Tail and Moment Bounds for Gaussian Chaoses in Banach Spaces

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(based on joint works with Radosław Adamczak and Rafał Meller)

online workshop Concentration of Measure Phenomena
October 19 2020

Introduction

Let $(F, \|\cdot\|)$ be a Banach space. A (homogeneous, tetrahedral) F-valued Gaussian chaos of order d is a random variable defined as

$$S = \sum_{1 \le i_1 < i_2 \ldots < i_d \le n} a_{i_1,\ldots,i_d} g_{i_1} \cdots g_{i_d}, \tag{1}$$

where $a_{i_1,\ldots,i_d} \in F$ and $g_1, \ldots, g_n$ are i.i.d.standard Gaussian variables.

We will discuss two-sided estimates on L_p-moments (defined as $\|S\|_p := (\mathbb{E}\|S\|^p)^{1/p}$) and tails of $\|S\|$.

We will also present bounds for arbitary quadratic forms and (if time permits) for general polynomials in Gaussian random variables.

The talk is based on two joint papers with Radosław Adamczak and Rafał Meller.

Symmetry conditions

Observe that

$$\sum_{1 \le i_1 < i_2 \ldots < i_d \le n} a_{i_1,\ldots,i_d} g_{i_1} \cdots g_{i_d} = \sum_{i_1,\ldots,i_d=1}^{n} \tilde{a}_{i_1,\ldots,i_d} g_{i_1} \cdots g_{i_d},$$

where $\tilde{a}_{i_1,\ldots,i_d} = \frac{1}{d!} a_{i_1^*,i_2^*,\ldots,i_d^*}$ if $i_1,\ldots,i_d$ are pairwise disjoint and $i_1^* < i_2^* < \ldots < i_d^*$ is the increasing permutation of $(i_1,\ldots,i_d)$ and $\tilde{a}_{i_1,\ldots,i_d} = 0$ if $i_j = i_k$ for some $1 \le j < k \le d$.

We say that a finite matrix $(a_{i_1,\ldots,i_d})$ is *tetrahedral* and *symmetric* (or *satisfies symmetry conditions*) if it is invariant with respect to permutation of indexes and $a_{i_1,\ldots,i_d} = 0$ if $i_j = i_k$ for some $1 \le j < k \le d$.

Moments vs tails

By Chebyshev's inequality we always have

$$\mathbb{P}(\|S\| \geq e\|S\|_p) \leq e^{-p}, \quad p \geq 1.$$

If additionally $\|S\|_{2p} \leq \lambda\|S\|_p$ then the Paley-Zygmund inequality yields

$$\mathbb{P}\Big(\|S\| \geq \frac{1}{C(\lambda)}\|S\|_p\Big) \geq \min\Big\{\frac{1}{C(\lambda)}, e^{-p}\Big\}, \quad p \geq 1.$$

In particular for a finite set $r_1, \ldots, r_n > 0$ the moment bound

$$\|S\|_p \sim a_0 + \sum_{k=1}^{n} a_k p^{r_k}, \quad p \geq 1$$

is equivalent to two tail bounds for $t \geq 0$:

$$\mathbb{P}\Big(\|S\| \geq \frac{1}{C} + t\Big) \geq \frac{1}{C}\exp\Big(-C\min_{1\leq k\leq n}\Big(\frac{t}{a_k}\Big)^{1/r_k}\Big),$$

$$\mathbb{P}(\|S\| \geq Ca_0 + t) \leq 2\exp\Big(-\frac{1}{C}\min_{1\leq k\leq n}\Big(\frac{t}{a_k}\Big)^{1/r_k}\Big).$$

Real linear forms ($d = 1$, $F = \mathbb{R}$)

In the real case ($F = \mathbb{R}$) we have $\sum_i a_i g_i \overset{d}{=} \|a\|_2 g_1$, so for $p \geq 1$

$$\Big\| \sum_i a_i g_i \Big\|_p = \|a\|_2 \|g_1\|_p \sim \sqrt{p}\|a\|_2$$

and

$$\mathbb{P}\Big(\Big|\sum_i a_i g_i\Big| \geq t\Big) = \mathbb{P}\Big(|g_1| \geq \frac{t}{\|a\|_2}\Big) \sim \frac{\|a\|_2}{t + \|a\|_2} \exp\Big(-\frac{t^2}{2\|a\|_2^2}\Big).$$

We use the notation $f \sim g$ if $\frac{1}{C} f \leq g \leq Cf$, by C we denote universal constants.
We write $f \sim_\lambda g$ if $\frac{1}{C(\lambda)} f \leq g \leq C(\lambda) f$ and $C(\lambda)$ depends only on the parameter λ.

Vector-valued linear forms ($d = 1$, arbitrary F)

Gaussian concentration and integration by parts yield that for a_i from a normed space F we have

$$\Big\|\sum_i a_i g_i\Big\|_p \sim \mathbb{E}\Big\|\sum_i a_i g_i\Big\| + \sup_{\varphi\in F^*,\|\varphi\|\le 1}\Big\|\sum_i \varphi(a_i) g_i\Big\|_p$$
$$\sim \mathbb{E}\Big\|\sum_i a_i g_i\Big\| + \sqrt{p}\sup_{x\in B_2^n}\Big\|\sum_i a_i x_i\Big\|.$$

Equivalently, in terms of tails we have for $t \geq 0$,

$$\mathbb{P}\Big(\Big\|\sum_i a_i g_i\Big\| \geq t + \frac{1}{C}\mathbb{E}\Big\|\sum_i a_i g_i\Big\|\Big) \geq \frac{1}{C}\exp\left(-\frac{Ct^2}{\sup_{x\in B_2^n}\big\|\sum_i a_i x_i\big\|^2}\right),$$
$$\mathbb{P}\Big(\Big\|\sum_i a_i g_i\Big\| \geq t + C\mathbb{E}\Big\|\sum_i a_i g_i\Big\|\Big) \leq C\exp\left(-\frac{t^2}{C\sup_{x\in B_2^n}\big\|\sum_i a_i x_i\big\|^2}\right).$$

Real centered quadratic forms ($d = 2$, $F = \mathbb{R}$)

