Aspects of Coulomb gases

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Concentration of Measure Phenomena
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Outline

Electrostatics

Gases

Dynamics for planar case

Conditioning

Jellium

Coulomb kernel in mathematical physics

- Coulomb kernel in $\mathbb{R}^d$, $d \geq 1$,

$$x \in \mathbb{R}^d \mapsto g(x) = \begin{cases} \log \dfrac{1}{|x|} & \text{if } d = 2 \\ \dfrac{1}{(d-2)|x|^{d-2}} & \text{if not} \end{cases}.$$

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$$\Delta g = -c_d \delta_0 \quad \text{where} \quad c_d = |\mathbb{S}^{d-1}| = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

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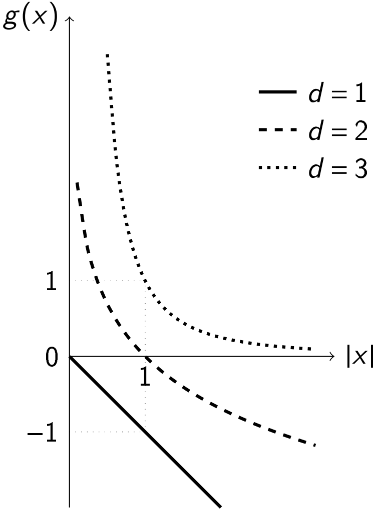
- Riesz kernel $|x|^{-s}$, if $s = d - \alpha$ then fractional Laplacian Δ_α

From now on $d \geq 2$



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- Integration by parts and "carré du champ", $\eta = \mu - \nu$,

$$\mathcal{E}(\eta) = \frac{1}{2}\int \mathrm{U}_\eta \mathrm{d}\eta = -\frac{1}{2c_d}\int \mathrm{U}_\eta\Delta\mathrm{U}_\eta \mathrm{d}x = \frac{1}{2c_d}\int |\nabla\mathrm{U}_\eta|^2\mathrm{d}x.$$

External field equilibrium measure

- External field potential $V:\mathbb{R}^d \to (-\infty,+\infty]$

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- $\mathrm{supp}(\mu_V)$ is compact if $\lim_{|x|\to\infty}(V(x) - \log|x| 1_{d=2}) = +\infty$

Convexity and Bochner positivity

- Convexity/Positivity for probability measures μ and ν

$$\frac{t\mathscr{E}_V(\mu)+(1-t)\mathscr{E}_V(\nu)-\mathscr{E}_V(t\mu+(1-t)\nu)}{t(1-t)}$$

$$=\mathscr{E}(\mu-\nu)=\frac{1}{2c_d}\int_{\mathbb{R}^d}|\nabla \mathrm{U}_{\mu-\nu}|^2\mathrm{d}x.$$

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- Euler–Lagrange variational characterization (quadratic form!)

$$\mathrm{U}_{\mu_V}+V\begin{cases}=c_V & \text{on } \mathrm{supp}(\mu_V)\\ \geq c_V & \text{outside}\end{cases}$$

quasi everywhere, where $c_V=\mathscr{E}(\mu_V)-\int V\mathrm{d}\mu_V$

Examples of equilibrium measures

Dimension d	Potential V	Equilibrium measure μ_V
≥ 1	$\infty 1_{\lvert\cdot\rvert > r}$	Uniform on sphere of radius r
≥ 1	$< \infty$ and $\mathscr{C}^2$	$c_d^{-1} \Delta V$ on interior of support
≥ 1	$\frac{1}{2} \lvert\cdot\rvert^2$	Uniform on unit ball
(Ginibre) 2	$\frac{1}{2} \lvert\cdot\rvert^2$	Uniform on unit disc
(Spherical) 2	$\frac{1}{2} \log(1 + \lvert\cdot\rvert^2)$	Heavy-tailed $\frac{1}{\pi(1+\lvert\cdot\rvert^2)^2}$

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(CUE) 2	$\infty 1_{([a,b]\times\{0\})^c}$	Arcsine $s\mapsto\frac{1_{s\in[a,b]}}{\pi\sqrt{(s-a)(b-s)}}$
(GUE) 2	$\frac{\lvert\cdot\rvert^2}{2}1_{\mathbb{R}\times\{0\}}+\infty 1_{(\mathbb{R}\times\{0\})^c}$	Semicircle $s\mapsto\frac{\sqrt{4-s^2}}{2\pi}1_{s\in[-2,2]}$

