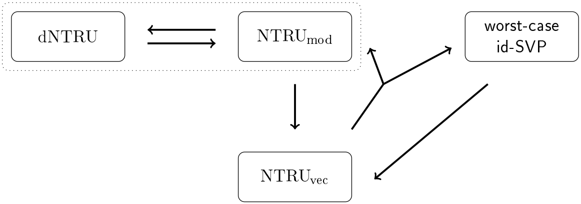
For dNTRU to $\mathrm{NTRU}_{\mathrm{mod}}$:

We do not have a perfect oracle

- need to handle distributions
- use the “oracle hidden center” framework [PRS17]

[PRS17] Peikert, Regev, and Stephens-Davidowitz. Pseudorandomness of ring-LWE for any ring and modulus. STOC.

Conclusion and open problems



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- Can we prove reduction from module problems with rank ≥ 2?
  - for instance, uSVP in modules of rank-2?

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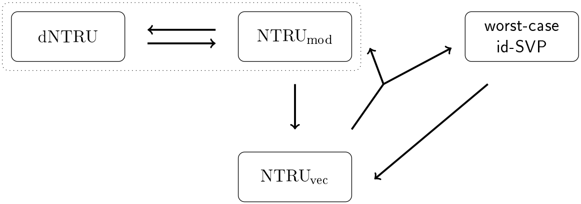

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