If $a_{ij} = a_{ji} \in \mathbb{R}$ then for $p \geq 1$ we have

$$\Big\| \sum_{ij} a_{ij}(g_i g_j - \delta_{ij}) \Big\|_p \sim \sqrt{p}\|A\|_{\mathrm{HS}} + p\|A\|_{\mathrm{op}}.$$

Equivalent tail bound for $t \geq 1$:

$$\begin{aligned} &\frac{1}{C}\exp\Big(-C\min\Big\{\frac{t^2}{\|A\|_{\mathrm{HS}}^2}, \frac{t}{\|A\|_{\mathrm{op}}}\Big\}\Big) \\ &\leq \mathbb{P}\Big(\Big|\sum_{ij} a_{ij}(g_i g_j - \delta_{ij})\Big| \geq t\Big) \\ &\leq C\exp\Big(-\frac{1}{C}\min\Big\{\frac{t^2}{\|A\|_{\mathrm{HS}}^2}, \frac{t}{\|A\|_{\mathrm{op}}}\Big\}\Big). \end{aligned}$$

The upper bound may be established in several ways - by concentration, rotational invariance of canonical Gaussian measure or by estimating the Laplace transform.

Borell-Arcones-Giné estimate

Application of the Gaussian concentration yields

Theorem (Borell'84, Arcones-Giné'93)

Let $(a_{ij})_{i,j\leq n}$ be a symmetric matrix with values in a normed space $(F, \|\ \|)$. Then for $p \geq 1$,

$$\left(\mathbb{E}\left\|\sum_{ij} a_{ij}(g_i g_j - \delta_{ij})\right\|^p\right)^{1/p} \sim \mathbb{E}\left\|\sum_{ij} a_{ij}(g_i g_j - \delta_{ij})\right\| + \sqrt{p}\,\mathbb{E}\sup_{\|x\|_2\leq 1}\left\|\sum_{ij} a_{ij} x_i g_j\right\| + p \sup_{\|x\|_2\leq 1, \|y\|_2\leq 1}\left\|\sum_{ij} a_{ij} x_i y_j\right\|.$$

Unfortunately the second term on the right hand side is usuallly difficult to estimate.

Easy lower bound

Proposition

For any symmetric F-valued matrix (a_{ij}) and $p \geq 1$,

$$\left(\mathbb{E}\left\|\sum_{ij} a_{ij}(g_i g_j - \delta_{ij})\right\|^p\right)^{1/p}$$

$$\geq \frac{1}{C}\left(\mathbb{E}\left\|\sum_{ij} a_{ij}(g_i g_j - \delta_{ij})\right\| + \sqrt{p}\sup_{\|x\|_2 \leq 1}\mathbb{E}\left\|\sum_{ij} a_{ij} x_i g_j\right\|\right.$$

$$\left. + \sqrt{p}\sup_{\|(x_{ij})\|_2 \leq 1}\left\|\sum_{ij} a_{ij} x_{ij}\right\| + p\sup_{\|x\|_2 \leq 1, \|y\|_2 \leq 1}\left\|\sum_{ij} a_{ij} x_i y_j\right\|\right).$$

Conjecture

The bound above may be reversed up to a universal constant.

Upper bound for moments

We are able to show the conjecture with an additional factor – $\mathbb{E}\|\sum_{i\neq j} a_{ij}g_{ij}\|$.

Theorem (Adamczak,L.,Meller'2020)

For any symmetric F-valued matrix (a_{ij}) and $p \geq 1$ we have

$$\left(\mathbb{E}\left\|\sum_{ij} a_{ij}(g_i g_j - \delta_{ij})\right\|^p\right)^{1/p} \leq C\Bigg(\mathbb{E}\left\|\sum_{ij} a_{ij}(g_i g_j - \delta_{ij})\right\|$$

$$+ \mathbb{E}\left\|\sum_{i\neq j} a_{ij}g_{ij}\right\| + \sqrt{p}\sup_{\|x\|_2\leq 1}\mathbb{E}\left\|\sum_{ij} a_{ij}x_i g_j\right\|$$

$$+ \sqrt{p}\sup_{\|(x_{ij})\|_2\leq 1}\left\|\sum_{ij} a_{ij}x_{ij}\right\| + p\sup_{\|x\|_2\leq 1,\|y\|_2\leq 1}\left\|\sum_{ij} a_{ij}x_i y_j\right\|\Bigg).$$

Tail bounds for Gaussian quadratic forms

Theorem (ALM'2020)

For any symmetric F-valued matrix (a_{ij}) and $t > 0$ we have

$$\mathbb{P}\Big(\Big\|\sum_{ij} a_{ij}(g_i g_j - \delta_{ij})\Big\| \geq t + C\Big(\mathbb{E}\Big\|\sum_{ij} a_{ij}(g_i g_j - \delta_{ij})\Big\| + \mathbb{E}\Big\|\sum_{i\neq j} a_{ij} g_{ij}\Big\|\Big)\Big)$$

$$\leq 2\exp\left(-\frac{1}{C}\min\left\{\frac{t^2}{U^2}, \frac{t}{V}\right\}\right),$$

where

$$U = \sup_{\|x\|_2 \leq 1} \mathbb{E}\Big\|\sum_{i\neq j} a_{ij} x_i g_j\Big\| + \sup_{\|(x_{ij})\|_2 \leq 1}\Big\|\sum_{ij} a_{ij} x_{ij}\Big\|,$$

$$V = \sup_{\|x\|_2 \leq 1, \|y\|_2 \leq 1}\Big\|\sum_{ij} a_{ij} x_i y_j\Big\|.$$

Moreover,

$$\mathbb{P}\Big(\Big\|\sum_{ij} a_{ij}(g_i g_j - \delta_{ij})\Big\| \geq t + \frac{1}{C}\mathbb{E}\Big\|\sum_{ij} a_{ij}(g_i g_j - \delta_{ij})\Big\|\Big)$$

$$\geq \frac{1}{C}\exp\left(-C\min\left\{\frac{t^2}{U^2}, \frac{t}{V}\right\}\right).$$

