Paving property for strongly Rayleigh measures

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Joint work with **Kasra Alishahi**

Geometry of Polynomials Reunion Workshop

Strong Rayleigh point processes

$\mathfrak{X} \subseteq [n]$ random subset & $\mathfrak{X} \sim \mu$

Alternatively, $\mathfrak{X}$ is a random $\{0,1\}$-vector

$\mathfrak{X}$ (or μ) is *strongly Rayleigh* if

$$p_{\mathfrak{X}}(\boldsymbol{z}) = \sum_{A \subseteq [n]} \mathbb{P}(\mathfrak{X} = A)\, \boldsymbol{z}^A$$

is (real) stable

$\boldsymbol{z}^A := \prod_{i \in A} z_i$

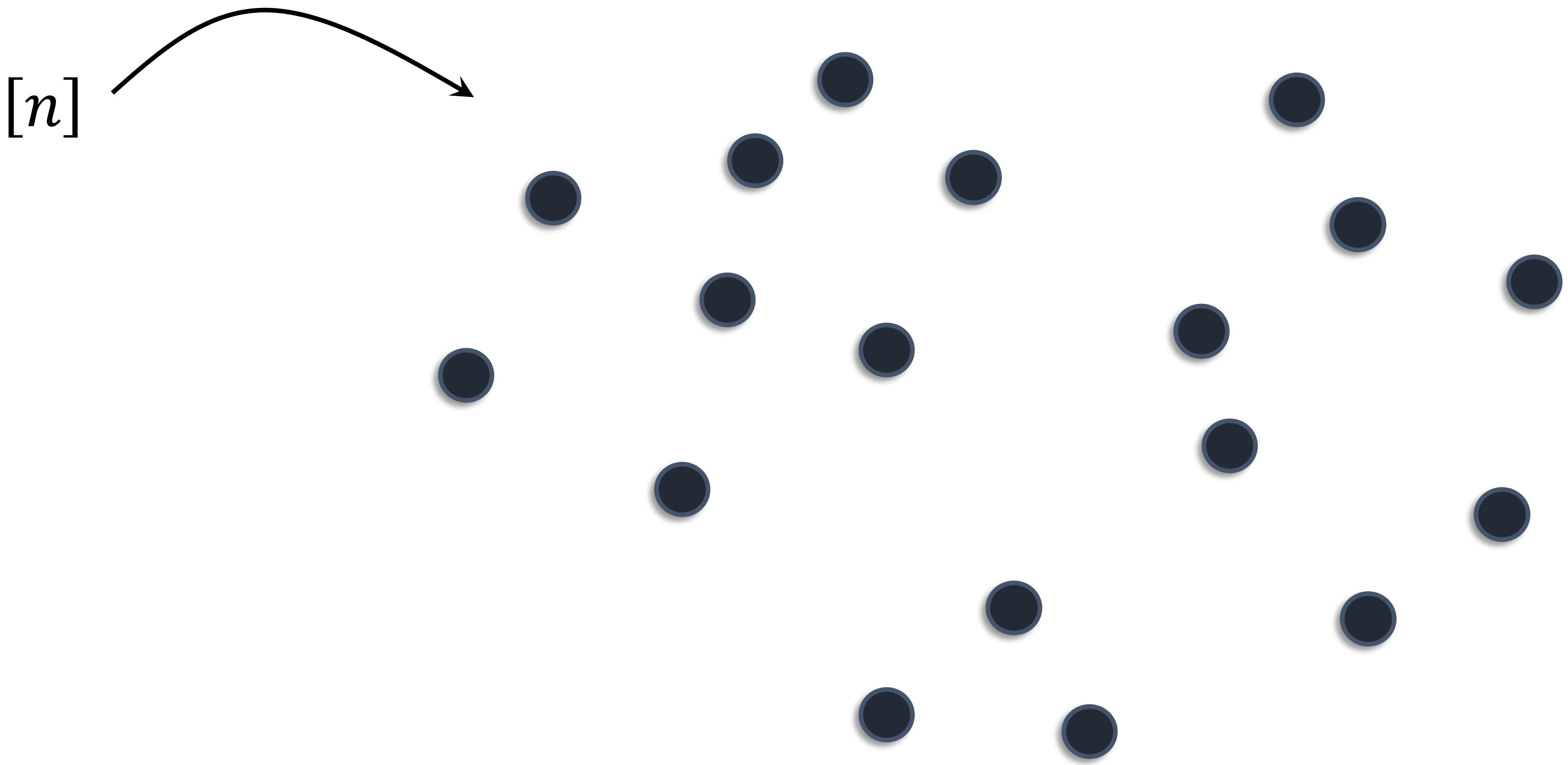
Borcea, Brändén, Liggett '08:

Strong Rayleighness is the most "natural" notion of negative dependence!

