Exercises on Matrix Rigidity

Satya Lokam

4 September 2014

1. Given any $n \times n$ matrix A, show that its rank can be reduced to $\leq r$ by changing at most $(n-r)^2$ entries of A.

2. (Midrijānis) The Sylvester matrix $S_k \in \{-1,+1\}^{2^k \times 2^k}$ is recursively defined by

$$S_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad S_k = \begin{bmatrix} S_{k-1} & S_{k-1} \\ S_{k-1} & -S_{k-1} \end{bmatrix}.$$

Let $n := 2^k$ and S_k be the $n \times n$ Sylvester matrix. Prove that, for r a power of 2, $\mathcal{R}_{S_k}(r) \geq \frac{n^2}{4r}$.

(Hint: Consider a tiling of S_k by $2r \times 2r$ Sylvester matrices and use an averaging argument.)

3. Show that all submatrices of a Cauchy matrix $((x_i + y_j)^{-1})$ are nonsingular, where x_i and y_j are all distinct and for all i, j, $x_i + y_j \neq 0$.

4. Let $V = (a_i^{j-1})_{i,j=1}^n$, where the a_i are distinct *positive* reals. Show that all submatrices of V are nonsingular.

(Hint: Descartes' rule of signs.)

5. Let $G = [\, I \mid A \,]$ be the generator matrix (in standard form) of a $[2n, n, d]$ error correcting code over $\mathbb{F}_q$ with $d \geq (1-\epsilon)n$, where q is a constant and $\epsilon > 0$ is a constant depending on q. Thus G is an $n \times 2n$ matrix and A is an $n \times n$ matrix over $\mathbb{F}_q$. Show that for every $t \geq \epsilon n + 1$, every $2t \times 2t$ submatrix of A must have rank at least t.

6. Show that if a matrix has an eigenvlaue of multiplicity k, its rank can be reduced to $(n-k)$ by changing at most k entries. Conclude that an $n \times n$ Hadamard matrix has rigidity at most $n/2$ for target rank $n/2$. Show a similar upper bound for the Discrete Fourier Transform matrix.

7. (Valiant) Let $\mathbb{F}$ be a finite field of order q. In this exercise, we will show that for "most" $n \times n$ matrices A over $\mathbb{F}$, we must change at least $\Omega((n-r)^2/\log n)$ entries to reduce the rank of A to r.

   (a) Show that the number of matrices of rank at most r over $\mathbb{F}$ is at most $q^{2nr-r^2} \cdot \binom{n}{r}$.

   (b) Observe $\mathcal{R}_A(r) \leq s$, if $A = B + C$ where $\text{rank}(B) \leq r$ and C has at most s nonzero entries. How many choices do we have for C? Using this and (a), show that the number of matrices of rigidity at most s is at most $q^{2nr-r^2+s+2s\log_q n + n\log_q 2}$.

   (c) If $r \leq n - c_1\sqrt{n}$ and $s < c_2(n-r)^2/\log n$, for some positive constants c_1 and c_2, show that the fraction of matrices A with $\mathcal{R}_A(r) \leq s$ is $O(1/n)$.

8. (Pudlák-Rödl) In this exercise, we will show that most $\mathbb{R}$ matrices with entries in $\{0,1\}$ have rigidity $\Omega(n^2)$ for target rank $r = O(n)$. As noted above, $\mathcal{R}_A(r) \leq s$ if $A = B + C$ where $\text{rank}(B) \leq r$ and C has at most s nonzero entries. We express A by the *sign-patterns* of a set of real polynomials.

(a) Show that $n \times n$ matrices of rank at most r can be parametrized by at most $2nr - r^2$ variables.

(b) Observe that each entry a_{ij} is a polynomial $p_{ij}(z_1, \ldots, z_m)$ over $\mathbb{R}$ on $m \leq 2nr - r^2 + s$ variables and degree at most 2. Since $a_{ij} \in \{0, 1\}$, the sign of $p_{ij}(z) - 1/2$ uniquely determines it. Hence the number of 0-1 matrices A such that $\mathcal{R}_A(r) \leq s$ is bounded above by the number of sign-patterns of such polynomials.

(c) Use the following theorem due to Warren to get an upper bound on the number of choices for A.

Theorem: Let $f = (f_1, \ldots, f_m)$ be a sequence of m polynomials of degree at most d in n variables over $\mathbb{R}$. Assume $m \geq n$ and $d \geq 1$. Then the number of sign-patterns of f is less than

$$\left(\frac{4\mathrm{e}md}{n}\right)^n.$$

For an elementary (using linear algebra) proof of this inequality, see L. Rónyai, L. Babai, and M. K. Ganapathy, "On the number of zero-patterns of a sequence of polynomials," *Journal of the American Mathematical Society*, vol. 14, no. 3, pp. 717 – 735, 2001.

(d) Comparing the number of choices for A obtained above to 2^{n^2} (total number of 0-1 $n \times n$ matrices), prove that most 0-1 matrices A must have $\mathcal{R}_A(\epsilon n) \geq \Omega(n^2)$ for sufficiently small $\epsilon > 0$.