Alternative moment bounds

We may eliminate additional term $\mathbb{E}\|\sum_{i\neq j} a_{ij}g_{ij}\|$ by cost of additional logarithms. Namely we have the following bounds:

$$\Big\|\sum_{ij} a_{ij}(g_i g_j - \delta_{ij})\Big\|_p$$

$$\leq C\Bigg(\mathbb{E}\Big\|\sum_{ij} a_{ij}(g_i g_j - \delta_{ij})\Big\| + \sqrt{p}\sup_{\|x\|_2\leq 1}\mathbb{E}\Big\|\sum_{ij} a_{ij}x_i g_j\Big\|$$

$$+\sqrt{p}\ln(ep)\sup_{\|(x_{ij})\|_2\leq 1}\Big\|\sum_{ij} a_{ij}x_{ij}\Big\| + p\sup_{\|x\|_2\leq 1,\|y\|_2\leq 1}\Big\|\sum_{ij} a_{ij}x_i y_j\Big\|\Bigg).$$

and

$$\Big\|\sum_{ij} a_{ij}(g_i g_j - \delta_{ij})\Big\|_p$$

$$\leq C\Bigg(\ln(ep)\mathbb{E}\Big\|\sum_{ij} a_{ij}(g_i g_j - \delta_{ij})\Big\| + \sqrt{p}\sup_{\|x\|_2\leq 1}\mathbb{E}\Big\|\sum_{ij} a_{ij}x_i g_j\Big\|$$

$$+\sqrt{p}\sup_{\|(x_{ij})\|_2\leq 1}\Big\|\sum_{ij} a_{ij}x_{ij}\Big\| + p\ln(ep)\sup_{\|x\|_2\leq 1,\|y\|_2\leq 1}\Big\|\sum_{ij} a_{ij}x_i y_j\Big\|\Bigg).$$

Gaussian $\alpha+$ property

One may eliminate the term $\mathbb{E}\|\sum_{i\neq j} a_{ij}g_{ij}\|$ if the space $(F, \|\ \|)$ satisfy the following property:
There exists constant $K < \infty$ such that for any symmetric F-valued matrix (a_{ij})

$$\mathbb{E}\Big\|\sum_{i\neq j} a_{ij}g_{ij}\Big\| \leq K\mathbb{E}\Big\|\sum_{i\neq j} a_{ij}g_ig_j\Big\|. \qquad (\alpha+)$$

This property appears in the literature under the name *Gaussian property* $(\alpha+)$ and is closely related to Pisier's contraction property. It has found applications, e.g., in the theory of stochastic integration in Banach spaces. It holds for Banach spaces of type 2, and for Banach lattices $(\alpha+)$ is equivalent to finite cotype. In particular L_r spaces satisfy $(\alpha+)$ with $K \leq C\sqrt{r}$.

Two-sided bound under Gaussian $\alpha+$ property

Corollary (ALM'2020)

Let (a_{ij}) be a symmetric matrix with values in a normed space $(F, \| \ \|)$ which satisfies $(\alpha+)$. Then

$$\Big(\mathbb{E}\Big\|\sum_{ij} a_{ij}(g_i g_j - \delta_{ij})\Big\|^p\Big)^{1/p}$$
$$\sim_K \mathbb{E}\Big\|\sum_{i\neq j} a_{ij} g_i g_j\Big\| + \sqrt{p} \sup_{\|x\|_2 \le 1} \mathbb{E}\Big\|\sum_{ij} a_{ij} x_i g_j\Big\|$$
$$+ \sqrt{p} \sup_{\|(x_{ij})\|_2 \le 1} \Big\|\sum_{ij} a_{ij} x_{ij}\Big\| + p \sup_{\|x\|_2 \le 1, \|y\|_2 \le 1} \Big\|\sum_{ij} a_{ij} x_i y_j\Big\|.$$

Two-sided bounds in L_r-spaces

$L_r(X,\mu)$-spaces satisfy $(\alpha+)$, moreover one has

$$\frac{1}{C}\Big\|\sqrt{\sum_{ij} a_{ij}^2}\Big\|_{L_r} \le \mathbb{E}\Big\|\sum_{ij} a_{ij} g_{ij}\Big\|_{L_r} \le C\sqrt{r}\Big\|\sqrt{\sum_{ij} a_{ij}^2}\Big\|_{L_r},$$

$$\frac{1}{C}\Big\|\sqrt{\sum_{j}\Big(\sum_i a_{ij} x_i\Big)^2}\Big\|_{L_r} \le \mathbb{E}\Big\|\sum_{ij} a_{ij} x_i g_j\Big\|_{L_r} \le C\sqrt{r}\Big\|\sqrt{\sum_{j}\Big(\sum_i a_{ij} x_i\Big)^2}\Big\|_{L_r},$$

$$\frac{1}{C\sqrt{r}}\Big\|\sqrt{\sum_{ij} a_{ij}^2}\Big\|_{L_r} \le \mathbb{E}\Big\|\sum_{ij} a_{ij}(g_i g_j - \delta_{ij})\Big\|_{L_r} \le Cr\Big\|\sqrt{\sum_{ij} a_{ij}^2}\Big\|_{L_r}.$$

As a consequence we have

$$\Big(\mathbb{E}\Big\|\sum_{ij} a_{ij}(g_i g_j - \delta_{ij})\Big\|_{L_r}^p\Big)^{1/p}$$

$$\sim_r \Big\|\sqrt{\sum_{ij} a_{ij}^2}\Big\|_{L_r} + \sqrt{p}\sup_{\|x\|_2\le 1}\Big\|\sqrt{\sum_j\Big(\sum_i a_{ij}x_i\Big)^2}\Big\|_{L_r}$$

$$+\sqrt{p}\sup_{\|(x_{ij})\|_2\le 1}\Big\|\sum_{ij} a_{ij} x_{ij}\Big\|_{L_r} + p\sup_{\|x\|_2\le 1,\|y\|_2\le 1}\Big\|\sum_{ij} a_{ij} x_i y_j\Big\|_{L_r}.$$