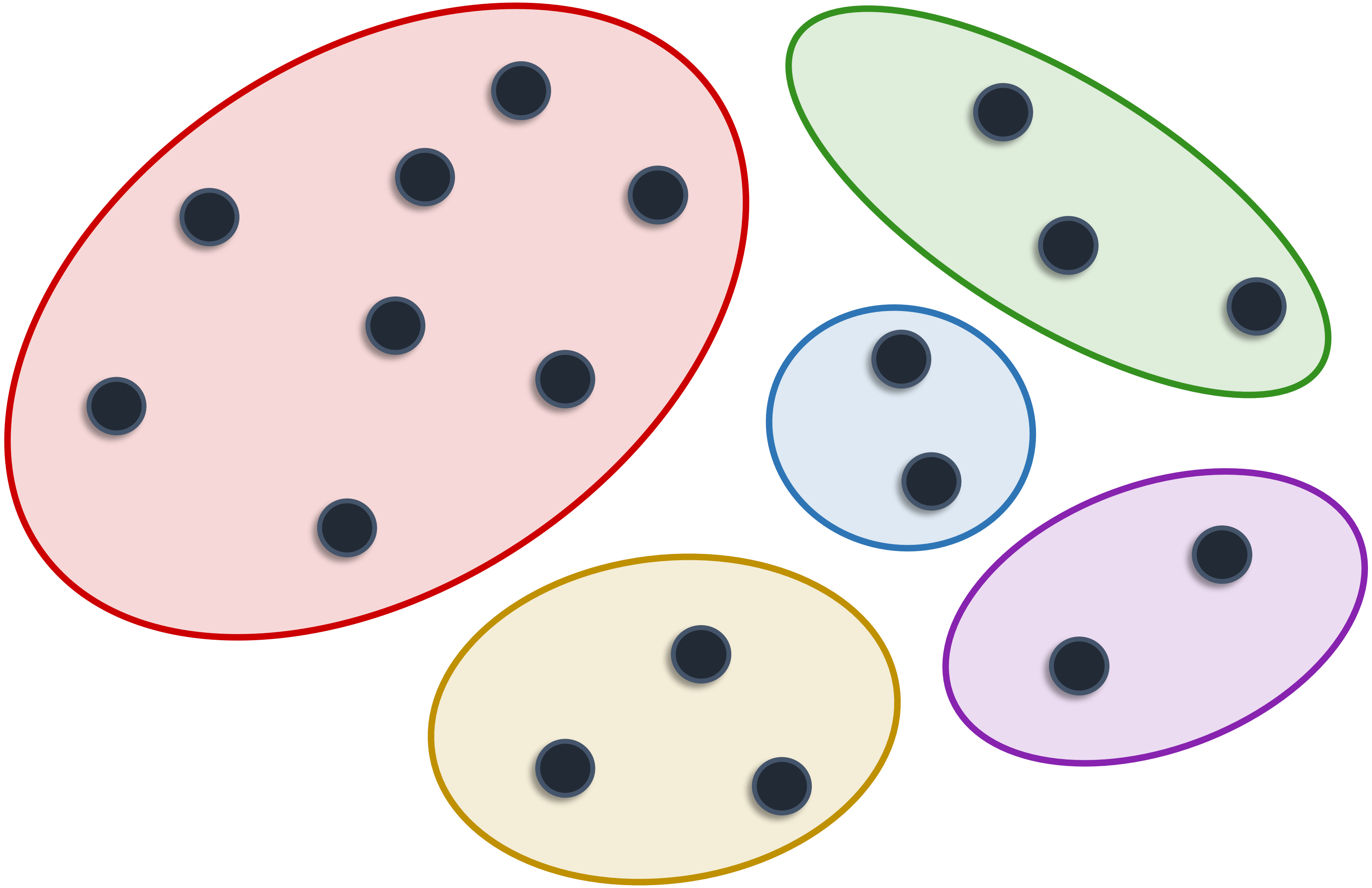
↓

- Relatively easy to check
- Includes almost all of the known examples
- Strong consequences
- Many applications

