Moments of the distance between independent random vectors (based on a joint work with Assaf Naor)

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Concentration of measure phenomena
Simons Institute, Berkeley (on-line)
October 22, 2020

A. Naor and K. Oleszkiewicz, *Moments of the distance between independent random vectors*, in: *Geometric Aspects of Functional Analysis, Israel Seminar (GAFA) 2017–2019*, Volume II, editors: Bo'az Klartag and Emanuel Milman, Lect. Notes in Math. 2266, Springer Nature Switzerland AG 2020, pages 229–256.

Starting point

Let F be a separable Banach space and let X and Y be **independent** F-valued random vectors.

If $\mathbb{E}\|X\|_F, \mathbb{E}\|Y\|_F < \infty$, then the pointwise triangle inequality

$$\|X - Y\|_F \leq \|X - z\|_F + \|Y - z\|_F,$$

true for every $z \in F$, implies

$$\mathbb{E}\|X - Y\|_F \leq \inf_{z \in F} \mathbb{E}\left(\|X - z\|_F + \|Y - z\|_F\right).$$

To what extent can this type of bound be reversed?

Independence of X and Y excludes $X \equiv Y$ (unless constant a.s.).

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Concentration

Let Z be an F-valued random vector with distribution defined by

$$\mathcal{L}(Z) = \frac{1}{2}\mathcal{L}(X) + \frac{1}{2}\mathcal{L}(Y),$$

so that

$$\mathbb{P}(Z \in A) = \big(\mathbb{P}(X \in A) + \mathbb{P}(Y \in A)\big)/2$$

for every Borel subet A of F.

Then $\mathbb{E}\|X - z\|_F + \mathbb{E}\|Y - z\|_F = 2\,\mathbb{E}\|Z - z\|_F$, i.e., the expression

$$\inf_{z \in F} \left(\mathbb{E}\|X - z\|_F + \mathbb{E}\|Y - z\|_F\right) = 2 \inf_{z \in F} \mathbb{E}\|Z - z\|_F$$

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B for barycenter

Given a separable Banach space $(F, \|\cdot\|_F)$, we define a number of geometric moduli.

To start with, given $p > 0$ (usually $p \geq 1$), let $b_p(F) = b_p(F, \|\cdot\|_F)$ be the infimum over those $b > 0$ for which every pair of independent F-valued random vectors X, Y with finite p-th moments (i.e., $\mathbb{E}\|X\|_F^p, \mathbb{E}\|Y\|_F^p < \infty$) satisfies

$$\inf_{z \in F} \mathbb{E}\left(\|X - z\|_F^p + \|Y - z\|_F^p\right) \leq b \cdot \mathbb{E}\|X - Y\|_F^p.$$

The use of letter "b" in this notation comes from Riemannian/Alexandrov geometry – it refers to the geometric barycenter in the context of Hadamard spaces (complete simply connected spaces whose Alexandrov curvature is nonpositive).

For every F and $p \geq 1$, there is $b_p \leq 2(3/2)^p$.
The bound cannot be improved in general.

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or, equivalently,

$$2\,\mathbb{E}\|Z - \mathbb{E}Z\|_F^p \leq m \cdot \mathbb{E}\|X - Y\|_F^p.$$

Obviously, $b_p(F) \leq m_p(F)$ for every F and $p \geq 1$.
To see it, choose $z = \mathbb{E}Z = (\mathbb{E}X + \mathbb{E}Y)/2$.

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Given $p > 0$ (usually $p \geq 1$), let $r_p(F) = r_p(F, \|\cdot\|_F)$ be the infimum over those $r > 0$ for which every pair of independent F-valued random vectors X, Y with finite p-th moments satisfies

$$\mathbb{E}\left(\|X - X'\|_F^p + \|Y - Y'\|_F^p\right) \leq r \cdot \mathbb{E}\|X - Y\|_F^p,$$

where X' and Y' are independent copies of X and Y, respectively.