Non-centered Gaussian quadratic forms

For non-centered Gaussian quadratic forms $S = \sum_{i,j} a_{ij} g_i g_j$ one has $\|S\|_p \sim \|\mathbb{E}S\| + \|S - \mathbb{E}S\|_p$, so we have symmetric vector-valued matrices (a_{ij}) and $p \geq 1$,

$$\frac{1}{C}\Bigg(\mathbb{E}\Big\|\sum_{ij} a_{ij} g_i g_j\Big\| + \sqrt{p}\sup_{\|x\|_2\leq 1}\mathbb{E}\Big\|\sum_{ij} a_{ij} x_i g_j\Big\|$$

$$+\sqrt{p}\sup_{\|(x_{ij})\|_2\leq 1}\Big\|\sum_{ij} a_{ij} x_{ij}\Big\| + p\sup_{\|x\|_2\leq 1,\|y\|_2\leq 1}\Big\|\sum_{ij} a_{ij} x_i y_j\Big\|\Bigg)$$

$$\leq \Big(\mathbb{E}\Big\|\sum_{ij} a_{ij} g_i g_j\Big\|^p\Big)^{1/p}$$

$$\leq C\Bigg(\mathbb{E}\Big\|\sum_{ij} a_{ij} g_i g_j\Big\| + \mathbb{E}\Big\|\sum_{i\neq j} a_{ij} g_{ij}\Big\| + \sqrt{p}\sup_{\|x\|_2\leq 1}\mathbb{E}\Big\|\sum_{ij} a_{ij} x_i g_j\Big\|$$

$$+\sqrt{p}\sup_{\|(x_{ij})\|_2\leq 1}\Big\|\sum_{ij} a_{ij} x_{ij}\Big\| + p\sup_{\|x\|_2\leq 1,\|y\|_2\leq 1}\Big\|\sum_{ij} a_{ij} x_i y_j\Big\|\Bigg).$$

Hanson-Wright inequality

One of the crucial inequalities related to bounds of Gaussian quadratic forms is the Hanson-Wright inequality.

Theorem (Hanson-Wright'71, Wright'73, Barthe-Milman'13, Rudelson-Vershynin'13)

For any sequence of independent mean zero α-subgaussian random variables $X_1, \ldots, X_n$ and any symmetric real-valued matrix $A = (a_{ij})_{i,j \le n}$ one has for $t > 0$

$$\mathbb{P}\left(\left| \sum_{i,j=1}^{n} a_{ij}(X_i X_j - \mathbb{E}(X_i X_j)) \right| \ge t \right) \le 2 \exp\left(-\frac{1}{C} \min\left\{ \frac{t^2}{\alpha^4 \|A\|_{\mathrm{HS}}}, \frac{t}{\alpha^2 \|A\|_{\mathrm{op}}} \right\} \right).$$

Recall that a r.v. X is *α-subgausian* if

$$\mathbb{P}(|X| \ge t) \le 2\exp(-t^2/2\alpha^2) \quad \text{for all } t > 0.$$

Normed space valued H-W inequality

Theorem (ALM'2020)

Let $X_1, X_2, \ldots, X_n$ be independent mean zero α-subgaussian random variables. Then for any symmetric matrix $(a_{ij})_{i,j\le n}$ with values in a normed space $(F, \|\cdot\|)$ and $t > C\alpha^2(\mathbb{E}\|\sum_{ij} a_{ij}(g_i g_j - \delta_{ij})\| + \mathbb{E}\|\sum_{i\ne j} a_{ij} g_{ij}\|)$ we have

$$\mathbb{P}\left(\left\|\sum_{ij} a_{ij}(X_i X_j - \mathbb{E}(X_i X_j))\right\| \ge t\right) \le 2\exp\left(-\frac{1}{C}\min\left\{\frac{t^2}{\alpha^4 U^2}, \frac{t}{\alpha^2 V}\right\}\right), \qquad \text{(HW)}$$

where

$$U = \sup_{\|x\|_2\le 1} \mathbb{E}\left\|\sum_{i\ne j} a_{ij} x_i g_j\right\| + \sup_{\|(x_{ij})\|_2\le 1}\left\|\sum_{ij} a_{ij} x_{ij}\right\|,$$

$$V = \sup_{\|x\|_2\le 1, \|y\|_2\le 1}\left\|\sum_{ij} a_{ij} x_i y_j\right\|.$$

How it works in the real case

It is not hard to check that in the case $F = \mathbb{R}$ we have $U \sim \|(a_{ij})\|_{\mathrm{HS}}$ and $V = \|(a_{ij})\|_{\mathrm{op}}$. Moreover,

$$\mathbb{E}\left\|\sum_{ij} a_{ij}(g_i g_j - \delta_{ij})\right\| + \mathbb{E}\left\|\sum_{i \neq j} a_{ij} g_{ij}\right\| \leq 2\|(a_{ij})\|_{\mathrm{HS}},$$

so the right hand side of (HW) is at least 1 for $t < C'(\mathbb{E}\|\sum_{ij} a_{ij}(g_i g_j - \delta_{ij})\| + \mathbb{E}\|\sum_{i \neq j} a_{ij} g_{ij}\|)$ and sufficiently large C. Hence (HW) holds for any $t > 0$ in the real case and is equivalent to the classical Hanson-Wright bound.

H-W bound in L_r-spaces, $1 \leq r < \infty$

Corollary (ALM'2020)

Let $X_1, X_2, \ldots, X_n$ be independent mean zero α-subgaussian random variables. Then for any symmetric matrix $(a_{ij})_{i,j\leq n}$ with values in $L_r = L_r(X,\mu)$, and $t > C\alpha^2 r \|\sqrt{\sum_{ij} a_{ij}^2}\|_{L_r}$,

$$\mathbb{P}\left(\left\|\sum_{ij} a_{ij}(X_iX_j - \mathbb{E}(X_iX_j))\right\|_{L_r} \geq t\right) \leq 2\exp\left(-\frac{1}{C}\min\left\{\frac{t^2}{\alpha^4 rU^2}, \frac{t}{\alpha^2 V}\right\}\right),$$

where

$$U = \sup_{\|x\|_2\leq 1}\left\|\sqrt{\sum_j\left(\sum_{i\neq j} a_{ij}x_i\right)^2}\right\|_{L_r} + \sup_{\|(x_{ij})\|_2\leq 1}\left\|\sum_{ij} a_{ij}x_{ij}\right\|_{L_r},$$

$$V = \sup_{\|x\|_2\leq 1, \|y\|_2\leq 1}\left\|\sum_{ij} a_{ij}x_iy_j\right\|_{L_r}.$$