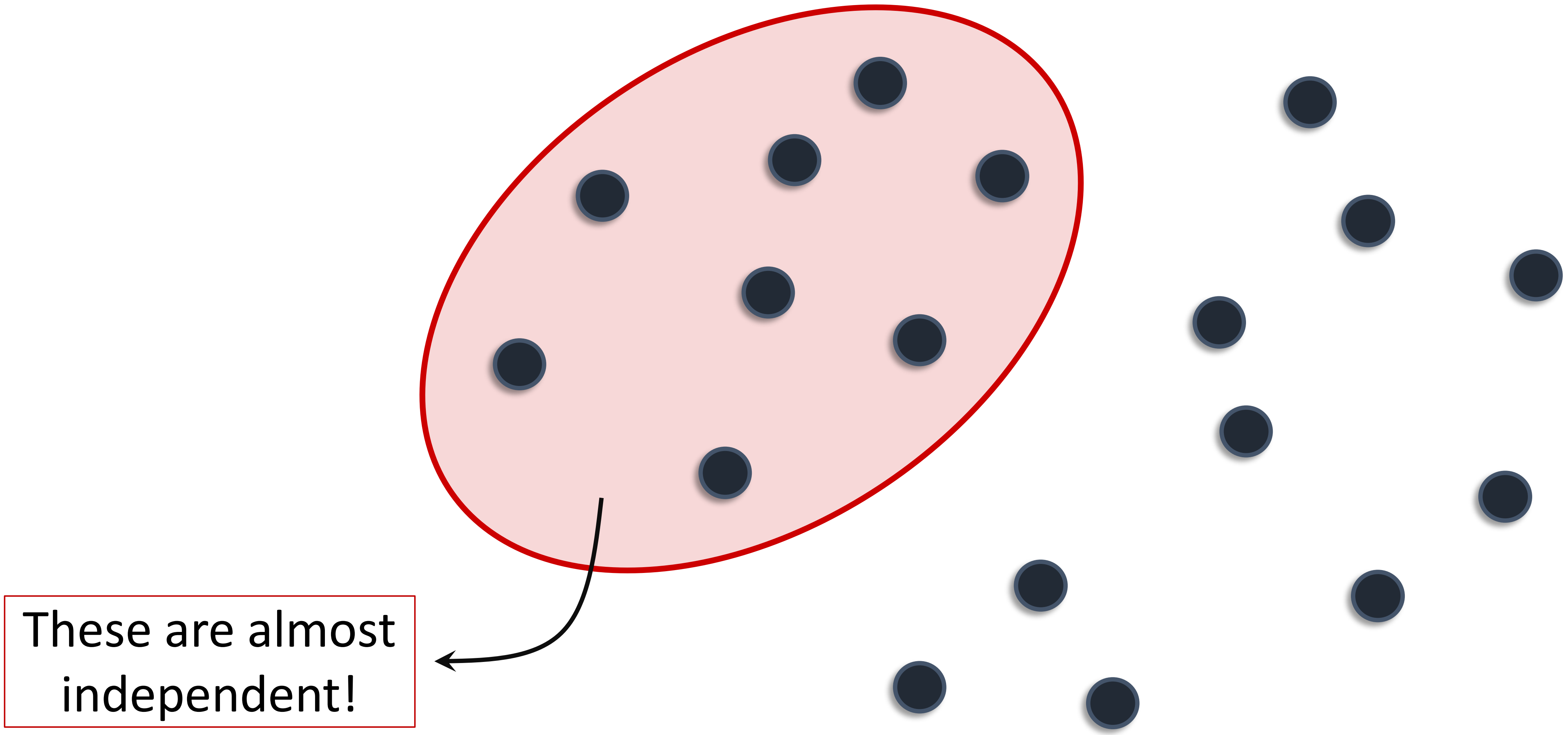
The idea



The idea



The idea



Claim

This resembles the paving conjecture!

The paving conjecture (Proved by Marcus, Spielman, Srivastava)

For every $\varepsilon > 0$ there is a $r = r(\varepsilon) \in \mathbb{N}$ such that every **Hermitian matrix** M with **zero diagonal** can be "(r, ε)-paved".