J for Jensen

Given $p \geq 1$, let $j_p(F) = j_p(F, \|\cdot\|_F)$ be the supremum over those $j > 0$ for which every pair of independent **and identically distributed** F-valued random vectors Z, Z' with finite p-th moments satisfies

$$j \cdot \mathbb{E}\|Z - \mathbb{E}Z\|_F^p \leq \mathbb{E}\|Z - Z'\|_F^p.$$

Obviously, by Jensen's inequality, always $j_p(F) \geq 1$.

For every $p \geq 1$ and every F, we have $m_p(F) \leq \frac{2+r_p(F)}{2j_p(F)}$:

$$j_p \cdot \left(\mathbb{E}\|X - \mathbb{E}Z\|_F^p + \mathbb{E}\|Y - \mathbb{E}Z\|_F^p\right) = 2j_p\,\mathbb{E}\|Z - \mathbb{E}Z\|_F^p \leq 2\,\mathbb{E}\|Z - Z'\|_F^p$$

$$= \mathbb{E}\|X - Y\|_F^p + \frac{\mathbb{E}\|X - X'\|_F^p + \mathbb{E}\|Y - Y'\|_F^p}{2} \leq \left(1 + \frac{r_p}{2}\right)\mathbb{E}\|X - Y\|_F^p.$$

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Exact values of $j_p(L_q)$, $b_p(L_q)$, and $r_p(L_q)$

Theorem: For every $p, q \in [1, \infty)$ we have $j_p(L_q) = 2^{c(p,q)}$, where

$$c(p,q) = \min\left(1, p-1, \frac{p}{q}, \frac{(q-1)p}{q}\right).$$

For every $1 \le p \le q \le 2$ we have $b_p(L_q) = m_p(L_q) = 2^{2-p}$.

Let $p, q \in (1, \infty)$.

If $\frac{p}{p-1} \le q \le p$, then $r_p(L_q) = 2^{p-1}$.

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Various bounds in the remaining cases.

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Complex interpolation

1. Express L_q as an interpolation space between L_2 and L_Q.
2. Choose the right operator. Prove that it satisfies a simple $L_\infty(L_Q) \to L_\infty(L_Q)$ norm bound and (via spectral methods) a more subtle $L_2(L_2) \to L_2(L_2)$ norm bound.
3. Bound the operator's $L_p(L_q) \to L_p(L_q)$ norm using complex interpolation methods.
4. Read from it a bound on the geometric modulus (here the choice of the "right operator" plays a crucial role).
5. If lucky enough, find an example indicating that the obtained bound on the geometric modulus cannot be improved (thus establishing the exact value of the modulus).
6. Try using first the interpolation trick for some other moduli and then taking advantage of some relation between moduli. Sometimes it works better than the straightforward approach described above.
7. Try to improve the obtained bounds, if they are not optimal, by using some other tricks (e.g. non-linear isometric embeddings).

The same for Schatten classes (noncommutative counterpart of L_q).

Interpolation picture

$$T : L_Q \overset{L_\infty \to L_\infty}{\longrightarrow} L_Q$$
$$T : L_q \overset{L_p \to L_p}{\longrightarrow} L_q \qquad L_q = [L_Q, L_2]_\theta, \quad \frac{1}{q} = \frac{1-\theta}{Q} + \frac{\theta}{2}, \quad \frac{1}{p} = \frac{1-\theta}{\infty} + \frac{\theta}{2}$$
$$T : L_2 \overset{L_2 \to L_2}{\longrightarrow} L_2$$

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Caveat: When playing with the interpolation parameter $\theta \in [0, 1]$, avoid $Q > \infty$ and $Q < 1$!

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Operator used in the proof of $j_p(L_q)$ bounds

Let $Z : (\Omega, \mu) \to \mathcal{H}$, where $\mathcal{H} \simeq L_2$ is a separable Hilbert space. For $L_2(\Omega \times \Omega \to \mathcal{H}, \mu \otimes \mu)$ let S denote the orthogonal projection to the closed linear subspace of functions of the form $f(x,y) = \Phi(x) - \Phi(y)$ for $\Phi : \Omega \to \mathcal{H}$ with $\int_\Omega \Phi \, \mathrm{d}\mu = 0$.