Outline

Coulomb gas or one-component plasma

- Particles subject to external field and singular pair repulsion

$$\begin{aligned}\beta \mathrm{E}_n(x_1,\ldots,x_n) &= \beta n^2\Big(\frac{1}{n}\sum_{i=1}^{n} V(x_i) + \frac{1}{n^2}\sum_{i<j} g(x_i - x_j)\Big)\\ &= \beta n^2 \mathcal{E}_V^{\neq}(\mu_{x_1,\ldots,x_n})\end{aligned}$$

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$$\mathcal{E}_V^{\neq}(\mu) = \int V \mathrm{d}\mu + \frac{1}{2}\iint_{\neq} g(u-v)\mathrm{d}\mu(u)\mathrm{d}\mu(v)$$

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- Boltzmann–Gibbs measure when $\mathrm{e}^{-n\beta(V - \log(1+|\cdot|)1_{d=2})} \in \mathrm{L}^1(\mathrm{d}x)$

$$\mathrm{dP}_n(x_1,\ldots,x_n) = \frac{\mathrm{e}^{-\beta \mathrm{E}_n(x_1,\ldots,x_n)}}{\mathrm{Z}_n}\mathrm{d}x_1\cdots\mathrm{d}x_n$$

Examples: exactly solvable RMT and Coulomb gases

- $d = 2$ gives

$$e^{-\beta(\sum_i V(x_i) - \sum_{i<j} g(x_i - x_j))} = e^{-\beta \sum_i V(x_i)} \prod_{i<j} |x_i - x_j|^\beta$$

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- $d = 2$ and $\beta = 2$ give determinantal structure

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- $M \in \mathscr{M}_{n,n}(\mathbb{C}),\ M \propto \mathrm{e}^{-\mathrm{Trace}(MM^*)}$

Ensemble	Random Matrix	Potential V $(d = \beta = 2)$
Ginibre	M	$\frac{1}{2}\lvert\cdot\rvert^2$
(GUE) Hermite	$\frac{1}{\sqrt{2}}(M + M^*)$	$\frac{1}{2}\lvert\cdot\rvert^2 \mathbb{1}_{\mathbb{R}\times\{0\}} + \infty \mathbb{1}_{(\mathbb{R}\times\{0\})^c}$
(LUE) Laguerre	MM^*	$\frac{1}{2}\lvert\cdot\rvert \mathbb{1}_{\mathbb{R}_+\times\{0\}} + \infty \mathbb{1}_{(\mathbb{R}_+\times\{0\})^c}$
Spherical	$M_1 M_2{}^{-1}$	$\frac{1}{2}\frac{n+1}{n}\log(1 + \lvert\cdot\rvert^2)$

Laplace method point of view

- Laplace point of view

$$\mathrm{dP}_n(x_1,\ldots,x_n) = \frac{\mathrm{e}^{-\beta n^2 \mathcal{E}_V^{\neq}(\mu_{x_1,\ldots,x_n})}}{\mathrm{Z}_n} \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

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- Large Deviation Principle (also works if $\beta = \beta_n$ with $n\beta_n \to \infty$)

$$\mathrm{P}_n(\mu_{x_1,\ldots,x_n} \in B) \underset{n\to\infty}{\approx} \mathrm{e}^{-\beta n^2 \inf_B(\mathcal{E}_V - \mathcal{E}_V(\mu_V))}$$

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- Law of Large Numbers : if $X \sim \mathrm{P}_n$ then almost surely

$$\mu_{X_1,\ldots,X_n} \xrightarrow[n\to\infty]{} \mu_V = \arg\min \mathcal{E}_V$$

..., Voiculescu, Ben Arous – Guionnet, Hiai–Petz,...

..., Serfaty et al, C.–Gozlan–Zitt, Dupuis, Berman, García-Zelada,...

Asymptotic analysis of fluctuations

- Gaussian structure: $\mathrm{P}_n \approx \mathrm{e}^{-\frac{n^2}{2}\langle -\beta c_d \Delta^{-1}\mu,\mu\rangle - n^2\beta\langle V,\mu\rangle}$

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- Asymptotics: Central Limit Theorem with Gaussian Free Field

$$\sum_{i=1}^{n} f(X_i) - \mathbb{E}(\cdots) \xrightarrow[n\to\infty]{\text{law}} \mathcal{N}\Big(0, \frac{1}{\beta c_d}\int_{\mathbb{R}^d} |\nabla f|^2 \mathrm{d}x + \cdots\Big)$$

. . . , Johansson, . . . , Rider–Virag, . . . , Serfaty et al, . . .