If M is $n \times n$ then $[n]$ can be partitioned into r sets $S_1, \dots, S_r$ such that

$$\forall k \in \{1, \dots, r\} : \|M[S_k]\| \leq \varepsilon \|M\|$$

(Discrete) Determinantal point processes

$\mathfrak{X} \subseteq [n]$ random subset

Kernel

$\mathfrak{X}$ is *determinantal* if there is a Hermitian K such that

$$\mathbb{P}(A \subseteq \mathfrak{X}) = \det K[A]$$

for all $A \subseteq [n]$

Some facts about DPPs

- K is kernel $\Leftrightarrow 0 \preccurlyeq K \preccurlyeq I$
- $\mathfrak{X}$ is independent $\Leftrightarrow K$ is diagonal
- $\mathfrak{X} \cap A$ is determinantal with kernel $K[A]$
- Is strongly Rayleigh with PGP $\det(KZ + I - K)$

$Z = \mathrm{Diag}(z_1, \ldots, z_n)$

Paving property for DPP

Apply the paving theorem to $K_0 = K - D$

$$\Downarrow$$

$$\begin{cases} \{1, \ldots, n\} = S_1 \sqcup \cdots \sqcup S_r \\ \forall k \in [r] : \|K[S_k] - D[S_k]\| \leq \varepsilon \end{cases}$$

$$\Downarrow ?$$

$\mathfrak{X} \cap S_k$ is almost independent

Problems

- How to deduce a probabilistic statement?
- How to deal with the strongly Rayleigh case?

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Extend the paving theorem to real stable polynomials!

Marcus-Spielman-Srivastava's paper

Interlacing polynomials + multivariate barrier method

$$\Downarrow$$

Weaver's vector balancing conjecture

$$\Downarrow$$

Paving conjecture

Marcus-Spielman-Srivastava's paper

Marcus, Spielman, Srivastava '15

$r \geq 1$ integer. Then for every **Hermitian matrix** A with **zero diagonal** there exists a partition $S_1 \sqcup \cdots \sqcup S_{r^2} = [n]$ such that

$$\forall k \in [r^2] : \|A[S_k]\| \leq \left(\frac{2\sqrt{2}}{\sqrt{r}} + \frac{2}{r}\right) \|A\|$$

A direct proof of the paving conjecture

Leake, Ravichandran '16

$r \geq 4$ integer. Then for every **Hermitian matrix** A with **zero diagonal** there exists partition $S_1 \sqcup \cdots \sqcup S_{r^2} = [n]$ such that

$$\forall k \in [r^2] : \|A[S_k]\| \leq \left(\frac{r-2}{r(r-1)} + 2\sqrt{\frac{r-2}{r(r-1)}} \right) \|A\|$$

Notations

$$\partial^A p \coloneqq \left(\prod_{i \in A} \partial_i\right) p$$

$$\bar{p}(x) = p(x, \ldots, x)$$

$$\mathrm{M}(q) \coloneqq \max_i |\lambda_i|$$

Paving real stable polynomials

Alishahi, B. '20

$r \geq 4$ integer. Then for every **multi-affine real stable polynomial** $p(z) = \sum_{A \subseteq [n]} a_A \boldsymbol{z}^A$ with $\boldsymbol{a}_\emptyset = \boldsymbol{1}$ and $\boldsymbol{a}_{\{\boldsymbol{i}\}^c} = \boldsymbol{0}$ **for all** $\boldsymbol{i} = \boldsymbol{1}, \dots, \boldsymbol{n}$ there exists a partition $S_1 \sqcup \cdots \sqcup S_{r^2} = [n]$ such that

$$\forall k \in [r^2] : \mathrm{M}\left(\overline{\partial^{S_k^c} p} \right) \leq \left(\frac{r-2}{r(r-1)} + 2\sqrt{\frac{r-2}{r(r-1)}} \right) \mathrm{M}(\bar{p})$$

Paving real stable polynomials

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$$p(\boldsymbol{z}) = \det(Z - A) \quad \Longrightarrow \quad \text{Leake-Ravichandran result}$$

Back to strongly Rayleigh measures

PGP is no good!

Comparison with determinantal measures

	✗	✓
Determinantal:	$\det(KZ + I - K)$	$\det(Z - K)$
Strong Rayleigh:	$p_{\mathfrak{X}}(\boldsymbol{z})$	?