Thus, $(Sf)(x,y) = (Tf)(x) - (Tf)(y)$ for some Tf. One can easily check that

$$(Tf)(x) = \int_\Omega \frac{f(x,z) - f(z,x)}{2} \, \mathrm{d}\mu(z),$$

so that $T : L_2(\Omega \times \Omega \to \mathcal{H}, \mu \otimes \mu) \to L_2(\Omega \to \mathcal{H}, \mu)$ is linear.

Since $(Tf)(Z)$ and $(Tf)(Z')$ are independent,

$$\|f\|^2_{L_2(\Omega\times\Omega\to\mathcal{H},\mu\otimes\mu)} \geq \|Sf\|^2_{L_2(\Omega\times\Omega\to\mathcal{H},\mu\otimes\mu)}$$

$$= \|(T \otimes \mathrm{Id}_{\mathcal{H}})f\|^2_{L_2(\Omega\times\Omega\to\mathcal{H},\mu\otimes\mu)} + \|(\mathrm{Id}_{\mathcal{H}} \otimes T)f\|^2_{L_2(\Omega\times\Omega\to\mathcal{H},\mu\otimes\mu)}$$

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$$(Tf)(x) = \int_\Omega \frac{f(x, z) - f(z, x)}{2} \, \mathrm{d}\mu(z),$$

so that $T : L_2(\Omega \times \Omega \to \mathcal{H}, \mu \otimes \mu) \to L_2(\Omega \to \mathcal{H}, \mu)$ is linear.

Since $(Tf)(Z)$ and $(Tf)(Z')$ are independent,

$$\|f\|^2_{L_2(\Omega\times\Omega\to\mathcal{H},\mu\otimes\mu)} \geq \|Sf\|^2_{L_2(\Omega\times\Omega\to\mathcal{H},\mu\otimes\mu)}$$

$$= \|(T \otimes \mathrm{Id}_{\mathcal{H}})f\|^2_{L_2(\Omega\times\Omega\to\mathcal{H},\mu\otimes\mu)} + \|(\mathrm{Id}_{\mathcal{H}} \otimes T)f\|^2_{L_2(\Omega\times\Omega\to\mathcal{H},\mu\otimes\mu)}$$

$$= 2\|Tf\|^2_{L_2(\Omega\to\mathcal{H},\mu)}.$$

Operator T norm bound

We have proved that, for T given by

$$(Tf)(x) = \int_\Omega \frac{f(x,z) - f(z,x)}{2}\, \mathrm{d}\mu(z),$$

we have

$$\|T\|_{L_2(\Omega\times\Omega\to\mathcal{H},\mu\otimes\mu)\to L_2(\Omega\to\mathcal{H},\mu)} \leq \sqrt{2}/2.$$

On the other hand, by the triangle inequality for **any** Banach space F we have, for T defined by the same formula,

$$\|T\|_{L_\infty(\Omega\times\Omega\to F,\mu\otimes\mu)\to L_\infty(\Omega\to F,\mu)} \leq 1$$

and

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It remains to interpolate the norm bound and apply it to $f(x,y) = Z(x) - Z(y)$, for which $Tf = Z - \mathbb{E}Z$.

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Teaser ($p = 0$)

Let $(F, \|\cdot\|_F)$ be a Banach space with $\dim_{\mathbb{R}}(F) \leq 3$. Then, for every pair of independent F-valued random vectors X, Y satisfying $\mathbb{E}\ln(1 + \|X\|_F) < \infty$ and $\mathbb{E}\ln(1 + \|Y\|_F) < \infty$, we have

$$\mathbb{E}\ln\|X - X'\|_F + \mathbb{E}\ln\|Y - Y'\|_F \leq 2\,\mathbb{E}\ln\|X - Y\|_F$$

and

$$\inf_{z\in F} \mathbb{E}\left(\ln\|X - z\|_F + \ln\|Y - z\|_F\right) \leq 2\,\mathbb{E}\ln\|X - Y\|_F,$$

where X' and Y' are independent copies of X and Y, respectively.

Kalton-Koldobsky-Yaskin-Yaskina:

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