H-W bound in type 2 spaces

Corollary (ALM'2020)

Let $X_1, X_2, \ldots, X_n$ be independent mean zero α-subgaussian random variables and let F be a normed space of type two constant λ. Then for any symmetric matrix $(a_{ij})_{i,j\le n}$ with values in F and $t > C\lambda^2\alpha^2\sqrt{\sum_{ij}\|a_{ij}\|^2}$ we have

$$\mathbb{P}\left(\left\|\sum_{ij} a_{ij}(X_iX_j - \mathbb{E}(X_iX_j))\right\| \ge t\right) \le 2\exp\left(-\frac{1}{C}\min\left\{\frac{t^2}{\alpha^4U^2}, \frac{t}{\alpha^2 V}\right\}\right),$$

where

$$U = \lambda \sup_{\|x\|_2\le 1}\sqrt{\sum_j\left\|\sum_i a_{ij}x_i\right\|^2} + \sup_{\|(x_{ij})\|_2\le 1}\left\|\sum_{ij} a_{ij}x_{ij}\right\|,$$

$$V = \sup_{\|x\|_2\le 1, \|y\|_2\le 1}\left\|\sum_{ij} a_{ij}x_iy_j\right\|.$$

Higher order real Gaussian chaoses

By $\mathcal{P}(U)$ we denote the family of (unordered) partitions of U into nonempty, pairwise disjoint sets. For $\mathcal{P} = \{I_1, \ldots, I_k\} \in \mathcal{P}([d])$ and $A = (a_{i_1,\ldots,i_d})$ we define

$$\|A\|_{\mathcal{P}} := \sup\Big\{\sum_{i_1,\ldots,i_d} a_{i_1,\ldots,i_d} \prod_{r=1}^{k} x^{(r)}_{i_{I_r}} \Big| \ \forall_{r\le k} \sum_{i_{I_r}} \Big(x^{(r)}_{i_{I_r}}\Big)^2 \le 1\Big\}.$$

Theorem (L'2006)

Assume that $A = (a_{i_1,\ldots,i_d})$ is a finite real tetrahedral symmetric matrix. Then for any $p \ge 1$,

$$\Big\|\sum_{i_1,\ldots,i_d} a_{i_1,\ldots,i_d} g_{i_1}\cdots g_{i_d}\Big\|_p \sim_d \sum_{\mathcal{P}\in\mathcal{P}([d])} p^{|\mathcal{P}|/2}\|A\|_{\mathcal{P}}.$$

and for $t \ge 0$,

$$\frac{1}{C_d}\exp\Big(-C_d \min_{\mathcal{P}\in\mathcal{P}([d])}\Big(\frac{t}{\|A\|_{\mathcal{P}}}\Big)^{2/|\mathcal{P}|}\Big)$$
$$\le \mathbb{P}\Big(\Big|\sum_{i_1,\ldots,i_d} a_{i_1,\ldots,i_d} g_{i_1}\cdots g_{i_d}\Big| \ge t\Big) \le 2\exp\Big(-\frac{1}{C_d}\min_{\mathcal{P}\in\mathcal{P}([d])}\Big(\frac{t}{\|A\|_{\mathcal{P}}}\Big)^{2/|\mathcal{P}|}\Big).$$

Moment bound for $d = 3$, $F = \mathbb{R}$

For $A = (a_{ijk})$ we have

$$\|A\|_{\{1,2,3\}} = \sup\left\{\sum_{ijk} a_{ijk}x_{ijk} \colon \sum_{ijk} x_{ijk}^2 \leq 1\right\} = \Big(\sum_{ijk} a_{ijk}^2\Big)^{1/2},$$

$$\begin{aligned}\|A\|_{\{1\}\{2,3\}} \sup\left\{\sum_{ijk} a_{ijk}x_i y_{jk} \colon \sum_i x_i^2 \leq 1, \sum_{jk} y_{jk}^2 \leq 1\right\} \\ = \sup\left\{\Big(\sum_{jk}\Big(\sum_i a_{ijk}x_i\Big)^2\Big)^{1/2} \colon \sum_i x_i^2 \leq 1\right\} \\ = \sup\left\{\Big(\sum_i\Big(\sum_{jk} a_{ijk}y_{jk}\Big)^2\Big)^{1/2} \colon \sum_{jk} y_{jk}^2 \leq 1\right\},\end{aligned}$$

$$\|A\|_{\{1\}\{2\}\{3\}} = \sup\left\{\sum_{ijk} a_{ijk}x_i y_j z_k \colon \sum_i x_i^2 \leq 1, \sum_j y_j^2 \leq 1, \sum_k z_k^2 \leq 1\right\}$$

and under symmetry assumptions (i.e. $a_{iik} = 0$ and $a_{ijk} = a_{ikj} = a_{jik} = a_{jki} = a_{kij} = a_{kji}$) we have for $p \geq 1$,

$$\Big\|\sum_{ijk} a_{ijk}g_i g_j g_k\Big\|_p \sim \sqrt{p}\|A\|_{\{1,2,3\}} + p\|A\|_{\{1\}\{2,3\}} + p^{\frac{3}{2}}\|A\|_{\{1\}\{2\}\{3\}}$$

Tail Bounds for real chaoses of order 3

Under the symmetry assumptions we have for $t \geq 0$,

$$\frac{1}{C}\exp\Bigg(-C\min\Bigg\{\Big(\frac{t}{\|A\|_{\{1,2,3\}}}\Big)^2,\Big(\frac{t}{\|A\|_{\{1\}\{2,3\}}}\Big),\Big(\frac{t}{\|A\|_{\{1\}\{2\}\{3\}}}\Big)^{2/3}\Bigg\}\Bigg)$$

$$\leq \mathbb{P}\Big(\Big|\sum_{ijk} a_{ijk}g_i g_j g_k\Big| \geq t\Big)$$

$$C\exp\Bigg(-\frac{1}{C}\min\Bigg\{\Big(\frac{t}{\|A\|_{\{1,2,3\}}}\Big)^2,\Big(\frac{t}{\|A\|_{\{1\}\{2,3\}}}\Big),\Big(\frac{t}{\|A\|_{\{1\}\{2\}\{3\}}}\Big)^{2/3}\Bigg\}\Bigg).$$

