Determinant vs Permanent : Exercises

1. Prove Ryser's formula : if $X = (x_{ij})$ is an $m \times m$ matrix,

$$perm_m(X) = \sum_{S \subset \{1,\ldots,m\}} (-1)^{m-|S|} \prod_{i=1}^{m} (\sum_{j \in S} x_{ij}).$$

2. Prove Glinn's formula :

$$perm_m(X) = 2^{1-m} \sum_{\epsilon \in \{\pm 1\}^m} (\prod_{k=1}^{m} \epsilon_k) \prod_{i=1}^{m} (\sum_{j=1}^{m} \epsilon_j x_{ij}).$$

3. Prove that for matrices of size $n \geq 3$, there is no way to change the signs of the entries of a generic matrix, in such a way that taking the determinant, one gets the permanent of the original matrix. (This holds true over any field of characteristic different from two.)

4. Let T be an invertible linear transformation of the space of $n \times n$ matrices, that preserves the space of rank one matrices. Show that there exist two matrices A and B such that either

$$T(X) = AXB \qquad or \qquad T(X) = A^t XB.$$

   Are A and B uniquely determined by T ? Hints :

   (a) Show that there are two families of maximal linear spaces of rank one matrices : those, denoted I_v, whose images are generated by a given vector v (or zero) ; those, denoted K_x, whose kernels contain a given hyperplane $x = 0$ (where x is a non zero linear form).

   (b) Suppose that T sends some I_v to some I_w. Show that there exist two transformations α and β such that T sends I_u to $I_{\alpha(u)}$ for any vector u, and K_x to $K_{\beta(x)}$ for any linear form x.

   (c) Conclude.

5. The goal of this exercise is to prove the following result of : There is no linear transformation T of the space of matrices such that the polynomial identity

$$perm(X) = det(T(X))$$

   does hold. Hints :

   (a) Show that T must be invertible.

(b) Show by descending induction on r that any subpermanent of size r of X is a linear combination of the r-minors of $T(X)$.

(c) Show that if X has all its subpermanents of size two equal to zero, then its non zero entries are all either on a line, a column, or a block of size two.

(d) Deduce that T preserves the space of rank one matrices. Conclude with the help of the previous exercise.

6. Use the same kind of techniques to characterize the linear transformations T that preserve the permanent.
   (a) Show that T must be invertible.
   (b) Show by descending induction on r that any subpermanent of size r of $T(X)$ is a linear combination of the subpermanents of size r of X.
   (c) Deduce that T preserves the space of rank one matrices.
   (d) According to Ex.1 there exist two matrices A and B such that either $T(X) = AXB$ or $T(X) = A^tXB$. Show that A and B must be diagonal up to permutations. Conclude.
7. Show that the permanent is determined (up to scalar) by its stabilizer. Hints :
   (a) Let P be a homogeneous polynomial of degree n of the space of $n \times n$ matrices, with the same stabilizer as the permanent. Using the diagonal part show that each monomial in P must be of the form $X_{1\sigma(1)} \cdots X_{n\sigma(n)}$ for some permutation σ.
   (b) Using the permutation part show that the coefficient of this monomial in P must be independent of σ.
8. Show that the determinant is determined (up to scalar) by its stabilizer.
9. The hypersurface $(\det = 0)$ contains very large linear spaces, for example the space K_v of matrices vanishing on some non zero fixed vector v.
   (a) Show that each K_v is a maximal linear space in the determinantal hypersurface.
   (b) Show that there exist linear spaces of singular matrices not contained in any of these maximal spaces (or their transpose).
10. Let P be any non constant polynomial in n variables $x_1, \ldots, x_n$. Prove that for any $d > 0$, the SL_{n+1}-orbit of the padded polynomial $x_0^d P$ is not closed.
11. Prove that the GL_{n^2}-orbit of the determinant contains the variety of polynomials of degree n which are products of n linear forms.