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- Quantitative: concentration of measure inequalities

$$\mathbb{P}\big(\mathrm{dist}(\mu_{X_1,\ldots,X_n},\mu_V) \geq r\big) \leq \mathrm{e}^{-a\beta n^2 r^2 + 1_{d=2}(\frac{\beta}{4} n\log n) + b\beta n^{2-2/d} + c(\beta)n}$$

..., Guionnet–Zeitouni, Rougerie–Serfaty, Hardy–C.–Maïda, Berman, ...

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- Coulomb specific: singularity, and $\mathcal{E} \leftrightarrow g \leftrightarrow (\mathrm{Sobolev})^{-1} \leftrightarrow \mathrm{W}_p$

Outline

Langevin dynamics

- Overdamped Langevin dynamics on $(\mathbb{R}^d)^n$: $X_t \xrightarrow[t\to\infty]{\text{law}} \mathrm{P}_n$

$$\mathrm{d}X_t = \sqrt{2\frac{\alpha}{\beta}}\mathrm{d}B_t - \alpha\nabla\mathrm{E}_n(X_t)\mathrm{d}t, \quad \mathrm{L} = \alpha(\beta^{-1}\Delta - \nabla\mathrm{E}_n\cdot\nabla)$$

Dyson, Bru, Lassalle, Rogers–Shi, Guionnet et al,. . . , Erdős–Yau et al. . .
Bolley–C.–Fontbona, C.–Lehec, Akkeman–Byun, . . .

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- Mean-field McKean–Vlasov limit: if $\sigma = \lim_{n\to\infty}\frac{\alpha_n}{\beta_n}$ then (?)

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. . . , Carrillo–McCann-Villani, . . . , Guionnet et al, . . . , Jabin et al, . . .

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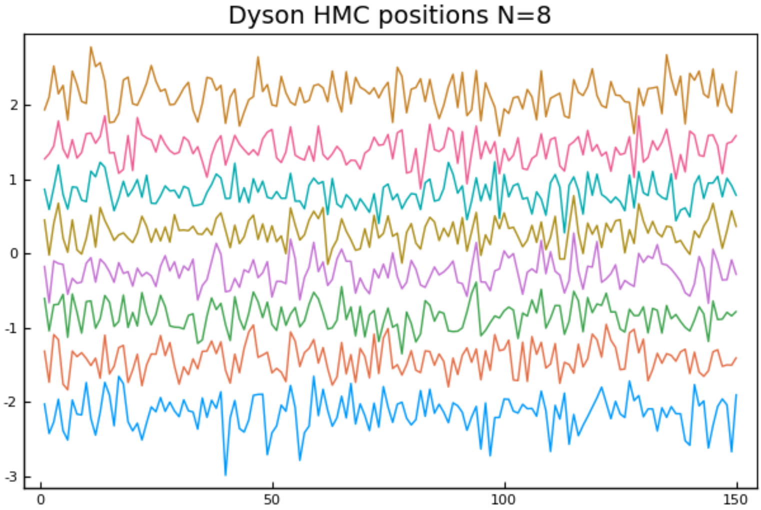
- Underdamped Langevin or kinetic Dyson–Ornstein–Uhlenbeck

$$\mathrm{d}X_t = \alpha Y_t\mathrm{d}t, \quad \mathrm{d}Y_t = -\alpha\nabla\mathrm{E}_n(X_t)\mathrm{d}t + \sqrt{2\frac{\gamma\alpha}{\beta}}\mathrm{d}B_t - \gamma\alpha Y_t\mathrm{d}t$$

Good for numerical simulation via Hamiltonian Monte Carlo

Ferré–C., Lu–Mattingly, Dolbeault et al

Example of kinetic Dyson HMC in 1D



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- $|X_{t,1}|^2+\cdots+|X_{t,n}|^2$ is a Cox–Ingersoll–Ross process
- As a consequence if $X_n\sim \mathrm{P}_n$ then

$$
\begin{aligned}
X_1+\cdots+X_n &\sim \mathcal{N}\left(0,\frac{\mathrm{I}_2}{\beta}\right)\\
|X_1|^2+\cdots+|X_n|^2 &\sim \mathrm{Gamma}\left(n+\beta\frac{n(n-1)}{4},\beta\frac{n}{2}\right)
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Outline