Notation for vector-valued chaoses of higher order

We write $(\mathcal{P}, \mathcal{P}') \in \mathcal{P}(U)$ if $\mathcal{P} \cup \mathcal{P}' \in \mathcal{P}(U)$ and $\mathcal{P} \cap \mathcal{P}' = \emptyset$. Let $\mathcal{P} = \{I_1, \ldots, I_k\}$, $\mathcal{P}' = \{J_1, \ldots, J_m\}$ be such that $(\mathcal{P}, \mathcal{P}') \in \mathcal{P}([d])$. We set

$$\|A\|_{\mathcal{P}'|\mathcal{P}} := \sup\Big\{\mathbb{E}\Big\| \sum_{i_1,\ldots,i_d} a_{i_1,\ldots,i_d} \prod_{r=1}^{k} x^{(r)}_{i_{I_r}} \prod_{l=1}^{m} g_{i_{J_l}} \Big\| \;\Big|\; \forall_{r\le k} \sum_{i_{I_r}} \Big(x^{(r)}_{i_{I_r}}\Big)^2 \le 1\Big\},$$

$$\|A\|_{\mathcal{P}} := \sup\Bigg\{\mathbb{E}\Bigg\| \sum_{i_1,\ldots,i_d} a_{i_1,\ldots,i_d} \prod_{r=1}^{k} x^{(r)}_{i_{I_r}} \prod_{l\in[d]\setminus(\bigcup \mathcal{P})} g_{i_l} \Bigg\| \;\Bigg| \; \forall_{r\le k} \sum_{i_{I_r}} \Big(x^{(r)}_{i_{I_r}}\Big)^2 \le 1\Bigg\}.$$

We do not exclude the situation that $\mathcal{P}'$ or $\mathcal{P}$ is an empty partition. In the first case $\|A\|_{\mathcal{P}} = \|A\|_{\mathcal{P}'|\mathcal{P}}$ is defined in non-probabilistic terms. Another case when $\|A\|_{\mathcal{P}} = \|A\|_{\mathcal{P}'|\mathcal{P}}$ is when $\mathcal{P}'$ consists of singletons only.

Moment bound for vector-valued chaoses of arbitrary order

Theorem (ALM'2020+)

Assume that $A = (a_{i_1,\dots,i_d})$ is a finite tetrahedral symmetric matrix with values in $(F, \|\ \|)$. Then for any $p \geq 1$,

$$\frac{1}{C(d)} \sum_{J \subset [d]} \sum_{\mathcal{P} \in \mathcal{P}(J)} p^{|\mathcal{P}|/2} \|A\|_{\mathcal{P}} \leq \Big\| \sum_{i_1,\dots,i_d} a_{i_1,\dots,i_d} g_{i_1} \cdots g_{i_d} \Big\|_p$$

$$\leq C(d) \sum_{(\mathcal{P},\mathcal{P}') \in \mathcal{P}([d])} p^{|\mathcal{P}|/2} \|A\|_{\mathcal{P}'|\mathcal{P}}.$$

Conjecture

Under the assumption of the theorem we have

$$\Big\| \sum_{i_1,\dots,i_d} a_{i_1,\dots,i_d} g_{i_1} \cdots g_{i_d} \Big\|_p \leq C(d) \sum_{J \subset [d]} \sum_{\mathcal{P} \in \mathcal{P}(J)} p^{|\mathcal{P}|/2} \|A\|_{\mathcal{P}}.$$

Examples for $d = 3$, $A = (a_{ijk})$

$$\|A\|_{\emptyset|\{1,2,3\}} = \|A\|_{\{1,2,3\}} = \sup_{\sum_{i,j,k} x_{ijk}^2 \le 1} \Big\| \sum a_{ijk} x_{ijk} \Big\|,$$

$$\|A\|_{\emptyset|\{1\},\{2,3\}} = \|A\|_{\{1\},\{2,3\}} = \sup_{\sum_i x_i^2 \le 1, \sum_{jk} y_{jk}^2 \le 1} \Big\| \sum_{ijk} a_{ijk} x_i y_{jk} \Big\|,$$

$$\|A\|_{\emptyset|\{1\},\{2\},\{3\}} = \|A\|_{\{1\},\{2\},\{3\}} = \sup_{\sum_i x_i^2 \le 1, \sum_j y_j^2 \le 1, \sum_k z_k^2 \le 1} \Big\| \sum_{ijk} a_{ijk} x_i y_j z_k \Big\|.$$

$$\|A\|_{\{1,2\},\{3\}|\emptyset} = \mathbb{E}\Big\| \sum_{i,j,k} a_{ijk} g_{ij} g_k \Big\|,$$

$$\|A\|_{\{1\}|\{2\},\{3\}} = \|A\|_{\{2\},\{3\}} = \sup_{\sum_j x_j^2 \le 1, \sum y_k^2 \le 1} \mathbb{E}\Big\| \sum_{i,j,k} a_{ijk} g_i x_j y_k \Big\|,$$

$$\|A\|_{\{1\},\{2\},\{3\}|\emptyset} = \|A\|_{\emptyset} = \mathbb{E}\Big\| \sum_{ijk} a_{ijk} g_i g_j g_k \Big\|,$$

$$\|A\|_{\{1\},\{2\}|\{3\}} = \|A\|_{\{3\}} = \sup_{\sum_j x_j^2 \le 1} \mathbb{E}\Big\| \sum_{i,j,k} a_{ijk} g_i g_j x_k \Big\|.$$

Tail bounds for vector-valued chaoses of arbitrary order

Theorem (ALM'2020+)