Kernel polynomial

$$? = g_{\mathfrak{X}}(z_1, \dots, z_n) = z_1 \dots z_n \, p_{\mathfrak{X}}\left(1 - \frac{1}{z_1}, \dots, 1 - \frac{1}{z_n}\right)$$

$$\Downarrow$$

$$g_{\mathfrak{X}}(\boldsymbol{z}) = \sum_{A \subseteq [n]} (-1)^{|A|} \mathbb{P}(A \subseteq \mathfrak{X}) \, \boldsymbol{z}^{A^c}$$

Properties of the kernel polynomial

- $g_{\mathfrak{X}}$ is (multi-affine) real stable
- $g_{\mathfrak{X}}$ is kernel $\Longleftrightarrow$ the roots of $\overline{g_{\mathfrak{X}}}$ in $[0,1]$
- & they play an important role (e.g. $|\mathfrak{X}| \sim I_{\lambda_1} + \cdots + I_{\lambda_n}$)
- $\mathfrak{X} \cap A$ is strongly Rayleigh with kernel $\partial^{A^c} g_{\mathfrak{X}}$

Proof (Part 1)

$$f_{\mathfrak{X}}(\boldsymbol{z}) \coloneqq g_{\mathfrak{X}}(z_1 + p_1, \ldots, z_n + p_n) = \sum_{A \subseteq [n]} a_A \boldsymbol{z}^A$$

$-p_i \coloneqq -\mathbb{P}(i \in \mathfrak{X})$ is the coefficient of $\boldsymbol{z}^{\{i\}^c}$

Proof (Part 1)

$$f_{\mathfrak{X}}(\boldsymbol{z}) \coloneqq g_{\mathfrak{X}}(z_1 + p_1, \dots, z_n + p_n) = \sum_{A \subseteq [n]} a_A \boldsymbol{z}^A$$

↓

- $f_{\mathfrak{X}}$ is multi-affine real stable
- $a_\emptyset = 1$
- $a_{\{i\}^c} = 0$ for all $i = 1, \dots, n$
- $\mathrm{M}\left(\overline{f_{\mathfrak{X}}}\right) \le 1$

Proof (Part 1)

Apply the paving theorem to $f_{\mathfrak{X}}$

$$\Downarrow$$

$$\begin{cases} \{1, \dots, n\} = S_1 \sqcup \cdots \sqcup S_r \\ \forall k \in [r] : \mathrm{M}\left(\overline{f_{\mathfrak{X} \cap S_k}} \right) \le \varepsilon \end{cases}$$

$$\Downarrow ?$$

$\mathfrak{X} \cap S_k$ is almost independent

Problems

- How to deduce a probabilistic statement?
- How to deal with the strongly Rayleigh case?

Connection between the entropy of $\mathfrak{X}$ and the roots of the kernel polynomial!

Entropy

$$H(X) = -\sum_x \mathbb{P}(X = x) \log \mathbb{P}(X = x)$$

$$\mathfrak{X} = (X_1, \dots, X_n)$$

$$\Downarrow$$

$$H(\mathfrak{X}) \leq \underbrace{H(X_1) + \cdots + H(X_n)}_{H(\hat{\mathfrak{X}})}$$

$H(\hat{\mathfrak{X}})$ → Independent version of $\mathfrak{X}$

An entropy lower bound

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$\mathfrak{X}$ strongly Rayleigh with kernel g and $\lambda_1, \dots, \lambda_n$ the roots of g. Then

$$H(\mathfrak{X}) \geq \sum_i h(\lambda_i)$$

$$\Downarrow$$

$$H(\mathfrak{X}) \geq H\left(I_{\lambda_1}, \dots, I_{\lambda_n}\right) \geq H(|\mathfrak{X}|)$$

Paving strongly Rayleigh processes

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For every $\delta > 0$ there exists an integer r such that for every SR process $\mathfrak{X}$ it is possible to partition its underlying space into r subsets $S_1, \ldots, S_r$ such that

$$\forall k \in [r] : \left| H(\mathfrak{X} \cap S_k) - H\left(\widehat{\mathfrak{X}} \cap S_k\right) \right| \leq |S_k| \delta$$

↓

$$\forall k \in [r] : \left| \overline{H}(\mathfrak{X} \cap S_k) - \overline{H}\left(\widehat{\mathfrak{X}} \cap S_k\right) \right| \leq \delta$$

Open Questions

- Applications?
- Is the distribution of $\mathfrak{X}$ majorized by the distribution of $I_{\lambda_1}, \dots, I_{\lambda_n}$?
  (True for determinantal measure)
- Is there a "fine" probabilistic interpretation for $I_{\lambda_1}, \dots, I_{\lambda_n}$?
  (We want a mixture like $\mathfrak{X} \sim \mathbb{E}_I[\mathfrak{X}_I]$, where $\mathfrak{X}_I$ are SR with fixed size and hopefully have a "nice form".)

Thank you!