Conditioned Coulomb gas

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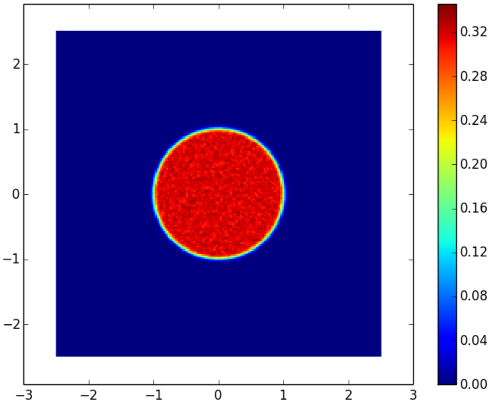

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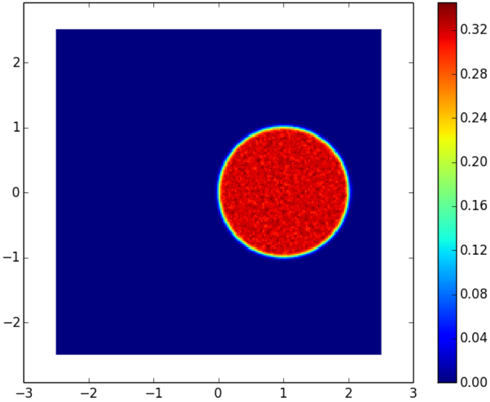

$$V(x) = |x|^2,\ \varphi(x) = a \cdot x + b$$

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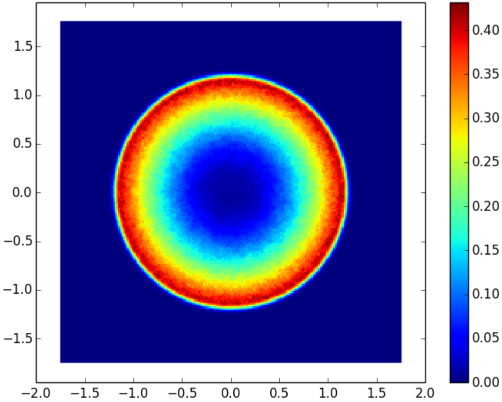

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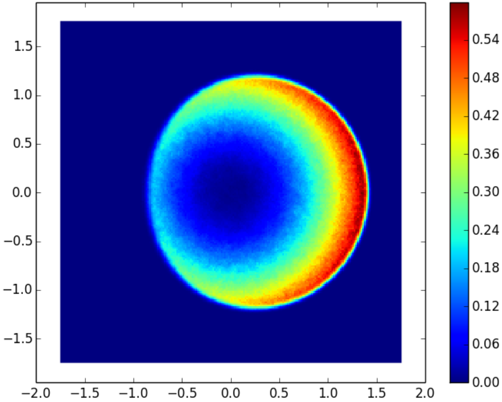

$$V(x) = |x|^4, \; \varphi(x) = ax_1 + b$$

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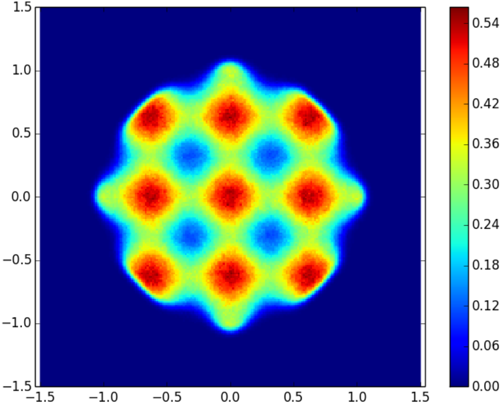

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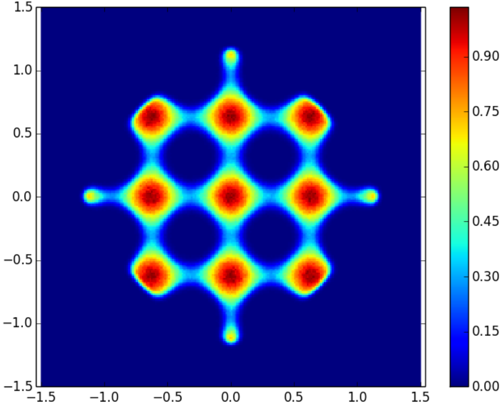

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- Coulomb gas is a Jellium with $\rho = \frac{n}{\alpha c_d}\Delta V$ on $S = \mathbb{R}^d$

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Gumbel if $\lambda > 1$ and Heavy-tailed if $\lambda = 1$.

García-Zelada–C.–Jung. More: Butez–García-Zelada

Thank you very much for your attention!