Under the symmetry assumptions we have for $t \geq 0$,

$$\mathbb{P}\Big(\Big\|\sum_{i_1,\ldots,i_d} a_{i_1,\ldots,i_d} g_{i_1}\cdots g_{i_d}\Big\| \geq t + C(d)\sum_{\mathcal{P}'\in\mathcal{P}([d])}\|A\|_{\mathcal{P}'|\emptyset}\Big)$$

$$\leq 2\exp\Big(-\frac{1}{C(d)}\min_{\substack{(\mathcal{P},\mathcal{P}')\in\mathcal{P}([d])\\ |\mathcal{P}|>0}}\Big(\frac{t}{\|A\|_{\mathcal{P}}}\Big)^{2/|\mathcal{P}|}\Big),$$

and

$$\mathbb{P}\Big(\Big\|\sum_{i_1,\ldots,i_d} a_{i_1,\ldots,i_d} g_{i_1}\cdots g_{i_d}\Big\| \geq \frac{1}{C(d)}\mathbb{E}\Big\|\sum_{i_1,\ldots,i_d} g_{i_1}\cdots g_{i_d}\Big\| + t\Big)$$

$$\geq \frac{1}{C(d)}\exp\Big(-C(d)\min_{\emptyset\neq J\subset[d]}\min_{\mathcal{P}\in\mathcal{P}(J)}\Big(\frac{t}{\|A\|_{\mathcal{P}'|\mathcal{P}}}\Big)^{2/|\mathcal{P}|}\Big).$$

Bounds for vector-valued chaoses of order 3

Under the symmetry assumptions we have

$$\frac{1}{C}S_1 \le \Big(\mathbb{E}\Big\|\sum_{ijk} a_{ijk}g_ig_jg_k\Big\|^p\Big)^{1/p} \le C(S_1+S_2),$$

$$\begin{aligned}
S_1 &:= \mathbb{E}\Big\|\sum_{ijk} a_{ijk}g_ig_jg_k\Big\| \\
&+ p^{1/2}\Big(\sup_{\|x\|_2\le 1}\mathbb{E}\Big\|\sum_{ijk} a_{ijk}g_ig_jx_k\Big\| + \sup_{\|x\|_2\le 1}\mathbb{E}\Big\|\sum_{ijk} a_{ijk}g_ix_{jk}\Big\| + \sup_{\|x\|_2\le 1}\Big\|\sum_{ijk} a_{ijk}x_{ijk}\Big\|\Big) \\
&+ p\Big(\sup_{\|x\|_2,\|y\|_2\le 1}\mathbb{E}\Big\|\sum_{ijk} a_{ijk}g_ix_jy_k\Big\| + \sup_{\|x\|_2,\|y\|_2\le 1}\Big\|\sum_{ijk} a_{ijk}x_{ij}y_k\Big\|\Big) \\
&+ p^{3/2}\sup_{\|x\|_2,\|y\|_2,\|z\|_2\le 1}\Big\|\sum_{ijk} a_{ijk}x_iy_jz_k\Big\| \\
S_2 &:= \mathbb{E}\Big\|\sum_{ijk} a_{ijk}g_{ijk}\Big\| + \mathbb{E}\Big\|\sum_{ijk} a_{ijk}g_{ij}g_k\Big\| + p^{1/2}\sup_{\|x\|_2\le 1}\mathbb{E}\Big\|\sum_{ijk} a_{ijk}g_{ij}x_k\Big\|.
\end{aligned}$$

Bounds under $(\alpha+)$ condition

Lemma

Assume that $(F, \| \ \|)$ *satisfies* $(\alpha+)$ *condition and* $(\mathcal{P}, \mathcal{P}') \in \mathcal{P}([d])$. *Then*

$$\|A\|_{\mathcal{P}'|\mathcal{P}} \leq K^{|\bigcup \mathcal{P}'| - |\mathcal{P}'|} \|A\|_{\mathcal{P}}.$$

Corollary (ALM'2020+)

Let $(F, \| \ \|)$ *satisfy* $(\alpha+)$ *condition. Then for* $p \geq 1$,

$$\Big\| \sum_{i_1,\ldots,i_d} a_{i_1,\ldots,i_d} g_{i_1} \cdots g_{i_d} \Big\|_p \sim_{d,K} \sum_{J \subset [d]} \sum_{\mathcal{P} \in \mathcal{P}(J)} p^{|\mathcal{P}|/2} \|A\|_{\mathcal{P}}.$$

Bounds in L_r-spaces

L_r spaces satisfy $(\alpha+)$ condition with $K = C\sqrt{r}$. Moreover as in the case $d = 2$ one may show that for a matrix $A = (a_{i_1,\ldots,i_d})$ with values in L_r, $J \subset [d]$ and $\mathcal{P} = \{(I_1,\ldots,I_k)\} \in \mathcal{P}([d] \setminus J)$ we have

$$\|A\|_{\mathcal{P}'|\mathcal{P}} \sim_{r,d} \|A\|_{\mathcal{P}}^{L_r} := \sup\left\{ \left\| \sqrt{\sum_{i_{[d]\setminus J}} \left(\sum_{i_J} a_{i_1,\ldots,i_d} \prod_{r=1}^{k} x_{i_{I_r}}^r \right)^2} \right\|_{L_q} \right| \\ \forall_{r\leq k} \sum_{i_{I_r}} \left(x_{i_{I_r}}^r \right)^2 \leq 1 \right\}.$$

Theorem (ALM'2020+)

Let $A = (a_{i_1,\ldots,i_d})$ be an $L_r(X,\mu)$-valued matrix satisfying symmetry assumptions, $1 \leq r < \infty$. Then $p \geq 1$,

$$\left\| \sum_{i_1,\ldots,i_d} a_{i_1,\ldots,i_d} g_{i_1} \cdots g_{i_d} \right\|_p \sim_{r,d} \sum_{J\subset[d]} \sum_{\mathcal{P}\in\mathcal{P}([J])} p^{\frac{|\mathcal{P}|}{2}} \|A\|_{\mathcal{P}}^{L_r}.$$

Arbitrary polynomials

The following proposition enables to reduce moment bounds for arbitrary polynomials to previously considered homogenous case.

