Christian Ikenmeyer

For a partition $\lambda \vdash dn$ with dn boxes let $p_\lambda(d[n])$ denote the multiplicity of the type λ in $\mathsf{Sym}^d\mathsf{Sym}^n V$ for a finite dimensional complex vector space V of high enough dimension. This is called a *plethysm coefficient*. For a partition triple (λ, μ, ν) with d boxes each let $k(\lambda; \mu; \nu)$ denote the multiplicity of the type (λ, μ, ν) in $\mathsf{Sym}^d \bigotimes^3 V = \mathsf{Sym}^d(A \otimes B \otimes C)$. This is called a *Kronecker coefficient*.

Exercise 1.
Prove that $p_{(3,1)}(2[2]) = 0$.

Exercise 2.
Prove that $p_{(2,2)}(2[2]) > 0$. Moreover, prove that $p_{(2,2)}(2[2]) = 1$.

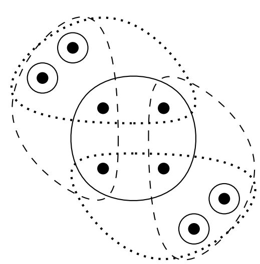
Exercise 3.
Prove that $k((2, 1, 1, 1, 1, 1, 1, 1); (7, 1, 1); (7, 1, 1)) = 0$.

Exercise 4.
Let $d, n \in \mathbb{N}$, $d > 1$, and let $\lambda = d \times n$ be the rectangular partition with d rows and n columns. Prove that

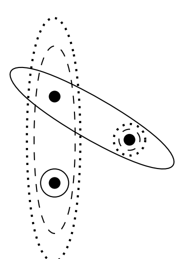
$$p_\lambda(d[n]) = \begin{cases} 1 \text{ for } n \text{ even} \\ 0 \text{ for } n \text{ odd} \end{cases} .$$

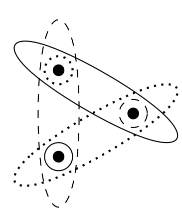
Exercise 5.
Partitions of the form $(a, 1, 1, \ldots, 1)$ are called *hook partitions*. Let $d, n \in \mathbb{N}$ and let $\lambda \vdash dn$ be a hook partition with at least 2 rows (so that λ is not only a single row). Prove that $p_\lambda(d[n]) = 0$.

Exercise 6.
Prove that the following hypergraph defines a polynomial in $\mathsf{Sym}^8 \bigotimes^3 \mathbb{C}^4$ that vanishes on all rank 4 tensors.



Exercise 7.
For the following two hypergraphs check if they give the zero polynomial in $\mathsf{Sym}^3 \bigotimes^3 \mathbb{C}^d$ or a nonzero polynomial?





Exercise 8.
Prove that the polynomial obtained from the hypergraph from exercise 3 also vanishes on rank 5 tensors.