Proposition (ALM'2020+,Adamczak-Wolff'15 for $F = \mathbb{R}$)

Let G be a standard Gaussian vector in $\mathbb{R}^n$ and $f: \mathbb{R}^n \to (F, \|\|)$ be a polynomial of degree D. Then for $p \geq 1$,

$$\|f(G) - \mathbb{E}f(G)\|_p \sim_D \sum_{d=1}^{D} \Big\| \sum_{i_1,\ldots,i_d=1}^{n} a^{(d)}_{i_1,\ldots,i_d} g^{(1)}_{i_1} \cdots g^{(d)}_{i_d} \Big\|_p,$$

where $(a^{(d)}_{i_1,\ldots,i_d})_{i_1,\ldots,i_d \leq n} := \mathbb{E}\nabla^d f(G)$.

Example Let f be a general polynomial with values in $(F, \| \ \|)$

$$f(G) = \sum_{i,j,k=1}^{n} a_{ijk} g_i g_j g_k + \sum_{i,j=1}^{n} b_{ij} g_i g_j + \sum_{i=1}^{n} c_i g_i + d,$$

the matrices $(a_{ijk})_{ijk}$, $(b_{ij})_{ij}$ are symmetric. Then

$$\mathbb{E}\nabla f(G) = \Big(c_i + 3\sum_{j=1}^{n} a_{ijj}\Big)_i, \quad \mathbb{E}\nabla^2 f(G) = 2(b_{ij})_{i,j}, \quad \mathbb{E}\nabla^3 f(G) = 6(a_{ijk})_{i,j,k}.$$

Crucial tool - decoupling

Let $(g_i^{(k)})_{i,k\geq 1}$ Be independent $\mathcal{N}(0,1)$ random variables and

$$S^{\mathrm{dec}} = \sum_{i_1,\ldots,i_d} a_{i_1,\ldots,i_d} g_{i_1}^{(1)} \cdots g_{i_d}^{(d)}$$

be the decoupled version of the Gaussian chaos

$$S = \sum_{i_1,\ldots,i_d} a_{i_1,\ldots,i_d} g_{i_1} \cdots g_{i_d}.$$

Theorem (Kwapień (1987), de la Peña, Montgomery-Smith (1994))

Let $(a_{i_1,\ldots,i_d})$ be tetrahedral and symmetric F-valued matrix. Then for $p \geq 1$,

$$C(d)^{-1}\|S^{\mathrm{dec}}\|_p \leq \|S\|_p \leq C(d)\|S^{\mathrm{dec}}\|_p.$$

Moreover for $t \geq 0$,

$$C(d)^{-1}\mathbb{P}(|S^{\mathrm{dec}}| \geq C(d)t) \leq \mathbb{P}(|S| \geq t) \leq C(d)\mathbb{P}(|S^{\mathrm{dec}}| \geq t/C(d)).$$

Reduction to a bound on supremum of Gaussian process

The crucial part of the proof (for $d = 2$) is to bound the quantity

$$\mathbb{E} \sup_{\|x\|_2 \le 1} \left\| \sum_{ij} a_{ij} g_i x_j \right\|.$$

We may assume that $F = \mathbb{R}^n$, choosing for T the unit ball in F^* we need to show that

$$\mathbb{E} \sup_{\|x\|_2 \le 1, t \in T} \left| \sum_{ijk} a_{ijk} g_i x_j t_k \right|$$

is essentially bounded by

$$\frac{1}{\sqrt{p}} \mathbb{E} \sup_{t \in T} \left| \sum_{ijk} a_{ijk} g_i g'_j t_k \right| + \sup_{\|x\|_2 \le 1} \mathbb{E} \sup_{t \in T} \left| \sum_{ijk} a_{ijk} g_i x_j t_k \right|$$

$$+ \sup_{t \in T} \left(\sum_{ij} \left(\sum_{k} a_{ijk} t_k \right)^2 \right)^{1/2} + \sqrt{p} \sup_{\|x\|_2 \le 1, t \in T} \left(\sum_{i} \left(\sum_{jk} a_{ijk} x_j t_k \right)^2 \right)^{1/2}.$$

Entropy bound

We have a Gaussian process on $V = B_2^n \times T$ with the L_2-distance

$$d((x,t),(x',t')) := \left(\sum_i \left(\sum_{jk} a_{ijk}(x_j t_k - x_j' t_k') \right)^2 \right)^{1/2} .$$

It is not hard to check that

$$\mathrm{diam}(V) \sim \sup_{\|x\|_2 \leq 1, t \in T} \left(\sum_i \left(\sum_{jk} a_{ij} x_j t_k \right)^2 \right)^{1/2}$$

Proposition

For $\delta > 0$ we have

$$\sqrt{\log N(V,d,\delta)} \leq C\Bigg(\delta^{-1/2} \left(\mathbb{E} \sup_{t \in T} \left| \sum_{ijk} a_{ijk} g_i g_j' t_k \right| \right)^{1/2}$$

$$+ \delta^{-1} \left(\sup_{\|x\|_2 \leq 1} \mathbb{E} \sup_{t \in T} \left| \sum_{ijk} a_{ijk} g_i x_j t_k \right| + \sup_{t \in T} \left(\sum_{ij} \left(\sum_k a_{ijk} t_k \right)^2 \right)^{1/2} \right) \Bigg)$$

Chaining argument

Unfortunately the Dudley bound

$$\mathbb{E} \sup_{v \in V} G_v \leq C \int_0^{\mathrm{diam}(V)} \sqrt{\log N(V, d, \delta)} d\delta$$

does not work.

So instead we need to use chaining. We use the classical Sudakov bound in $\mathbb{R}^{n^2}$:

$$\sup_{\delta > 0} \delta \log^{1/2} N(\mathcal{A}, d_{HS}, \delta) \leq C\mathbb{E} \sup_{(a_{ij}) \in \mathcal{A}} \sum_{ij} a_{ij} g_{ij},$$

but then we have to add the additional term $\mathbb{E}\| \sum_{ij} a_{ij} g_{ij} \|$.

Alternatively, we may use Talagrand Sudakov-type bounds for suprema of Gaussian chaoses:

$$\sup_{\delta > 0} \delta \log^{1/4} N(\mathcal{A}, d_{HS}, \delta) \leq C\mathbb{E} \sup_{(a_{ij}) \in \mathcal{A}} \sum_{ij} a_{ij} g_i g_j'$$

but then additional logarithmic factors appear.

Bibliograhy

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Thank you for your attention!